Exercises 4

(1) Show that if the second fundamental form of a surface $r : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is identically 0, then the surface is a plane.

A surface is a plane if and only if it has a constant normal vector.

Now, we have
$$e = r_{uu} \cdot n \equiv 0 = -r_u \cdot n_u$$
$$f = r_{uv} \cdot n \equiv 0 = -r_u \cdot n_v = -r_v \cdot n_u$$
$$g = r_{vv} \cdot n \equiv 0 = -r_v \cdot n_v$$

We further have $n \cdot n \equiv 1 \Rightarrow n \cdot n_u,\ n \cdot n_v \equiv 0.$

So n_u, n_v are always tangent vectors.

Thus:
$$n_u = (n_u \cdot r_u) r_u + (n_u \cdot r_v) r_v$$
$$= 0$$
$$n_v = (n_v \cdot r_u) r_u + (n_v \cdot r_v) r_v$$
$$= 0$$

$n_u, n_v \equiv 0 \Rightarrow n$ is constant

$\Rightarrow r$ must be a plane.

(2) Show that the Gauss curvature of the graph $(u,v) \mapsto (u, v, F(u,v))$ is given by:

$$K = \frac{f_{uu} f_{vv} - (f_{uv})^2}{\left(1 + (f_u)^2 + (f_v)^2\right)^2}$$

We use the fact that $K = \kappa_1 \kappa_2$ where κ_1, κ_2 are the principal curvatures [the eigenvalues of the shape operator].

As a linear map, the shape operator is given by

$$S_p = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

The determinant of a matrix is invariant under conjugation (in this case changing the basis to the basis of eigenvectors), so we have:

$$\left| \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \right| = \kappa_1 \kappa_2$$

i.e. $$K = \frac{eg - f^2}{EG - F^2}$$

Now:

$$\left.\begin{aligned} r_u &= (1, 0, f_u) \\ r_v &= (0, 1, f_v) \end{aligned}\right\} \Rightarrow E = 1 + f_u^2 \quad F = f_u f_v \quad G = 1 + f_v^2$$

$$r_u \times r_v = (-f_u, -f_v, 1) \quad \} \Rightarrow EG - F^2 = \|r_u \times r_v\|^2 = f_u^2 + f_v^2 + 1$$

$$\eta = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

$$\left.\begin{aligned} r_{uu} &= (0, 0, f_{uu}) \\ r_{uv} &= (0, 0, f_{uv}) \\ r_{vv} &= (0, 0, f_{vv}) \end{aligned}\right\} \Rightarrow e = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}} \quad f = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}} \quad g = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}$$

$$\text{so: } \quad K = \frac{f_{uu}f_{vv} - f_{uv}^2}{1 + f_u^2 + f_v^2} \bigg/ f_u^2 + f_v^2 + 1$$

$$= \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

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(3) If K is the Gauss curvature of the surface $r: U \to \mathbb{R}^3$, show that: $n_u \times n_v = K(r_u \times r_v)$

We use the identity $(a \times b) \cdot (v \times w) = (a \cdot v)(b \cdot w) - (b \cdot v)(a \cdot w)$

n is given by $\dfrac{r_u \times r_v}{\| r_u \times r_v \|}$.

and we note that:
$$e = n \cdot r_{uu} = -n_u \cdot r_u$$
$$f = n \cdot r_{uv} = -n_u \cdot r_v = -n_v \cdot r_u$$
$$g = n \cdot r_{vv} = -n_v \cdot r_v$$

Now, it is clear that $n_u \times n_v$ and $r_u \times r_v$ are linearly independent - the tangent plan is the span of $\begin{cases} n_u, n_v \\ r_u, r_v \end{cases}$.

Therefore we require to show that:

$$(n_u \times n_v) \cdot (r_u \times r_v) = K \| r_u \times r_v \|^2$$

(By the identity above)
$$(n_u \cdot r_u)(n_v \cdot r_v) - (n_v \cdot r_u)(n_u \cdot r_v) = K(EG - F^2)$$

$$(-e)(-g) - (-f)(-f) = K(EG - F^2)$$

or that $\quad K = \dfrac{eg - f^2}{EG - F^2} \quad$ - which we know.

(4) Derive the formula for mean curvature

$$H = \frac{Ge + Eg - 2Ff}{2(EG - F^2)}$$

By definition, $H = \frac{1}{2}(\kappa_1 + \kappa_2)$, where κ_1, κ_2 are the principal curvatures. As was observed in Q2 we note that the trace of a matrix is preserved under conjugation

$$\Rightarrow \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2} \operatorname{Tr}\left(\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \right)$$

$$= \frac{1}{2(EG - F^2)} \cdot (eG - fF - fF + gE)$$

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(5) Let $r: U \to \mathbb{R}^3$ be a parametrised surface, and let r_t be the <u>normal displacement</u>: $r_t(u,v) = r(u,v) - t\,n(u,v)$.

We assume this is a regular surface for sufficiently small t. Show that:

$$\frac{d}{dt}\Big|_{t=0} A(r_t(U)) = 2\int_U H \, dA$$

$$\frac{d}{dt} A(r_t(U)) = \frac{d}{dt} \int_U dA_t$$

$$= \int_U \frac{d}{dt} \sqrt{E_t G_t - F_t^2} \, du \, dv$$

Now $\frac{\partial r_t}{\partial u} = \frac{\partial r}{\partial u} - t \frac{\partial n}{\partial u}$ and $\frac{\partial r_t}{\partial v} = \frac{\partial r}{\partial v} - t \frac{\partial n}{\partial v}$

So $E_t = E - 2t \frac{\partial r}{\partial u} \cdot \frac{\partial n}{\partial u} + t^2 \left\| \frac{\partial n}{\partial u} \right\|^2$

$F_t = F - t \frac{\partial n}{\partial u} \cdot \frac{\partial r}{\partial v} - t \frac{\partial n}{\partial v} \cdot \frac{\partial r}{\partial u} + t^2 \frac{\partial n}{\partial u} \cdot \frac{\partial n}{\partial v}$

$G_t = G - 2t \frac{\partial r}{\partial v} \cdot \frac{\partial n}{\partial v} + t^2 \left\| \frac{\partial n}{\partial v} \right\|^2$

i.e. $E_t = E + 2te + O(t^2)$

$F_t = F + 2tF + O(t^2)$

$G_t = G + 2tg + O(t^2)$

And so $dA_t = \sqrt{EG - F^2 + 2t(eG + gE - 2FF) + O(t^2)}$

$$\frac{d}{dt} dA_t = \frac{eG + gE - 2FF + O(t) \; du dv}{\sqrt{EG - F^2 + 2t(eG + gE - 2FF) + O(t^2)}}$$

evaluated at $t = 0$:

$$= \frac{eG + gE - 2FF \; dudv}{\sqrt{EG - F^2}} = \frac{eG + gE - 2FF}{EG - F^2} \sqrt{EG - F^2} \, dudv$$

$$= 2H \, dA.$$

i.e. $\left.\frac{d}{dt}\right|_{t=0} A(r_t(u)) = 2 \int_u H \, dA$

(6) Suppose $r : U \longrightarrow \mathbb{R}^3$ is a surface, and that the coordinate chart is conformal (i.e. the first fundamental form is given by:

$$\lambda(u,v)\,(du^2 + dv^2)$$

for some smooth $\lambda : U \longrightarrow \mathbb{R}$)

- Show that $r : U \longrightarrow \mathbb{R}^3$ is minimal iff $r_{uu} + r_{vv} = 0$
- Deduce that the catenoid and helicoid are minimal surfaces
- Are the intermediate surfaces in the "morph" from catenoid to helicoid described in [Exercises 3 Q2] minimal?

We have that:

$$E = \lambda = \|r_u\|^2 \qquad F = 0 = r_u \cdot r_v \qquad G = \lambda = \|r_v\|^2$$

$$n = \frac{r_u \times r_v}{\|r_u \times r_v\|} = \frac{r_u \times r_v}{\sqrt{EG - F^2}} = \frac{r_u \times r_v}{\lambda}$$

So $e = r_{uu} \cdot \dfrac{r_u \times r_v}{\lambda}$, $f = r_{uv} \cdot \dfrac{r_u \times r_v}{\lambda}$ $\quad g = r_{vv} \cdot \dfrac{r_u \times r_v}{\lambda}$

- r is minimal iff $H \equiv 0$.

$$H = \frac{Ge + Eg - 2fF}{2(EG - F^2)} = \frac{\lambda(e+g)}{2\lambda^2}$$

$$= \frac{e+g}{2\lambda}$$

$$= \frac{1}{2}(r_{uu} + r_{vv}) \cdot n$$

~~r_{uu} and r_{vv} are~~ $\therefore$ $r_{uu} + r_{vv}$ is not a tangent vector (*)

$\therefore \quad H = 0 \iff r_{uu} + r_{vv} = 0$

Recall that the family $r_\theta(u,v) = \begin{pmatrix} \cos\theta \cosh u \cos v + \sin\theta \sinh u \sin v \\ \cos\theta \cosh u \sin v - \sin\theta \sinh u \cos v \\ u\cos\theta + v\sin\theta \end{pmatrix}$

defines the Helicoid for $\theta = \pi/2$
— " — Catenoid for $\theta = 0$

and that $\forall \theta$, $E = G = \cosh^2 u$ $\quad F = 0$. So r_θ is a conformal family.

Now, $$\frac{\partial^2}{\partial u^2} r_\theta = \begin{pmatrix} \cos\theta \cosh u \cos v + \sin\theta \sinh u \sin v \\ \cos\theta \cosh u \sin v - \sin\theta \sinh u \cos v \\ 0 \end{pmatrix}$$

$$\frac{\partial^2}{\partial v^2} r_\theta = \begin{pmatrix} -\cos\theta \cosh u \cos v - \sin\theta \sinh u \sin v \\ -\cos\theta \cosh u \sin v + \sin\theta \sinh u \cos v \\ 0 \end{pmatrix}$$

so it is clear to see that for each θ, r_θ is a minimal surface.

(*) If it were, then:

$$r_{uu} + r_{vv} = (r_{uu} + r_{vv})\cdot r_u \, r_u + (r_{uu} + r_{vv})\cdot r_v \, r_v$$

But: $r_u \cdot r_u = r_v \cdot r_v = \lambda$ $\quad$ and $\quad$ $r_u \cdot r_v = 0$

$\Rightarrow r_{uu}\cdot r_u = r_{vu}\cdot r_v$ $\quad\quad$ $\Rightarrow r_{uu}\cdot r_v = -r_u \cdot r_{vu}$

$\quad r_{vv}\cdot r_v = r_{uv}\cdot r_u$ $\quad\quad\quad$ $r_{vv}\cdot r_u = -r_v\cdot r_{uv}$

and so both coefficients are 0.