MA3D9: Geometry of curves and surfaces

Exercises 4.

(1) Show that if the second fundamental form of a surface $\mathbf{r} : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$ is identically 0, then the surface is a plane.

(2) Show that the Gauss curvature of the graph $(u, v) \mapsto (u, v, f(u, v))$ (at any point) is given by:

$$\kappa = \frac{f_{uu} f_{vv} - (f_{uv})^2}{(1 + (f_u)^2 + (f_v)^2)^2}.$$

(3) If κ is the Gauss curvature of the surface $\mathbf{r} : U \longrightarrow \mathbf{R}^3$, show that

$$\mathbf{n}_u \wedge \mathbf{n}_v = \kappa(\mathbf{r}_u \wedge \mathbf{r}_v).$$

(4) Derive the formula for mean curvature:

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)}.$$

(5) Let $\mathbf{r} : U \longrightarrow \mathbf{R}^3$ be a parameterised surface, and let $\mathbf{r}(t)$ be the normal displacement, (i.e. $\mathbf{r}(t) = \mathbf{r}(t)(u, v) = \mathbf{r}(u, v) - t\mathbf{n}(u, v)$). We assume that this is a regular surface for sufficiently small t.
Show that:

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{area}(\mathbf{r}(t)(U)) = 2 \int_U H \, dA$$

where H denotes mean curvature, and $dA = \sqrt{EG - F^2}\, dudv$ is the area element. (You may assume that differentiation with respect to t commutes with integration with respect to u, v.)

(6) Suppose that $\mathbf{r} : U \longrightarrow \mathbf{R}^3$ is a surface, and that the co-ordinate chart is conformal. (That is, the first fundamental form is given by $\lambda(u, v)(du^2 + dv^2)$ for some smooth function $\lambda : U \longrightarrow \mathbf{R}$.)
Show that $\mathbf{r}$ is a minimal surface if and only if $\mathbf{r}_{uu} + \mathbf{r}_{vv} = 0$.
[Show that the mean curvature is given by $\frac{1}{2\lambda}(\mathbf{r}_{uu} + \mathbf{r}_{vv}).\mathbf{n}$.]
Deduce that the catenoid and helicoid are minimal surfaces. Is there a more direct way to see that the helicoid is minimal?
Are the intermediate surfaces in the “morph” from a catenoid to a helicoid described in Example sheet 3 minimal surfaces?