MA4H4 Geometric Group Theory

Exercise sheet 2

February 10, 2017

1. Let F_n be generated by the letters $\{x_1, x_2, \ldots, x_n\}$. Show that $[F_n, F_n]$ consists exactly of those elements whose reduced word representative contains an equal number of x_i's as x_i^{-1}'s, for each i.

2. Show that $F_n/[F_n, F_n] \cong \mathbb{Z}^n$.

3. Let $S = \{x_i\}_{i \in \mathbb{N}}$ be a countably infinite set indexed by $\mathbb{N}$. Let R be the set of relations of the form $x_{i+1} = x_j x_i x_j^{-1}$, for $i, j \in \mathbb{N}$ with $j < i$. Let $T = \langle S \mid R \rangle$ be the "Thompson group". (This is actually Thompson group F - there are also Thompson's group T and V). Show that:

$$T = \langle x_0, x_1 \mid [x_0^{-1} x_1, x_0 x_1 x_0^{-1}] = [x_0^{-1} x_1, x_0^2 x_1 x_0^{-2}] = 1 \rangle.$$

   (Hint: First show that all the x_n, for $n \geq 2$, can be expressed in terms of x_0 and x_1. Write some expressions for x_3 and x_4 and conjugate the appropriate ones by x_0 or x_0^2 to show that the required commutativity relations hold. Finally, show that these imply the original relations).

   This shows that T is indeed finitely presented.

4. Draw the Cayley graph of the dihedral group D_n with presentation

$$\langle a, b \mid a^n = b^2 = abab = 1 \rangle$$

   and the infinite dihedral group D_∞ with presentation

$$\langle a, b \mid b^2 = abab = 1 \rangle.$$

5. Draw the Cayley graph of the alternating group A_4 with generators (123) and $(12)(34)$.

6. The *triangle group* $\Delta(p, q, r)$ has presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle.$$

   (a) Draw the Cayley graph of the icosahedral group $\Delta(2, 3, 5)$.

   (b) Classify the triples (p, q, r) for which $1/p + 1/q + 1/r = 1$. What do their Cayley graphs look like?

   (c) Try to find a relationship between the Cayley graph of $\Delta(p, q, r)$ and a tessellation of a suitable space with congruent triangles. Why should the triangle groups with $1/p + 1/q + 1/r > 1$ be finite?

7. Let S be a finite generating set for G and suppose $S' \subseteq S$. We can form a subgraph $\Delta(G; S') \subseteq \Delta(G; S)$. Describe the connected components of $\Delta(G; S')$.

8. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ (with S_1 and S_2 disjoint). The *free product* $G_1 * G_2$ has presentation $\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$. (Check that the definition doesn't actually depend on the presentations of G_1 and G_2).

   (a) What is $\mathbb{Z} * \mathbb{Z}$?

   (b) Draw the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_3$ (using their standard presentations).

   (c) Describe (informally) the Cayley graph of a free product. (You may take the factor subgroups to be finite for concreteness).

   (d) Discuss whether the following statement should be true: every element of $G_1 * G_2$ can be written uniquely as an alternating product of nontrivial elements of G_1 and G_2.