The Triple Linkage:

A chaos-machine

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Flick the edge of one of the disks and see what happens

Be careful not to get your hair or fingers caught in the machine!

Lesson 1: Euclidean Geometry

What is its configuration space Σ?

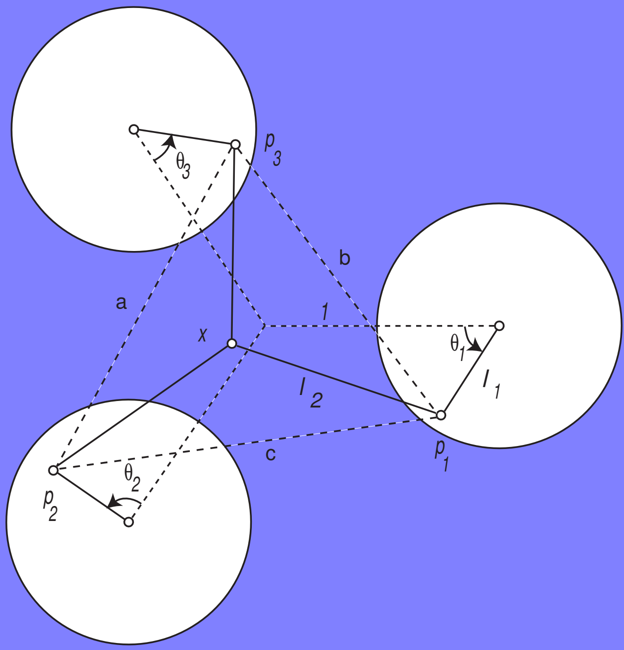


Figure 1: Notation for the parameters and configuration of the triple linkage

Σ is the set of triples $(\theta_1, \theta_2, \theta_3)$ of angles such that the circumradius of triangle $P_1P_2P_3$ is l_2, i.e.

$$\frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}} = l_2$$

where a, b, c are the lengths of its sides, which are given by Cartesian geometry, e.g.

$$a^2 = 3 + 2l_1^2 + l_1(\sqrt{3}\sin\theta_3 - 3\cos\theta_3 - \sqrt{3}\sin\theta_2 - 3\cos\theta_2)$$

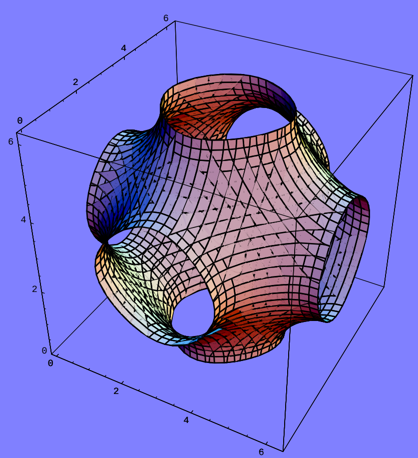


Figure 2: For $l_1 = \dfrac{19}{180}, l_2 = \dfrac{181}{180}$, the configuration space Σ is an orientable surface of genus 3.

Lesson 2: Topology (cf. Thurston & Weeks)

Can we find the topological type of Σ without drawing it?

Yes, let $A = \{$Allowed positions x of central pivot$\}$. Then A is the intersection of 3 annuli A_i, centred on c_i, each of inner radius $r = |l_1 - l_2|$ and outer radius $R = l_1 + l_2$, e.g.

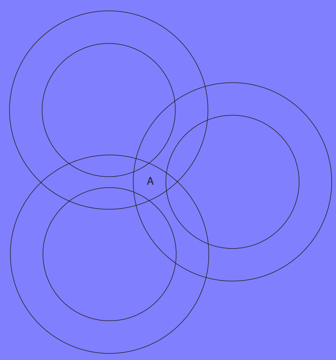
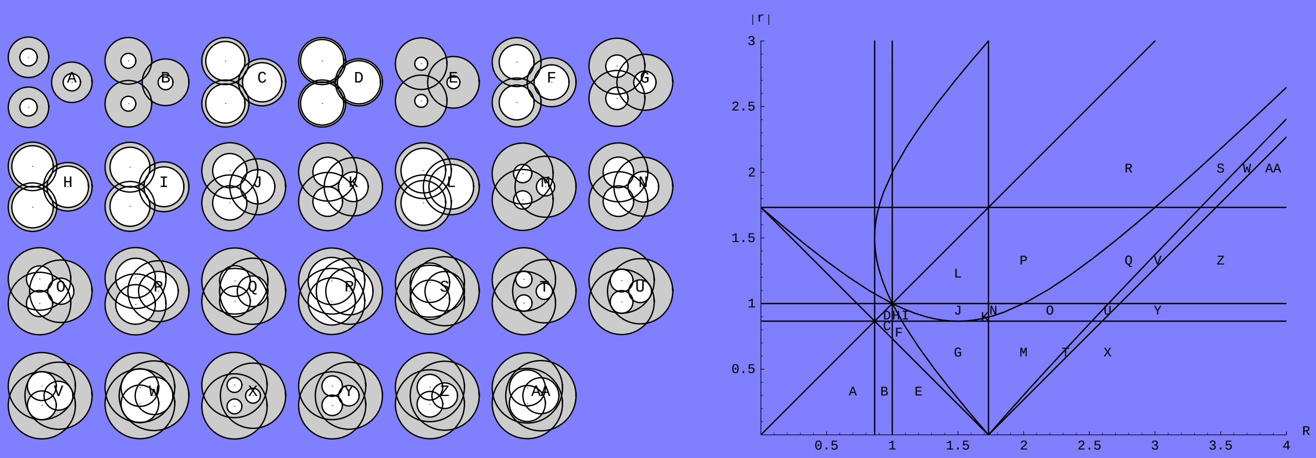


Figure 3: For $l_1 = 0.2, l_2 = 1.0$, A is a curvy hexagon.

To each x in the interior of A there correspond 8 configurations, joining in pairs on the edges and in fours at the vertices. So Σ is made of $F = 1 \times 8 = 8$ faces, meeting along $E = 6 \times 4 = 24$ edges, and at $V = 6 \times 2 = 12$ vertices. Thus Σ has Euler characteristic $\chi = F - E + V = -4$.

For most (l_1, l_2), Σ is an orientable compact surface, so it is homeomorphic to a disjoint union of copies of orientable surfaces X_g of genus $g \geq 0$. For $l_1 = 0.2, l_2 = 1.0$, Σ is connected, so it consists of just one X_g and then $\chi = 2 - 2g$ implies $g = 3$, agreeing with Figure 2.



$I, G, M : X_3 \quad H, F, E, J, N : X_0 \quad X : X_{12} \quad Z, AA : X_7 \quad Q, V, S : 3X_0$

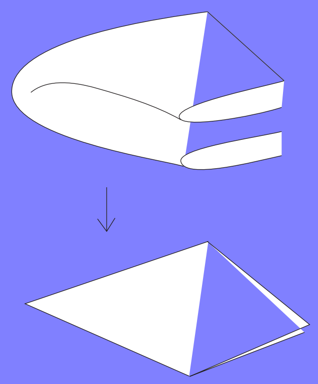
$Y : X_7 \cup X_0 \quad W : 6X_0 \quad K, O : 4X_0 \quad U : 7X_0 \quad T : X_3 \cup 3X_0$

Figure 4: All the stable ways the three annuli can intersect as (l_1, l_2) vary, and the resulting topological type of Σ (A, B, C, D, L, P and R and all regions above the diagonal are unattainable).

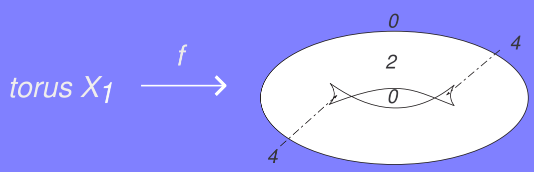
Lesson 3: Singularity Theory

For generic (l_1, l_2), Σ is a smooth surface. How is this compatible with the view as 8 hexagons glued together, for example?

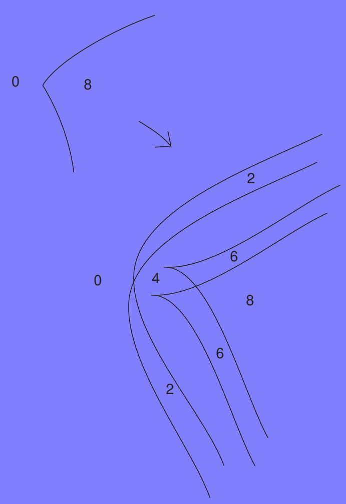
Curvy hexagon A is an exceptional image of a smooth X_3. It is like viewing a folded towel from directly above:



Generic image of a smooth surface X under a smooth map $f : X \to \mathbb{R}^2$ is a region bounded by pieces of fold curves, which can intersect transversely or meet at cusp points, e.g.



For a generic perturbation of $x : \Sigma \to A$, the corners of the hexagon deform like:



A Global Result [Levine]: For generic $f : \Sigma \to \mathbb{R}^2$, for each component c of the singular set of f in Σ, define $r(c) = \#$ half-revolutions made by the unoriented tangent to its image, following c in the direction keeping extra preimages on left. Then $\Sigma_c r(c) = \chi(\Sigma)$.

Lesson 4: Group Theory

Which cyclic changes in configuration can be deformed to each other, and which concatenations of cyclic changes are equivalent?

Take Σ connected (or take one connected component).

Choose a point $\theta^0 \in \Sigma$. Say 2 (oriented) curves on Σ from θ^0 to θ^0 are homotopic if they can be continuously deformed into each other, keeping the end points fixed.

Let $\Gamma = \{\text{homotopy classes}\}$. It is a group under concatenation, and we get isomorphic groups for different θ^0: "fundamental group" $\pi_1(\Sigma)$.

Can describe Γ by generators and relations, e.g.

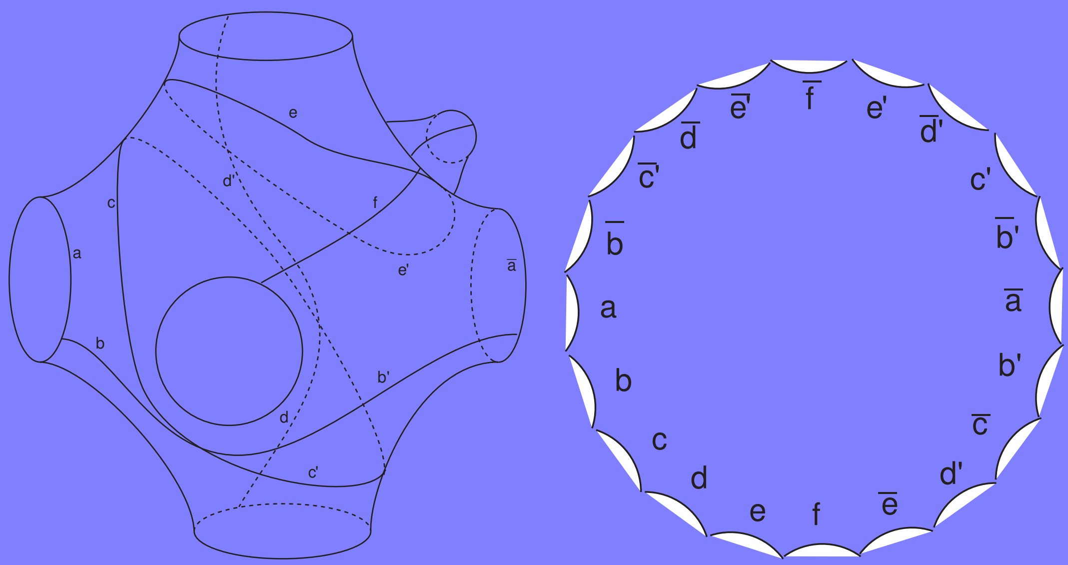


Figure 5: Cut X_3 along 6 simple closed curves as shown to obtain a 20-gon. [Koebe]

Given $\theta^0 \in$ interior of 20-gon, can label any curve from θ^0 to θ^0 by its "cutting sequence". Every sequence can occur. 2 curves are homotopic iff their cutting sequences can be obtained from each other by inserting or deleting words of the form $x\bar{x}$ or $wxyz$ for going round any of the 5 vertices. Thus

$$\Gamma \simeq \langle a, b, c, d, e, f, \bar{e}, d', \ldots | a\bar{a}, \ldots, cb'c'\bar{b}, \ldots \rangle$$

Can deduce that $\#$ homotopy classes represented by a sequence of length $\leq N$ grows exponentially with N.

"Free" homotopy classes of closed curves, where θ^0 is allowed to vary, correspond to conjugacy classes $C_\gamma = \{g^{-1}\gamma g : g \in \Gamma\}$ in Γ. The number of conjugacy classes represented by a sequence of length $\leq N$ also grows exponentially with N.

Lesson 5: Mechanics

What are the equations of motion?

If we ignore friction, the motion is given by Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad j = 1, 2$$

for any local coordinate system q_1, q_2, where $L = T - V$; $T(q, \dot{q})$ = kinetic energy; $V(q)$ = potential energy

e.g. if centre of mass c of each rod is on the line joining its pivots, then

$$T = \frac{1}{2}(\mu_1|\dot{x}|^2 + \mu_2\dot{x}\sum \dot{p}_i + \mu_3\sum|\dot{p}_i|^2) \text{ with}$$

$$\begin{aligned}
\mu_1 &= \tfrac{3}{l_2^2}(I_c + m|p - c|^2) \\
\mu_2 &= -\tfrac{2}{l_2^2}(I_c + m(x - c).(p - c)) \\
\mu_3 &= \tfrac{I}{l_1^2} + \tfrac{1}{l_2^2}(I_c + m|x - c|^2) \text{ where}
\end{aligned}$$

$$\begin{aligned}
m &= \text{mass of rod} \\
I_c &= \text{moment of inertia of rod about its centre of mass} \\
I &= \text{moment of inertia of disk}
\end{aligned}$$

No obvious nice global coordinate system (though could use redundant set θ).

Instead note that energy $E = T + V$ is conserved, and up to a reparametrisation of time, the motion at energy E is equivalent to the geodesic flow ("straight lines") of Riemannian metric

$$ds = 2\sqrt{(E - V(q))T_q(dq)} \qquad \text{[Jacobi \& Maupertuis]}$$

on the part of Σ with $V(q) \leq E$, i.e. paths q making

$$\int_{\tau_1}^{\tau_2} 2\sqrt{(E - V(q(\tau)))T_{q(\tau)}(\dot{q}(\tau))}\, d\tau$$

stationary with respect to variations fixing $q(\tau_1), q(\tau_2)$ (recover true time-parametrisation t by $\frac{dt}{d\tau} = \frac{1}{2\sqrt{E-V}}$).

Focus on $E > \max_{q\in\Sigma} V(q)$, even better, on $V \equiv 0, E > 0$.

Lesson 6: Hyperbolic Geometry

How do the straight lines of the kinetic energy metric on Σ look?

Start with an idealised metric: one can view X_3 as the quotient $\mathring{D}/\Gamma$ where $\mathring{D} = \{z \in \mathbb{C} : |z| < 1\}$ with Poincaré metric $P : ds = \dfrac{2|dz|}{1-|z|^2}$ (which has curvature -1 everywhere and whose straight lines are the arcs of circles perpendicular to the boundary), and Γ acts on $\mathring{D}$ by a subgroup of isometries of $\mathring{D}$ of the form

$$z \mapsto \frac{\alpha z + \beta}{\bar\beta z + \bar\alpha} \qquad \text{with } |\alpha|^2 - |\beta|^2 = 1 \text{ and } \mathrm{Re}\ \alpha > 1 \text{ (except for the identity of } \Gamma)$$

These are "translations", moving everything in $\mathring{D}$ away from one fixed point on $\partial\mathring{D}$ towards another.

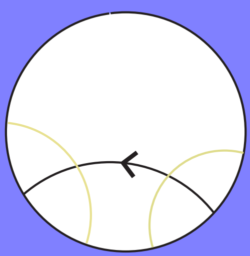


Figure 6: The action of a translation on $\mathring{D}$.

Can choose a 20-gon in $\mathring{D}$ whose sides are straight lines, and a translation for each generator x of Γ that takes side $\bar{x}$ to side x, e.g. the case of X_2:

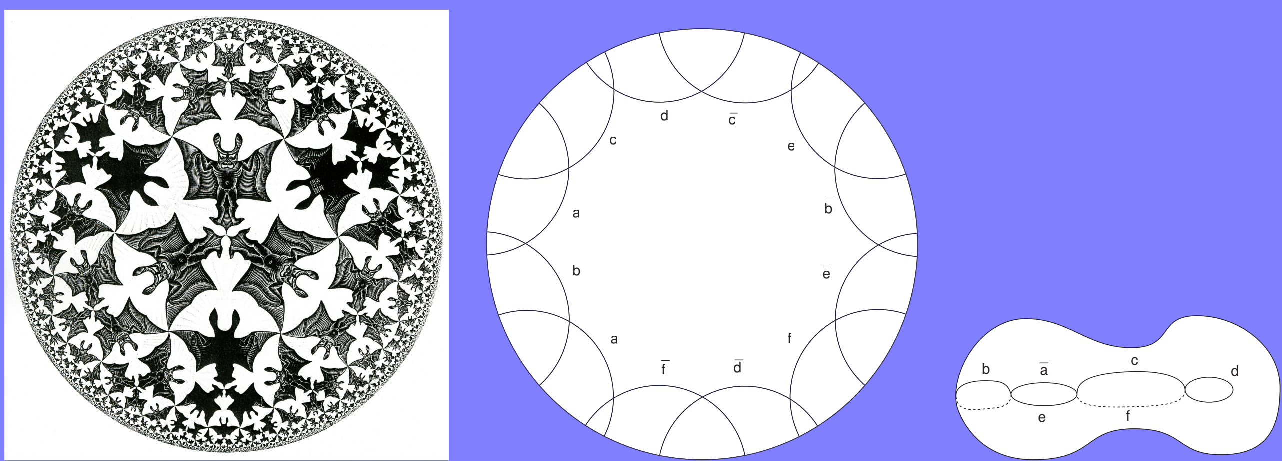
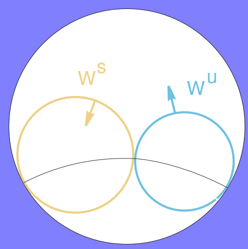


Figure 7: Escher's Cirkellimiet IV demonstrates that $X_2 \simeq \mathring{D}/\Gamma$, where Γ has generators a, b, c, d, e, f and relations $a\bar{a}, b\bar{b}$, etc., $abe\bar{b}$, $acef$, $cdf\bar{d}$

The geodesics for the Poincaré metric on $\mathring{D}/\Gamma$ are the quotient of the straight lines by Γ, e.g. γ/Γ is a closed geodesic on $\mathring{D}/\Gamma$ iff γ is the "axis" of a (non-identity) element of Γ, so there is precisely one closed geodesic for each (non-identity) conjugacy class in Γ. More generally, there is precisely one geodesic for each equivalence class (under the relations) of doubly infinite sequences of generators of Γ: "symbolic dynamics"

Most geodesics starting from nearby initial unit tangent vectors separate exponentially: "positive Lyapunov exponent". The sets of unit tangent vectors whose orbits converge together with a given one in forwards or backwards time are called the stable and unstable manifolds of the unit tangent vector. For the Poincaré metric they are the sets of unit normals to its "horocycles".



Now allow the real kinetic energy metric.

Morse: For any Riemannian metric R on $\Sigma = \mathring{D}/\Gamma$ there is a closed subset Y of $T_1\Sigma$ and continuous surjection $h : Y \to T_1\Sigma$ carrying geodesics of R to geodesics of the Poincaré metric (generalised by Denvir and MacKay to surfaces with $\chi < 0$ and geodesically convex boundary).

Thus for each geodesic of the Poincaré metric on X_g, the triple linkage at $E > \max V$ on any connected component of configuration space with genus ≥ 2 has at least one motion which does qualitatively the same.

But when is the dynamics topologically equivalent to the geodesic flow of the Poincaré metric? A sufficient conditions is that the Jacobi metric have everywhere negative curvature, so...

Lesson 7: Riemannian Geometry

What is the curvature for the true kinetic metric?

The linearised equations about a geodesic (which determine its Lyapunov exponents) are

$$\frac{d^2}{ds^2}\xi_\perp + \kappa\xi_\perp = 0, \frac{d^2}{ds^2}\xi_{||} = 0, \; [Jacobi]$$

where $\xi_\perp, \xi_{||}$ are the perpendicular and parallel components of a small displacement and $\kappa(q)$ = Gaussian curvature at q.

Curvature κ measures deviation from flatness locally, e.g. length of set of points at distance r from $q \sim 2\pi(r - \frac{\kappa}{6}r^3)$, area of set within distance $r \sim \pi(r^2 - \frac{\kappa}{12}r^4)$, for small r.

Brioschi formula: if $ds^2 = \begin{bmatrix} du & dv \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$ in local coordinates u, v then

$$\kappa = \frac{1}{4(EG-F^2)^2} \left(\begin{vmatrix} 2E_{vv} + 4F_{uv} - 2G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_v & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix} \right)$$

Gauss-Bonnet: $\int_\Sigma \kappa dS = 2\pi\chi\Sigma$, so $\chi = -4 \implies \kappa < 0$ on average.

If $\kappa < 0$ everywhere then Jacobi equation implies exponential separation of most pairs of geodesics and hence topological equivalence to the case of the Poincaré metric.

So Tim searched parameter space $(l_1, l_2, \mu_1, \mu_2, \mu_3)$ for a case with $\kappa < 0$ everywhere, and found one:

$$l_1 = \frac{7}{40}, l_2 = \frac{41}{40}, \mu_1 = \frac{11}{50}, \mu_2 = \frac{3}{100}, \mu_3 = \frac{23}{100} \qquad \text{taking } V = 0$$

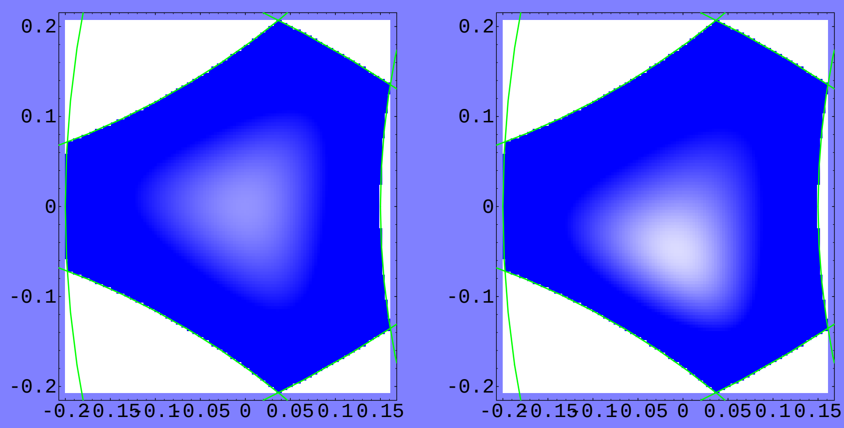


Figure 8: Curvature (darker = more negative) on two of the eight hexagons

This suggested studying the limiting case $l_2 = 1 + bl_1$, for some $b, l_1 \to 0, m, I_c \to 0$.

Then Σ has formula $\Sigma_i \cos\theta_i = -3b$, the kinetic energy is just $T = \frac{I}{2}\Sigma_i \dot\theta_i^2$, and κ is just the product of the principal curvatures of Σ as a surface embedded in $\mathbb{T}^3$, which evaluates to

$$\kappa = \left(\frac{9}{2}b^2 + 3b \prod \cos\theta_i - \frac{1}{2}\sum \cos^2\theta_i \right) / \sum \sin^2\theta_i$$

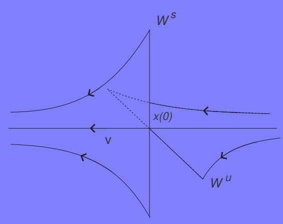
For $|b| < \frac{1}{3}$, $\Sigma \simeq X_3$; and for $b = 0, \kappa \leq 0$ with equality only if all $\theta_i \in \{\pm\frac{\pi}{2}\}$. The case $b = 0$ is the Schwarz P-surface from the theory of minimal surfaces (soap films), i.e. sum of principal curvatures $= 0$. It is also the Fermi surface for a half-filled simple cubic tight-binding model of electrons in a crystal.

Lesson 8: Anosov Systems

Is everywhere negative curvature necessary for such chaotic behaviour?

No, an invariant set Λ for an autonomous system $\dot{x} = v(x)$ on a manifold M is uniformly hyperbolic if $\exists K$ s.t. the linearised equations $\dot{\xi}_\perp = P_\perp Dv_{x(t)}\xi_\perp + f_\perp(t)$ for the $\perp$ component $\xi_\perp$ of a displacement from any orbit x in Λ under any bounded forcing function $f_\perp$ have a unique bounded solution $\xi_\perp$ and $\|\xi_\perp\| \leq K\|f_\perp\|$.

It follows that the set of points whose forward orbit converges with $x(t)$ is a smooth submanifold $W^s(x(0))$, and the same for backwards: $W^u(x(0))$, and the orbits converge together at least like $Ce^{-\lambda|t|}$ for some C, λ.



If $\tilde{v}$ is a small perturbation of v then $\exists\tilde{\Lambda}$ near to Λ, invariant and uniformly hyperbolic under $\tilde{v}$, with topologically equivalent flow.

The case where the whole of M is uniformly hyperbolic is called Anosov.

For geodesic flow on a surface, the uniform hyperbolicity condition is equivalent to $\exists K$ s.t. $\ddot{\xi}_\perp + \kappa(q(t))\xi_\perp = f_\perp(t)$ has a unique bounded solution $\xi_\perp$, and $\|\xi_\perp\| \leq K\|f_\perp\|$. If $\kappa < 0$ everywhere, the geodesic flow is Anosov (because the Green function decays exponentially both ways). The same is true if $\kappa \leq 0$ and $= 0$ at only a finite set of points. Thus the geodesic flow on the Schwarz P-surface is Anosov. But then all nearby vector fields on $T_1\Sigma$ are topologically equivalent and Anosov. This includes the geodesic flow for the triple linkage for all l_1, b, m, I_c small enough, small deviations from D_3 symmetry, and for variations in V sufficiently small compared with E. Anosovness is preserved under time reparametrisation, so the true motion is Anosov on each (high enough) energy level.

Anosov systems have other wonderful properties, e.g. a "Markov Partition": a finite partition of M into regions $(R_i)_{i=1}^N$ such that (except for ambiguities on the boundaries) there is a 1-1 correspondence between orbits and doubly infinite paths in an associated graph G with nodes $1, \ldots, N$ and the graph has a node with at least 2 paths back to it:"chaotic symbolic dynamics".

An explicit Markov partition for any Anosov geodesic flow on Σ: take the 6 curves a, b, c, d, e, f to have minimal length in their free homotopy classes, for each pair of the 20 symbols $x = a, \bar{a}, b, \bar{b}, c, \bar{c}$ etc., let R_{xy} be the set of unit tangents whose geodesic has crossing sequence beginning xy. Pairs of the form $x\bar{x}$ don't occur, and pairs corresponding to passing opposite sides of a vertex are regarded as equivalent. So we have 360 regions. Put an edge from xy to yz for all x, y, z except those corresponding to going round a vertex more than 180^0.

Lesson 9: Probability Theory

What do typical orbits do?

For a Hamiltonian system H, Liouville measure $\delta(H-E)\prod_i dp_i dq_i$ (Boltzmann's "ergode") is invariant on energy level $H^{-1}(E)$. For a geodesic flow on a surface this reduces to $dS\,d\varphi$, where dS and $d\varphi$ are the elements of surface area and direction of tangent vectors.

For an Anosov energy level, Liouville measure is ergodic, meaning there is no decomposition into 2 invariant sets of positive measure. It follows that for almost every orbit the fraction of time it spends in any subset is proportional to its measure.

Actually for any transitive Anosov system there is a (unique) invariant (ergodic) probability measure μ^+ (called SRB) s.t. almost every orbit w.r.t. any Borel measure spends fraction $\mu^+(A)$ of forward time in A. It can be obtained as the unique Gibbs state for symbol sequences with energy function $\int \lambda^u(\gamma(t))dt$, where $\lambda^u(\gamma(t))$ is the expansion rate along $W^u(\gamma(t))$ and γ is the orbit corresponding to the given symbol sequence.

The triple linkage is not just ergodic (on energy levels), but also mixing: $\forall$ measurable subsets A, B

$$\mu(\varphi_t A \cap B) \overset{t\to+\infty}{\longrightarrow} \mu(A)\mu(B) \qquad \text{(normalising } \mu(M)=1\text{)}$$

This is because for a transitive Anosov flow, either there is a surface of section with constant return time, or it is mixing, thus all Anosov geodesic flows are mixing, and so are all flows near an Anosov geodesic flow and in fact any Anosov flow on $T_1X_g, g \geq 2$.

Under certain conditions on a mixing Anosov flow (which hold for Anosov geodesic flows on surfaces and near-constant time-rescaling rates), the correlations of Hölder continuous functions decay exponentially:

$$\int f(\varphi_t x) g(x)\; d\mu(x) \leq C_{fg} e^{-\mu t} \qquad [Dolgopyat]$$

In particular, for any basis h_i of first cohomology of M, the velocity-autocorrelations

$$C_{ij}(t) = \int h_i(v(\varphi_t x)) h_j(v(x))\; d\mu(x)$$

are integrable: $D_{ij} = \int_0^\infty C_{ij}(t)dt < \infty$, and D is positive definite. It follows that the motion in any Abelian cover (like the lifting of Σ to the triply periodic extension of Figure 2, so we keep track of the numbers of revolutions of the discs) is "diffusive" on a large scale. (Note: the first Betti number $\beta_1(T_1X_3) = 7$ so there are 4 additional Abelian directions to those evident in Figure 2.)

Lesson 10: Partial Differential Equations

Given a kinetic energy function T on a surface Σ, can you design a potential V to make the Jacobi metric for given energy have constant negative curvature? ("Yamabe Problem")

Yes, for $g \geq 2$: $\Delta\phi - e^{2\phi} = \kappa$ has a unique solution $\phi : \Sigma \to \mathbb{R}$, where κ and Δ are the curvature and Laplacian of T. Then choose $V = E - \frac{1}{2}e^{2\phi}$. Hence energy level $H^{-1}(E)$ (and all nearby ones) are Anosov and mixing.

Lesson 11: Normally hyperbolic theory

Real system suffers from friction. Can we add a driving force to obtain a (non-trivial) uniformly hyperbolic attractor?

Yes, choose E sufficiently $> V_{max}$, let $\tilde{K} = \frac{1}{2}\sum I\dot{\theta}_j^2$ and apply feedback torques $\Gamma_j = -\gamma(\tilde{K} - E)\dot{\theta}_j$ for some $\gamma > \sqrt{\frac{I}{2E}}$ (or any feedback law close on a neighbourhood of $\tilde{K}^{-1}(E)$). Then $\tilde{K}^{-1}(E)$ is "normally hyperbolic" and attracting for the limit system, and the motion on it is unchanged. So for all smooth perturbations it persists to a invariant attracting normally hyperbolic submanifold, and the dynamics on it is a small perturbation of the frictionless case, so is topologically equivalent and Anosov and supports a mixing measure.

Lesson 12: Partially hyperbolic theory

If instead of the high gain feedback required for normal hyperbolicity, we use only low gain feedback (or even just a constant or time-periodic driving force), what can we get?

The unperturbed system is "partially hyperbolic" on the whole of $T\Sigma$:

$$T_\gamma(T\Sigma) = E_\gamma^+ \oplus E_\gamma^- \oplus \underbrace{\mathbb{R}X_H(\gamma) \oplus \mathbb{R}\nabla H(\gamma)}_{\text{"centre subspace" with weak (actually linear) expansion or contraction}}$$

Perturbation keeps partial hyperbolicity $T(T\Sigma) = \tilde{E}^+ \oplus \tilde{E}^- \oplus \tilde{E}^c$. There are robustly transitive partially hyperbolic attractors. Can we make one using friction and a weak driving force?