MA3D9. Curves and surfaces.

Exercises 2.

(1) Calculate the Frenet frame, curvature and torsion of the curve $[t \mapsto (t, at^2, bt^3)]$ at the origin. Verify the Frenet-Serret formula in this case.

(2) (The helix). Consider the curve $\gamma : \mathbf{R} \longrightarrow \mathbf{R}^3$ given by

$$\gamma(s) = (a\cos(\omega s), a\sin(\omega s), b\omega s),$$

where $a > 0$, b and ω are constant.

Calculate ω, if s is an arc length parameter. Calculate the Frenet frame, curvature and torsion of γ at an arbitrary point. Show that the curvature and torsion are constant. Verify the Frenet-Serret formula in this case.

Show that, for suitable a, b we can find such a curve with any given constant torsion and constant positive curvature.

(3) Suppose that $\gamma : I \longrightarrow \mathbf{R}^3$ is a regular curve (not necessarily parameterised by arc length). Show that the curvature and torsion are given respectively by

$$\kappa = \frac{|\gamma' \wedge \gamma''|}{|\gamma'|^3}$$

and

$$\tau = \frac{(\gamma' \wedge \gamma'').\gamma'''}{|\gamma' \wedge \gamma''|^2}.$$

(4) Suppose that $\beta, \gamma : I \longrightarrow \mathbf{R}^3$ are two unit speed smooth curves. Suppose that the curvatures and torions are everywhere positive, and that $\mathbf{B}_\beta(s) = \mathbf{B}_\gamma(s)$ for all s. Show that there is a fixed $\mathbf{p} \in \mathbf{R}^n$ such that for all $s \in I$, $\gamma(s) = \beta(s) + \mathbf{p}$.

(5) Let γ is a unit-speed curve with non-zero curvature and torsion. Let $r = 1/\kappa$ and $t = 1/\tau$.

(a) If the image of γ lies in the unit sphere, show that $r^2 + (r't)^2 \equiv 1$.

[Differentiate $||\gamma(s)||^2 = 1$ three times to show that $\gamma = -r\mathbf{N} - r't\mathbf{B}$.]

(b) If $r^2 + (r't)^2 \equiv 1$, show that γ lies in a unit sphere (about some fixed point).

[Let $\beta = \gamma + r\mathbf{N} + r't\mathbf{B}$. Show that $\beta' = 0$, so that β is constant.]