MA3D9: Geometry of curves and surfaces

Exercises 5.

(1) Suppose that A is a 2×2 positive semidefinite symmetric matrix (i.e. $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbf{R}^2$). Show that $\det A \geq 0$.

Suppose that B is another symmetric matrix with $B - A$ positive semidefinite. Show that B is positive semidefinite, and that $\det B \geq \det A$. (For example, diagonalise B.)

Suppose that $f, g : \mathbf{R}^2 \longrightarrow \mathbf{R}$ are smooth functions, with $g(u, v) \geq f(u, v) \geq 0$ for all $u, v \in \mathbf{R}$ and with $f(0, 0) = g(0, 0) = 0$. Let κ_f, κ_g be respectively the Gauss curvatures of the graphs of f and g at the origin. Show that $\kappa_g \geq \kappa_f \geq 0$.

(2) Let S be a regular surface, and $p \in S$. Suppose that the normal at p has non-zero component in the z-direction. Show that there is a chart $\mathbf{r} : U \longrightarrow S$ with $p \in \mathbf{r}(U)$ and with $\mathbf{r}(u, v) = (u, v, f(u, v))$ for all $(u, v) \in U$, where $f : U \longrightarrow \mathbf{R}$ is a smooth function.

Suppose that S lies on one side of its the tangent plane at p. Show that the Gauss curvature of S at p is non-negative.

Suppose that there is some $a \in \mathbf{R}^3$ such that p is a furthest point of S from a. That is, $||a - q|| \leq ||a - p||$ for all $q \in S$. Show that the Gauss curvature of S at p is at least $1/||a - p||^2$.

(3) Consider the quadratic form $\mathbf{x} \mapsto \mathbf{x}^T P\mathbf{x}$ on $\mathbf{R}^2$ given by the matrix

$$P = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Show that that maximal absolute value attained by the form for $||\mathbf{x}|| = 1$ is equal to $\max\{|\lambda|, |\mu|\}$.

Let S be a surface, and $p \in S$. Deduce that the maximal value of $|\mathbf{e}.\nabla_{\mathbf{e}}\mathbf{n}|$ for $\mathbf{e} \in T_p(S)$ with $||\mathbf{e}|| = 1$ is equal to $\max\{|\kappa_1|, |\kappa_2|\}$, where κ_1 and κ_2 are the principal curvatures.

If $\kappa_1, \kappa_2 \geq 0$, show that the minimal value is $\min\{\kappa_1, \kappa_2\}$.

Suppose that γ is a unit speed curve in S with $\gamma(t) = p$, and that the Gauss curvature of S at p is positive. Show that $|\gamma''(t).\mathbf{n}| \geq \min\{|\kappa_1|, |\kappa_2|\}$. Deduce that the curvature of γ at p is at least $\min\{|\kappa_1|, |\kappa_2|\}$.

(How does this relate to the case of the sphere in Ex. Sheet 2?)

(4) Let γ be a smooth unit-speed curve in a regular surface S. Write $\mathbf{T}$, $\mathbf{N}_S$ and $\mathbf{n}$ respectively for the tangent to γ, the normal to γ in S and the normal to S in $\mathbf{R}^3$ (so that $\{\mathbf{T}, \mathbf{N}_S, \mathbf{n}\}$ is an orthonormal basis). Let Π denote the second fundamental form on $T_{\gamma(t)}(S)$, and let γ_S be the geodesic curvature of γ in S. Show that:

$$\begin{aligned} \mathbf{T}' &= \kappa_S \mathbf{N}_S + \Pi(\mathbf{T}, \mathbf{T})\mathbf{n} \\ \mathbf{N}_S' &= -\kappa_S \mathbf{T} + \Pi(\mathbf{T}, \mathbf{N}_S)\mathbf{n} \\ \mathbf{n}' &= -\Pi(\mathbf{T}, \mathbf{T})\mathbf{T} - \Pi(\mathbf{T}, \mathbf{N}_S)\mathbf{N}_S. \end{aligned}$$