MA4H4 Geometric Group Theory

Exercise sheet 1

May 19, 2011

1. Show that $\mathbb{Z}^n \cong \mathbb{Z}^m$ implies $n = m$.

2. Show that a finitely generated torsion-free abelian group is isomorphic to $\mathbb{Z}^n$, for some $n \geq 0$.

3. Prove that any finitely generated subgroup of $(\mathbb{Q}\ , +)$ is either trivial or infinite cyclic.

4. Prove the following statements:

   (a) If $N \triangleleft \Gamma$ and Γ is f.g., then Γ/N is f.g.

   (b) If $N \triangleleft \Gamma$, N is f.g. and Γ/N is f.g., then Γ is f.g.

   (c) If $G \leq \Gamma$ and $[\Gamma : G] < \infty$ ("finite index") then Γ is f.g. if and only if G is f.g.

5. Show that if $F(B)$ is f.g. then B is also finite.

6. Prove that every element in $F(S)$ can be written uniquely as a reduced word in $S \sqcup S^{-1}$.

7. Let $F_2 = \langle a, b \rangle$. Let $S = \{b^n a b^{-n} \mid n \in \mathbb{N}\}$. Show (combinatorially) that $\langle S \rangle$ is freely generated by S. Deduce that $\langle S \rangle$ is not f.g. (Note that $\langle S \rangle$ is isomorphic to $F(S)$, since they are both freely generated by S.)

8. Show that there are only countably many f.p. groups up to isomorphism.

9. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ be matrices in $\mathrm{SL}(2, \mathbb{R})$. Let

$$X = \{(x, y) \in \mathbb{R}^2 \mid |x| > |y|\} \text{ and } Y = \{(x, y) \in \mathbb{R}^2 \mid |x| < |y|\}.$$

(a) Describe the sets $A^n(Y)$ and $B^n(X)$ for $n \neq 0$. How do these sets relate to X and Y?

(b) Show that

$$W = A^{n_1} B^{n_2} A^{n_3} \ldots A^{n_{k-2}} B^{n_{k-1}} A^{n_k}$$

is not the identity, where $k \geq 1$ and $n_i \neq 0$ for all i. (Hint: Look at $W(Y)$).

(c) Deduce that all non-trivial reduced words in A and B do not represent the identity. (There are 4 cases: the words must begin with either an A or B and end with either an A or B).

(d) Hence, show that A and B generate a free subgroup of rank 2 in $\mathrm{SL}(2, \mathbb{R})$.

Remark: This is an example of the *table tennis lemma* or (ping-pong lemma) - a method often used in geometric group theory to prove that a certain set of elements of a group freely generate a free subgroup. Formulate a statement and prove it. Try to generalise to the case of n generators.