Exercises 3

(1) Compute the first fundamental forms of the following parametrised surfaces:

Ellipsoid: $r(u,v) = (a\cos u\cos v,\ b\sin u\cos v,\ c\sin v)$

$r_u = (-a\sin u\cos v,\ b\cos u\cos v,\ 0)$
$r_v = (-a\cos u\sin v,\ -b\sin u\sin v,\ c\cos v)$

$[\text{FFF} = E\,du^2 + 2F\,du\,dv + G\,dv^2]$

$E = \cos^2 v\,(a^2\sin^2 u + b^2\cos^2 u)$
$F = (a^2-b^2)\cos u\cos v\sin u\sin v$
$G = \sin^2 v\,(a^2\cos^2 u + b^2\sin^2 u) + c^2\cos^2 v$

Hyperbolic Paraboloid: $r(u,v) = (au\cosh v,\ bu\sinh v,\ u^2)$

$E = a^2\cosh^2 v + b^2\sinh^2 v + 4u^2$
$F = u(a^2+b^2)\cosh v\sinh v$
$G = u^2(a^2\sinh^2 v + b^2\cosh^2 v)$

Elliptic Paraboloid: $r(u,v) = (au\cos v,\ bu\sin v,\ u^2)$

$E = a^2\cos^2 v + b^2\sin^2 v + 4u^2$
$F = u(b^2-a^2)\cos v\sin v$
$G = u^2(a^2\sin^2 v + b^2\cos^2 v)$

Two-Sheeted Hyperboloid: $r(u,v) = (a\sinh u\cos v,\ b\sinh u\sin v,\ c\cosh u)$

$E = \cosh^2 u\,(a^2\cos^2 v + b^2\sin^2 v) + c^2\sinh^2 u$
$F = (b^2-a^2)\cosh u\sinh u\cos v\sin v$
$G = \sinh^2 u\,(b^2\cos^2 v + a^2\sin^2 v)$

(2) Fix $\theta \in \mathbb{R}$ and let r_θ be the parametrised defined by: $r_\theta(u,v) = \begin{pmatrix} \cos\theta\cosh u\cos v + \sin\theta\sinh u\sin v \\ \cos\theta\cosh u\sin v - \sin\theta\sinh u\cos v \\ u\cos\theta + v\sin\theta \end{pmatrix}$

Calculate the first fundamental form of r_θ, and show r_θ is isometric to r_ϕ for all θ, ϕ.
What are the surfaces when $\theta = 0, \frac{\pi}{2}$?

$E = \cosh^2 u \qquad F = 0 \qquad G = \cosh^2 u$

[lots of calculation missed out here!]

Note that the coefficients do not depend on θ. So each surface r_θ is isometric to r_ϕ.

When $\theta = 0$ we have the Catenoid
$\theta = \pi/2$ we have the Helecoid.

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(3) Let r be a regular surface. The "coordinate curves" are curves of the form $\begin{cases} u \mapsto r(u,v) & \text{for fixed } v \\ v \mapsto r(u,v) & \text{for fixed } u. \end{cases}$

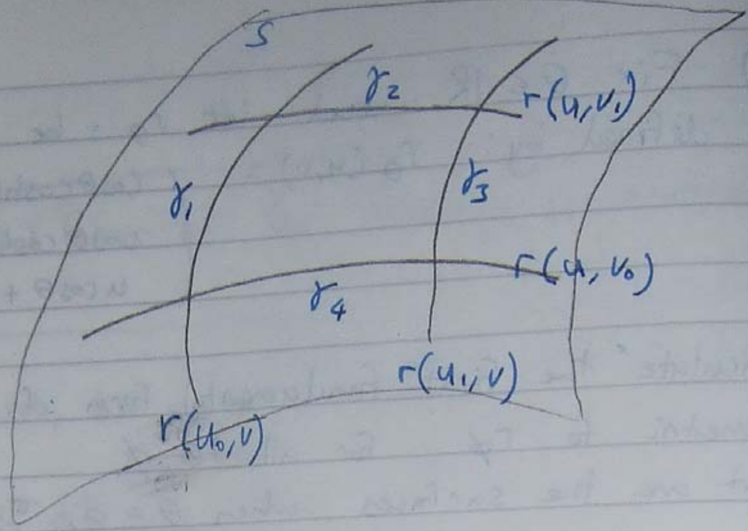
Show that the coordinate curves are regular.
Show that $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0 \iff$ any quadrilateral formed out of coordinate curves has opposite sides of equal length.
In this case, show that we can reparametrise so that the first fundamental form is $du^2 + 2\cos\theta\, du\, dv + dv^2$

A regular curve has nowhere stationary derivative.
A regular surface has linearly independent partial derivatives; in particular r_u, r_v are nowhere 0.

It is clear then, that the coordinate curves are regular.

We need to see when the conditions

$\ell(\gamma_1) = \ell(\gamma_3)$ and
$\ell(\gamma_2) = \ell(\gamma_4)$ hold.



$$\ell(\gamma_1) = \int_{v_0}^{v_1} \left\| \frac{\partial r}{\partial v}(u_0, v) \right\| dv$$

$$\ell(\gamma_3) = \int_{v_0}^{v_1} \left\| \frac{\partial r}{\partial v}(u_1, v) \right\| dv$$

So equality holds iff $\left\| \frac{\partial r}{\partial v}(u,v) \right\|$ is independent of u.

But $\left\| \frac{\partial r}{\partial v} \right\| = \sqrt{\frac{\partial r}{\partial v} \cdot \frac{\partial r}{\partial v}} = \sqrt{G}$.

The condition then becomes $\frac{\partial}{\partial u} \sqrt{G} = 0$

Similarly, $\ell(\gamma_2) = \ell(\gamma_4)$ iff $\frac{\partial}{\partial v} \sqrt{E} = 0$

Finally, if $E = E(u)$, $G = G(v)$ then we are free to reparametrise so that $E = G = 1$.

$$F = \frac{\partial r}{\partial \tilde{u}}(\tilde{u}, \tilde{v}) \cdot \frac{\partial r}{\partial \tilde{v}}(\tilde{u}, \tilde{v}) = \left\| \frac{\partial r}{\partial \tilde{u}} \right\| \left\| \frac{\partial r}{\partial \tilde{v}} \right\| \cos\theta$$

$$= \cos\theta$$

(4) show the ellipsoid $\{ (x,y,z) : \frac{x^2}{a^2} + y^2/b^2 + z^2/c^2 = 1 \}$
$a, b, c \in \mathbb{R}$

is an embedded regular surface by describing explicit charts.

Ditto (for $c \neq 0$) $\{(x,y,z) : x^2+y^2-z^2 = c\}$

$r(u,v) = (a \cos u \cos v, \; bu \sin u \cos v, \; c \sin v)$ is a chart.

[a chart in the GC & S context]
Elsewhere - eg. Manifolds - a chart is $\varphi : U \longrightarrow \mathbb{R}^2$, $U \subset S$

In this case there is no universal chart. Would need to take 2 charts, for example, defined by stereographic projection.

When $c > 0$ $(c = c_1^2)$

$r(u,v) = (c_1 \cosh u \cos v, \; c_1 \cosh u \sin v, \; c_1 \sinh u)$

When $c < 0$ $(c = -c_2^2)$

$r(u,v) = (c_2 \sinh u \cos v, \; c_2 \sinh u \sin v, \; c_2 \cosh u)$

Note that when $c=0$, we get a cone.

This is clearly not a regular surface, as removing the origin in disconnects it!

(5) Show any two ellipsoids are diffeomorphic

We will show any ellipsoid is diffeomorphic to the unit sphere S^2.

$$S^2 \longrightarrow \text{Ellipsoid}$$
$$(x,y,z) \longmapsto (ax, by, cz)$$

This is clearly smooth, and has smooth inverse:

$$\text{Ellipsoid} \longrightarrow S^2$$
$$(\bar{x},\bar{y},\bar{z}) \longmapsto \left(\tfrac{\bar{x}}{a}, \tfrac{\bar{y}}{b}, \tfrac{\bar{z}}{c}\right)$$

(6) Show the image of a surface of revolution of an injective curve is an embedded surface in $\mathbb{R}^3$. Show any two such images are diffeomorphic.

Let $\alpha(u)$ be the injective curve. Note $\frac{d}{du}\alpha$ is never 0 [or else α not injective]. $\alpha = (x(u), y(u))$

The surface of revolution is given by:

$$r_\alpha(u,v) = (x(u)\cos v,\ x(u)\sin v,\ y(u))$$

$$\frac{\partial r_\alpha}{\partial u} = (x'(u)\cos v,\ x'(u)\sin v,\ y'(u))$$

$$\frac{\partial r_a}{\partial v} = (-x(u)\sin v,\ x(u)\cos v,\ 0)$$

These partial derivatives are non-zero when $(x,y) \neq (0,0)$ and $(x',y') \neq (0,0)$.

Further, they are only linearly dependent if: $y'(a) = 0$

$\Rightarrow x'(u) \neq 0$

$\Rightarrow \dfrac{x(u)}{x'(u)} = \dfrac{-\cos v}{\sin v} \cdot \lambda$ and $\dfrac{x(u)}{x'(u)} = \dfrac{\sin v \, \lambda}{\cos v}$

i.e. when $-\frac{1}{\tan v} = \tan v$

$\iff -1 = \tan^2 v$

i.e. never.

So the surface of revolution is an embedded surface!

Finally, it is known that any two smooth injective curves in $\mathbb{R}^2$ are diffeomorphic. — the implicit function theorem guarantees we can write a curve $\alpha = (x(u), y(u))$ in the form $\alpha = (x(u), y(x))$

or $(x(y), y(u))$

and so a diffeo:

$\alpha \ni (x, y_1(x)) \longmapsto \beta \ni (x, y_2(x))$

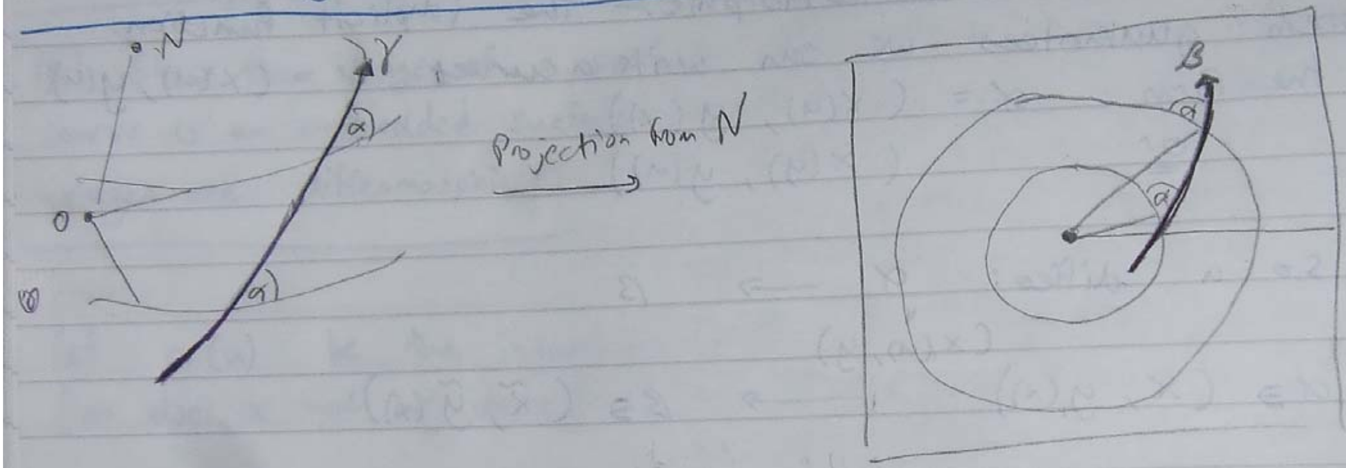
exists as y_1, y_2 are both diffeos $\mathbb{R} \to \mathbb{R}$.

(7) A loxodrome on the sphere is a path that makes a constant non-zero angle $\alpha \in (-\pi, \pi)$ with every latitude θ. [i.e. constant compass bearing].

Show that the image of a loxodrome under stereographic projection is a logarithmic spiral.

Calculate the total length of a loxodrome pole to pole.

Show that the interior angles of a triangle made from 3 loxodrome segments adds up to π.



N.B. That stereographic projection is conformal – it preserves angles.

In polar coordinates, $\beta = \pi_N(\gamma)$ is of the form $(r(\theta), \theta)$

Calculate $\frac{dr}{d\theta}$:

$$\beta(\theta) = (r\cos\theta, r\sin\theta)$$

$$\beta'(\theta) = (-r\sin\theta + r'\cos\theta, r\cos\theta + r'\sin\theta)$$

$$\|\beta'(\theta)\| = \sqrt{r^2 + r'^2}$$

The tangent vector makes a constant angle with the vector $\hat{\theta} = (-\sin\theta, \cos\theta)$

i.e. $\hat{T} \cdot \hat{\theta} = \cos\alpha$

$$\cos\alpha = \frac{1}{\sqrt{r^2 + r'^2}} \left(r' \sin^2\theta - r' \sin\theta\cos\theta + r\cos^2\theta + r' \sin\theta\cos\theta \right)$$

$$= \frac{r}{\sqrt{r^2 + r'^2}}$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{r^2}{\cos^2\alpha} - r^2$$

$$\text{so } \frac{dr}{d\theta} = \frac{r}{\cos\alpha}\sqrt{1-\cos^2\alpha} = r\tan\alpha$$

$$\Rightarrow r(\theta) = e^{\tan\alpha\,\theta}$$

So the image is a logarithmic spiral.

We apply the inverse stereographic projection to gain an equation for the loxodrome γ

$$\gamma(\theta) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right)$$

$$= \frac{1}{1+r^2} \left(2r\cos\theta, 2r\sin\theta, r^2-1 \right)$$

$$\gamma'(\theta) = \left[\frac{(2r\cos\theta, 2r\sin\theta, r^2-1) - 2rr'}{(1+r^2)^2} + \frac{\begin{pmatrix} 2r'\cos\theta & 2r'\sin\theta & \\ -2r\sin\theta, & +2r\cos\theta, & 2rr' \end{pmatrix}}{1+r^2} \right.$$

$$\|\gamma'(\theta)\|^2 = \frac{4r^2(r')^2}{(1+r^2)^4}\left[4r^2 + (r^2-1)^2\right]$$

$$\pm \frac{8rr'}{(1+r^2)^3}\left[4rr' \qquad + 2rr'(r^2-1)\right]$$

$$+ \frac{1}{(1+r^2)^2}\left[4(r^2+(r')^2 + (rr')^2)\right]$$

$$= \frac{4(r^2+(r')^2)}{(1+r^2)^2}$$

So length $= 2\int_{-\infty}^{\infty} \frac{e^{\tan\alpha\theta}\sqrt{1+\tan^2\alpha}}{1+e^{2\tan\alpha\theta}}\,d\theta$

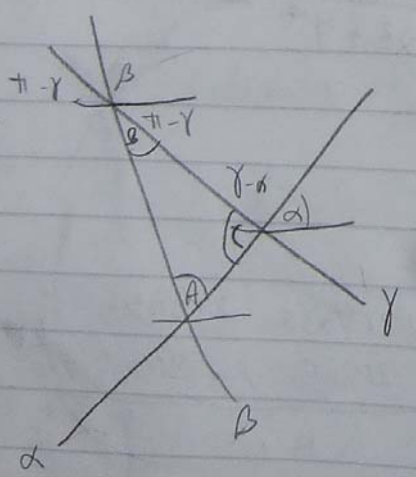
$e^{\tan\alpha\theta} = u \qquad \frac{du}{d\theta} = \tan\alpha\, e^{\tan\alpha\theta}$

$\theta = \frac{\log u}{\tan\alpha}$

$$= 2\int_{-\infty}^{\infty} \frac{u \cdot \frac{1}{\cos\alpha}}{1+u^2}\,\frac{du}{\tan\alpha\, u}$$

$$= \frac{2}{\sin\alpha}\int_{-\infty}^{\infty}\frac{du}{1+u^2} = \frac{2}{\sin\alpha}\arctan u\,\Big|_{-\infty}^{\infty}$$

$$= \frac{2\pi}{\sin\alpha}$$



Angle $A = \beta - \alpha$

Angle $B = \gamma - \beta$

Angle $C = \pi + \alpha - \gamma$

Sum $= \pi$