MA3D9: Geometry of curves and surfaces

Exercises 6.

(1) Let $\mathbf{x}, \mathbf{y}$ be tangential vector fields to the surface S. Show that $D_{\mathbf{x}}\mathbf{y} - D_{\mathbf{y}}\mathbf{x}$ is also tangential, and deduce that this is equal to $\nabla_{\mathbf{x}}\mathbf{y} - \nabla_{\mathbf{y}}\mathbf{x}$.
Write $[\mathbf{x}, \mathbf{y}] = \nabla_{\mathbf{x}}\mathbf{y} - \nabla_{\mathbf{y}}\mathbf{x}$. (The "Lie bracket".)
Show that if $(u, v) \mapsto \mathbf{r}(u, v)$ is a chart, show that $[\mathbf{r}_u, \mathbf{r}_v] = 0$.

Suppose that $f : S \longrightarrow \Sigma$ is a a diffeomorphism, and f_* is the derivative of f at a point p. Show that $f_*[\mathbf{x}, \mathbf{y}] = [f_*\mathbf{x}, f_*\mathbf{y}]$.

(2) Suppose that $\mathbf{x}$ is a tangential vector field on S. Show that if $\nabla_{\mathbf{x}} f = 0$ for every smooth function $f : S \longrightarrow \mathbf{R}$, then $\mathbf{x} = 0$ everywhere. [You may want to use the construction of a smooth function on $\mathbf{R}$ that is positive on $(-1, 1)$ and 0 everywhere else.] Deduce that $\mathbf{x}$ is determined by the operation of $\nabla_{\mathbf{x}}$ on smooth functions.
Show that if $\mathbf{x}, \mathbf{y}$ are smooth vector fields, and f a smooth function, then

$$\nabla_{[\mathbf{x},\mathbf{y}]} f = \nabla_{\mathbf{x}}\nabla_{\mathbf{y}} f - \nabla_{\mathbf{y}}\nabla_{\mathbf{x}} f.$$

Suppose that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vector fields. Show that:

$$[[\mathbf{x}, \mathbf{y}], \mathbf{z}] + [[\mathbf{y}, \mathbf{z}], \mathbf{x}] + [[\mathbf{z}, \mathbf{x}], \mathbf{y}] = 0.$$

(3) Given three tangential vector fields, $\mathbf{x}, \mathbf{y}, \mathbf{z}$, write

$$R(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \nabla_{\mathbf{x}}\nabla_{\mathbf{y}}\mathbf{z} - \nabla_{\mathbf{y}}\nabla_{\mathbf{x}}\mathbf{z} - \nabla_{[\mathbf{x},\mathbf{y}]}\mathbf{z}.$$

Show that if λ, μ, ν smooth functions on S, then $R(\lambda\mathbf{x}, \mu\mathbf{y}, \nu\mathbf{z}) = \lambda\mu\nu R(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

(4) Let

$$\mathbf{r}(u, v) = (\lambda(u)\cos v, \lambda(u)\sin v, \mu(u))$$

be a surface of revolution.
Show that the first and second fundamental forms are given respectively by

$$((\lambda')^2 + (\mu')^2)du^2 + \lambda^2 dv^2$$

$$\frac{1}{\sqrt{(\lambda')^2 + (\mu')^2}}((\lambda'\mu'' - \lambda''\mu')du^2 + \lambda\mu'\, dv^2).$$

Calculate the matrix for the shape operator. Show that the principal curvatures are given by

$$\frac{\lambda''\mu' - \lambda'\mu''}{((\lambda')^2 + (\mu')^2)^{3/2}} \qquad \frac{-\mu'}{\lambda((\lambda')^2 + (\mu')^2)^{1/2}},$$

and that the Gauss curvature is given by:

$$\kappa = \frac{\lambda'\mu'\mu'' - \lambda''(\mu')^2}{\lambda((\lambda')^2 + (\mu')^2)}.$$