1. Notation and Elementary Results

1.1. **Chain Complexes.**

Definition 1.1.1. *A* chain complex *is a family* $\{K_n\}$ *of* R*-modules equipped with* boundary maps, $\partial_n : K_n \to K_{n-1}$*, which are* R*-module homomorphisms for* $-\infty < n < \infty$*, such that* $\partial_n \partial_{n+1} = 0$*.*

The condition is equivalent to $\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$.

$$K : \ldots \longleftarrow K_{-2} \xleftarrow{\partial_{-1}} K_{-1} \xleftarrow{\partial_0} K_0 \xleftarrow{\partial_1} K_1 \xleftarrow{\partial_2} K_2 \longleftarrow \ldots$$

Definition 1.1.2. *The* homology $H(K)$ *of a chain complex* K *is the family of modules*

$$H_n(K) = Ker \partial_n \big/ Im \partial_{n+1}.$$

The elements of $C_n(K) = \operatorname{Ker} \partial_n$ are called *n-cycles*, the elements of $\partial_{n+1} K_{n+1}$, *n-boundaries.*

1.2. **Chain Transformations.**

Definition 1.2.1. *Given complexes* K, K'*, a chain tranformation* $f : K \to K'$ *is a family of module homomorphisms* $f_n : K_n \to K'_n$ *such that* $\partial'_n f_n = f_{n-1} \partial_n$*.*

$$\begin{array}{ccccccccc}
K : \ldots & \longleftarrow & K_{n-1} & \xleftarrow{\partial_n} & K_n & \xleftarrow{\partial_{n+1}} & K_{n+1} & \longleftarrow & \ldots \\
 & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \\
K' : \ldots & \longleftarrow & K'_{n-1} & \xleftarrow[\partial'_n]{} & K'_n & \xleftarrow[\partial'_{n+1}]{} & K'_{n+1} & \longleftarrow & \ldots
\end{array}$$

Chain transformations are simply morphisms in the category of chain complexes.

We can define a function $H_n(f) = f_*$ by $f_* : H_n(K) \to H_n(K') : c + \partial K_{n+1} \mapsto fc + \partial K'_{n+1}$ between homology classes. This makes H_n a covariant functor between the category of chain complexes and the category of R-modules, for $H_n(f)$ is an R-module homomorphism.

1.3. **Chain Homotopies.**

Definition 1.3.1. *A* chain homotopy s *between two chain transformations* $f : K \to K'$ *and* $g : K \to K'$ *is a family of module homomorphisms* $s_n : K_n \to K'_{n+1}$ *such that*

$$\partial'_{n+1} s_n + s_{n-1} \partial_n = f_n - g_n.$$

When such a homotopy exists we write $s : f \simeq g$.

Theorem 1.3.2. *If* $s : f \simeq g : K \to K'$ *then* $H_n(f) = H_n(g)$ *as maps between the homologies* $H_n(K)$ *and* $H_n(K')$*.*

Definition 1.3.3. *A* chain equivalence *is a chain transformation* $f : K \to K'$ *with an inverse* $h : K' \to K$*, such that there are homotopies* $s : hf \simeq 1_K$ *and* $t : fh \simeq 1_{K'}$*.*

Theorem 1.3.4. *If* $f : K \to K'$ *is a chain equivalence then the homologies are isomorphic under* $H_n(f) : H_n(K) \cong H_n(K')$*.*

Theorem 1.3.5. *Chain homotopies can be composed, i.e: for* $s : f \simeq g : K \to K'$ *and* $s' : f' \simeq g' : K' \to K''$*, the map* $f's + s'g : f'f \simeq g'g : K \to K''$ *is a chain homotopy.*

1.4. **Subcomplexes and Quotient Complexes.**

Definition 1.4.1. *A* subcomplex *S of a complex K is a family of submodules $S_n < K_n$ with $\partial S_n \subseteq S_{n-1}$.*

Since ∂^2 is zero on all of K it is certainly zero on S, thus S is itself a complex with boundary map ∂_S the restriction of ∂_K to S.

The injection $i : S \to K$ is a chain transformation.

Definition 1.4.2. *Given S a subcomplex of K, the* quotient complex *K/S is the family K_n/S_n of quotient modules with boundary map $\partial : K_n/S_n \to K_{n-1}/S_{n-1}$ induced by ∂_K.*

The natural projection $K \to K/S$ is a chain transformation. We can now write an exact sequence of complexes

$$0 \longrightarrow S \longrightarrow K \longrightarrow K/S \longrightarrow 0,$$

meaning that there is a short exact sequence of corresponding modules for each dimension n.

For a general chain transformation $f : K \to K'$ we can write $\operatorname{Ker} f = \{\operatorname{Ker} f_n\}$ and $\operatorname{Im} f = \{f_n K_n\}$ for the subcomplexes of K and K' respectively, given by the kernels and images of the f_n's respectively.

Thus the following sequence is always exact

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow K \xrightarrow{\ f\ } K' \longrightarrow \operatorname{Coker} f \longrightarrow 0$$

where $\operatorname{Coker} f = K' \big/ \operatorname{Im} f$ is the appropriate quotient complex.

1.5. **Cochain Complexes.**

Sometimes it is convenient to write a complex in reverse order so that the arrows progress to the right. When we do this we renumber the component modules, replacing K_{-n} with K^n so that the map $\partial_{-n} : K_{-n} \to K_{-n-1}$ now becomes $\delta^n : K^n \to K^{n+1}$

$$\ldots \longrightarrow K^{-2} \longrightarrow K^{-1} \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow \ldots$$

Note that the renumbering has ensured that the negative indices are still to the left. We still denote this complex K, as the complex itself has not changed, but we say that it is written in upper indices.

Definition 1.5.1. *A chain complex is said to be* positive *if $K_n = 0$ for $n < 0$. Clearly it has its homology $H_n(K) = 0$ for $n < 0$. The homology is also said to be positive.*

Similarly a chain complex is *negative* if $K_n = 0$ for $n > 0$.

Definition 1.5.2. *A negative complex written in upper indices*

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow \ldots$$

(thus positive in upper indices) is called a cochain *complex. The homology of a cochain complex can be written $H^n(K) = Ker\delta^n \big/ \delta K^{n-1}$.*

Similarly, a cochain homotopy is nothing but a chain homotopy written in upper indices $s^n : K^n \to K'^{n-1}$ with $\delta s + s\delta = f - g$.

1.6. **Complexes Over a Module.**

Definition 1.6.1. *Given a module A, a* complex over A *is a positive complex K with a chain transformation $\varepsilon : K \to A$ to a* trivial complex A, *i.e: with $A_0 = A$ and $A_n = 0$ for all $n \neq 0$.*

This definition is equivalent to the existence of a module homomorphism $\varepsilon : K_0 \to A$ such that $\varepsilon \partial_1 = 0 : K_1 \to A$.

Definition 1.6.2. *A* contracting homotopy *for a complex over A, $\varepsilon : K \to A$, is a chain transformation back the other way, $f : A \to K$ with $\varepsilon f = 1_A$ and a homotopy $s : 1_K \simeq f\varepsilon$.*

If $\varepsilon : K \to A$ has a contracting homotopy, the homology of K is fixed, since $\varepsilon_* : H_0(K) \cong A$ is an isomorphism and $H_n(K) = 0$ for all other n.

1.7. **Cohomology.**

Definition 1.7.1. *Given a complex K we call the elements of K_n* n-chains *and for an R-module G, we call the elements $f : K_n \to G$ of $Hom_R(K_n, G)$* n-cochains *of K.*

Definition 1.7.2. *We define the* coboundary *map by*

$$\delta^n f = (-1)^{n+1} f\, \partial_{n+1} \tag{1}$$

taking an n-cochain f to an $(n+1)$-cochain.

Now $\delta^n \delta^{n-1} = 0$ and so the sequence

(2)

$$\ldots \to \mathrm{Hom}_R(K_{n-1}, G) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_R(K_n, G) \xrightarrow{\delta^n} \mathrm{Hom}_R(K_{n+1}, G) \to \ldots$$

is a complex of Abelian groups which we write $\mathrm{Hom}_R(K, G)$.

We usually write the groups of homomorphisms that appear in this complex with upper indices, $\mathrm{Hom}^n(K, G) = \mathrm{Hom}(K_n, G)$.

Note: if K is positive in lower indices, $\mathrm{Hom}(K, G)$ is positive in upper indices.

Definition 1.7.3. *The* cohomology *of K with coefficients in G is the homology of the sequence* (2) *above. In upper indices it is denoted*

$$H^n(K, G) = H^n(Hom(K, G)) = Ker\,\delta^n \,\big/\, \delta\, Hom(K_{n-1}, G). \tag{3}$$

Definition 1.7.4. *Elements of $\delta Hom(K_{n-1}, G)$ are called* n-coboundaries *and elements of $Ker\,\delta^n$ are called* n-cocycles.

We emphasize that the K_n appearing in (2) are the usual modules with lower indices, but $\mathrm{Hom}(-, G)$ induces the map δ, automatically making (2) a cochain complex, hence the notation $\mathrm{Hom}^n(K, G) = \mathrm{Hom}(K_n, G)$ for its modules. Thus we see it is natural to associate the prefix "co-" to the cohomology of a complex as for all complexes with arrows to the right. However we recognize that we have cohomology, not because of upper indices or arrows to the right (for homology can be written as a cochain complex and thus satisfy these criteria) but because of the presence of the coefficient group G.

Note that a cocycle can be thought of as a homomorphism $f : K_n \to G$ with $f\partial = 0 : K_{n+1} \to G$.

A chain transformation $h : K \to K'$ induces a chain transformation $h^* = \mathrm{Hom}(h, 1) : \mathrm{Hom}(K', G) \to \mathrm{Hom}(K, G)$. Thus we can think of $\mathrm{Hom}(K, G)$ and the cohomology

$H^n(K,G)$ as bifunctors from complexes and Abelian groups to Abelian groups, contravariant in K and covariant in G.

Any homotopy $s : h \simeq g$ induces a map s^* which satisfies $s_n^* \partial_{n+1}^* + \partial_n^* s_{n-1}^* = h_n^* - g_n^*$, then $t_{n+1} = (-1)^{n+1} s_n^*$ is a homotopy $t : h^* \simeq g^*$, i.e: $t^{n+1}\delta^n + \delta^{n-1}t^n = h_n^* - g_n^*$.

2. Exact Homology Sequences

2.1. **Long Exact Homology Sequence.** Given an exact sequence of chain complexes

$$E : 0 \to K \xrightarrow{\chi} L \xrightarrow{\sigma} M \to 0 \tag{4}$$

we consider what happens when we take the homology of the complexes in the sequence. We clearly get the induced maps

$$H_n(K) \xrightarrow{\chi_*} H_n(L) \xrightarrow{\sigma_*} H_n(M) \tag{5}$$

but χ_* is no longer an injection in general.

Let m be a cycle in M_{n+1}, i.e: such that $\partial m = 0$. Since σ is an epimorphism, there is an $l \in L_{n+1}$ such that $\sigma l = m$. Since σ is a chain transformation, and $\partial m = 0$ one has $\partial \sigma l = \sigma \partial l = 0$. Thus ∂l is in the kernel of σ. Now since E is exact and thus $\operatorname{Ker}\sigma = \operatorname{Im}\chi$, there must be a unique $c \in K_n$ with $\chi c = \partial l$. We check easily that $\operatorname{cls}(c) \in H_n(K)$ does not depend on our choice of l. It only depends on the homology class of m and is additive in m. Thus $\partial_E(\operatorname{cls} m) = \operatorname{cls} c$ is a homomorphism.

Definition 2.1.1. *We define the* connecting homomorphism ∂_E *for the exact sequence* E

$$\partial_E : H_{n+1}(M) \to H_n(K) \tag{6}$$

by $\partial_E(\operatorname{cls} m) = \operatorname{cls} c$ *where* $\chi c = \partial l$ *and* $\sigma l = m$ *for some* l.

We are now able to extend the sequence (5) in both directions.

Theorem 2.1.2. *The long sequence*

$$\begin{aligned} \ldots \to H_{n+1}(M) &\xrightarrow{\partial_E} H_n(K) \xrightarrow{\chi_*} \\ H_n(L) &\xrightarrow{\sigma_*} H_n(M) \xrightarrow{\partial_E} H_{n-1}(K) \to \ldots \end{aligned} \tag{7}$$

of homology groups, is exact.

We can now read off that the kernel of χ_* is $\partial_E H_{n+1}(M)$.

Theorem 2.1.3. *In the category of short exact sequences of complexes, the connecting homomorphism* ∂_E *is natural for each short exact sequence* E.

The morphisms of the category of exact sequences of complexes are triples (f, g, h) of chain transformations such that the following diagram commutes

$$\begin{array}{ccccccccc} E : 0 & \longrightarrow & K & \xrightarrow{\chi} & L & \xrightarrow{\sigma} & M & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ E' : 0 & \longrightarrow & K' & \xrightarrow{\chi'} & L' & \xrightarrow{\sigma'} & M' & \longrightarrow & 0 \end{array} \tag{8}$$

Thus we can think of $H_n(K)$, $H_n(L)$ and $H_n(M)$ as functors from the category of sequences of chain complexes to modules.

To say ∂_E is natural means therefore that the folowing diagram commutes

$$\begin{array}{ccc} H_{n+1}(M) & \xrightarrow{\partial_E} & H_n(K) \\ \Big\downarrow{\scriptstyle H_{n+1}(h)} & & \Big\downarrow{\scriptstyle H_n(f)} \\ H_{n+1}(M') & \xrightarrow{\partial_{E'}} & H_n(K') \end{array} \tag{9}$$

In fact, more is true.

Theorem 2.1.4. *A morphism (f, g, h) of sequences E to E' induces a morphism of exact homology sequences*

$$\begin{array}{ccccccccc} \ldots \to H_{n+1}(L) & \xrightarrow{\sigma_*} & H_{n+1}(M) & \xrightarrow{\partial_E} & H_n(K) & \xrightarrow{\chi_*} & H_n(L) \to \ldots \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle h_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle g_*} \\ \ldots \to H_{n+1}(L') & \xrightarrow{\sigma'_*} & H_{n+1}(M') & \xrightarrow{\partial_{E'}} & H_n(K') & \xrightarrow{\chi'_*} & H_n(L') \to \ldots \end{array} \tag{10}$$

2.2. **Long Exact Cohomology Sequences.** There is a similar construction for the cohomology groups of an exact sequence of complexes. This time we require the exact sequence to split, i.e: for each dimension n, K_n is a direct summand of L_n in the exact sequence

$$0 \to K_n \to L_n \to M_n \to 0. \tag{11}$$

Theorem 2.2.1. *If G is an R-module and E a short exact sequence of complexes which splits, then there is a connecting homomorphism*

$$\delta_E : H^n(K, G) \to H^{n+1}(M, G) \tag{12}$$

which is natural and such that the following long sequence is exact

(13)

$$\ldots \to H^n(M, G) \xrightarrow{\sigma^*} H^n(L, G) \xrightarrow{\chi^*} H^n(K, G) \xrightarrow{\delta_E} H^{n+1}(M, G) \to \ldots$$

Proof (partial): Since E is short exact and splits, then

$$E^* : 0 \to \mathrm{Hom}\,(M, G) \to \mathrm{Hom}\,(L, G) \to \mathrm{Hom}\,(K, G) \to 0 \tag{14}$$

is a short exact sequence of complexes. Thus it has a connecting homomorphism $\partial_{E^*} : H_{-n+1}(\mathrm{Hom}(K, G)) \to H_{-n}(\mathrm{Hom}(M, G))$ by the previous theorem. Simply writing this in upper indices gives us the connecting homomorphism for the cohomology, $\delta_E : K^n(K, G) \to H^{n+1}(M, G)$. □

We can describe δ_E more explicitly. Consider an n-cocycle of K as a homomorphism $f : K_n \to G$. Since E^* is exact, f can be written $f = g\chi$ for some $g : L_n \to G$, a cochain of L. But now since f is a cocycle, i.e: $f\partial = 0$, then $g\partial\chi = g\chi\partial = f\partial = 0$. But this means that $g\partial$ factors through σ by the universal properties of the split exact sequence, i.e: $g\partial = h\sigma$ for some $h : M_{n+1} \to G$. Now $h\partial\sigma = h\sigma\partial = g\partial\partial = 0$ and since σ is an surjection, it must be that $h\partial = 0$, i.e: h is a cocyle of M. Thus we define

Definition 2.2.2. *Let $\delta_E(cls\, f) = cls\, h$ where $h\sigma = g\partial$ and $g\chi = f$ for some g.*

Of course the definition does not depend on the choice of g.

Now there is a second long exact sequence of cohomology groups that we can define. This time it comes from a short exact sequence of coefficient modules

$$S : 0 \to G' \xrightarrow{\lambda} G \xrightarrow{\tau} G'' \to 0. \tag{15}$$

As usual λ induces homomorphisms $\lambda_* : H^n(K, G') \to H^n(K, G)$ for any complex K. We have the following

Theorem 2.2.3. *If K is a complex of projective modules K_n, then there is a connecting homomorphism $\delta_S : H^n(K, G'') \to H^{n+1}(K, G')$ which is natural when either considered as a transformation between covariant functors on S, or contravariant functors of K, and which gives the long exact sequence*

(16)
$$\ldots \to H^n(K, G') \xrightarrow{\lambda_*} H^n(K, G) \xrightarrow{\tau_*} H^n(K, G'') \xrightarrow{\delta_S} H^{n+1}(K, G') \to \ldots$$

Proof (partial): Since the K_n are projective, then

(17) $$S_* : 0 \to \mathrm{Hom}(K, G') \to \mathrm{Hom}(K, G) \to \mathrm{Hom}(K, G'') \to 0$$

is exact. Thus it has a connecting homomorphism $\delta_S = \partial_{S_*}$ with the usual change to upper indices.

We construct δ_E explicitly. Let $f : K_n \to G''$ be a cocycle. Since s_* is exact, we can write $f = \tau g$ for some $g : K_n \to G$. Since F is a cocycle, $0 = f\partial = \tau g\partial$. Thus $g\partial = \lambda h$ for some cocycle $h : K_{n+1} \to G'$ by the exactness of S, and since $0 = g\partial\partial = \lambda h\partial$ and λ is an injection. We therefore define

Definition 2.2.4. *Let $\delta_S(cls\, f) = cls\, h$ where $\lambda h = g\partial$ and $\tau g = f$ for some g.*

3. Extensions

3.1. The Abelian Group Ext(A, C).

Definition 3.1.1. *Given R-modules A and C, an* extension *of A by C is a short exact sequence*

(18) $$E = (\chi, \sigma) : 0 \longrightarrow A \xrightarrow{\chi} B \xrightarrow{\sigma} C \longrightarrow 0$$

of R-modules and R-module homomorphisms.

Definition 3.1.2. *A* morphism *of extensions is a triple $\Gamma = (\alpha, \beta, \gamma)$ of module homomorphisms such that the following diagram commutes*

(19)
$$\begin{array}{cccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\ \Gamma\downarrow & & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ E': & 0 & \longrightarrow & A' & \xrightarrow{\chi'} & B' & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \end{array}$$

Definition 3.1.3. *We say that two extensions of A by C are* congruent *$(E \equiv E')$ if there is a morphism $(1_A, \beta, 1_C) : E \to E'$.*

Lemma 3.1.4. *If E and E' are two congruent extensions of A by C then β is an isomorphism of B with B'.*

We often speak of B itself as the extension. It is unique up to isomorphism. More formally we speak of (B, Θ) where A is a submodule of B and Θ an isomorphism $B/A \cong C$.

Definition 3.1.5. *An extension is said to be* split *if it is congruent to the direct sum*

(20) $$0 \longrightarrow A \xrightarrow{\chi} A \oplus C \xrightarrow{\sigma} C \longrightarrow 0$$

This is only the case if χ has a left inverse, or σ has a right inverse.

Notation 3.1.6. *Let $Ext_R(C, A)$ denote the set of congruence classes of extensions of A by C.*

Any extension by a projective module P is split, so $\mathrm{Ext}_R(P, A)$ has only a single element.

We will show that $\mathrm{Ext}_R(C, A)$ is always an Abelian group. But in order to define the addition for this group we first need to define compositions of an extension with a homomorphism.

This can be done best in the context of showing Ext to be a functor on the category of modules to the category of sets, covariant on A and contravariant on C.

For the latter, we'll want to define for each homomorphism $\gamma : C' \to C$ an extension $\gamma^* E \in \mathrm{Ext}_R(C', A)$, which we will denote $E\gamma$, such that $E1_C \equiv E$ and $E(\gamma\gamma') \equiv (E\gamma)\gamma'$. This follows from the following lemma.

Lemma 3.1.7. *Given an extension $E \in Ext_R(C, A)$ and a homomorphism of modules $\gamma : C' \to C$, there exists an extension $E\gamma \in Ext(C', A)$ and a morphism $\Gamma = (1_A, \beta, \gamma) : E\gamma \to E$ which are unique up to a congruence of $E\gamma$.*

In fact $E\gamma$ is the extension which makes the following diagram commute up to congruence

(21)
$$\begin{array}{ccccccccc}
E\gamma : 0 & \longrightarrow & A & \xrightarrow{\chi'} & ? & \xrightarrow{\sigma'} & C' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
E : 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0
\end{array}$$

We also note the following

Lemma 3.1.8. *Each morphism of extensions $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1) : E_1 \to E$ with $\gamma_1 = \gamma$ can be written uniquely as*

(22)
$$E_1 \xrightarrow{(\alpha_1, \beta', 1)} E\gamma \xrightarrow{(1, \beta, \gamma)} E$$

Similarly for covariance of Ext in A we define a composite αE for each $\alpha : A \to A'$.

Lemma 3.1.9. *For $E \in Ext(C, A)$ and $\alpha : A \to A'$ there is an extension $\alpha E \in Ext(C, A')$ and a morphism $\Gamma = (\alpha, \beta, 1_C)$ which are unique up to a congruence of αE.*

The required extension fills in the following commutative diagram

(23)
$$\begin{array}{ccccccccc}
E : 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\
\alpha E : 0 & \longrightarrow & A' & \xrightarrow{\chi'} & ? & \xrightarrow{\sigma'} & C & \longrightarrow & 0
\end{array}$$

Again we have

Lemma 3.1.10. *Any morphism $\Gamma_1 = (\alpha_1, \beta_1, \gamma_1) : E \to E_1$ with $\alpha_1 = \alpha$ can be factored through $E \to \alpha E$*

(24)
$$E \xrightarrow{(\alpha, \beta, 1)} \alpha E \xrightarrow{(1, \beta', \gamma_1)} E_1$$

In fact $\mathrm{Ext}(C, A)$ is a bifunctor, for

Lemma 3.1.11. *For homomorphisms α, γ*

(25)
$$\alpha(E\gamma) \equiv (\alpha E)\gamma.$$

Theorem 3.1.12. *For any extension $E = (\chi, \sigma)$ the extensions χE and $E\sigma$ split, i.e: in the first case*

$$\begin{array}{ccccccccc} E: 0 & \longrightarrow & A & \xrightarrow{\chi} & B & \xrightarrow{\sigma} & C & \longrightarrow & 0 \\ & & \downarrow \chi & & \downarrow \nu & & \| & & \\ E': 0 & \longrightarrow & B & \longrightarrow & B \oplus C & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{26}$$

Theorem 3.1.13. *If $\Gamma_1 = (\alpha, \beta, \gamma) : E \to E'$ is a morphism of extensions, then $\alpha E \equiv E'\gamma$.*

We now wish to see that $\mathrm{Ext}_R(C, A)$ is an Abelian group. The homomorphisms we will need are

Definition 3.1.14. *The* diagonal *homomorphism is defined by*

$$\Delta_C : C \to C \oplus C : c \mapsto (c, c). \tag{27}$$

The codiagonal *homomorphism is defined by*

$$\nabla_A : A \oplus A \to A : (a_1, a_2) \mapsto a_1 + a_2. \tag{28}$$

Note that we can define the usual sum of homomorphisms f, $g : C \to A$ using these homomorphisms, by

$$f + g = \nabla_B (f \oplus g) \Delta_C. \tag{29}$$

Here we are considering $A \oplus C$ to be a covariant bifunctor of A and C. For, given homomorphisms $\alpha : A \to A'$ and $\gamma : C \to C'$ we can define a homomorphism $\alpha \oplus \gamma : A \oplus C \to A' \oplus C' : (a, c) \mapsto (\alpha a, \gamma c)$. We note the required properties hold, $(\alpha \oplus \gamma)(\alpha' \oplus \gamma') = \alpha\alpha' \oplus \gamma\gamma'$ and $1_A \oplus 1_C = 1_{A \oplus C}$.

We also make the definition

Definition 3.1.15. *The direct sum of two extensions*

$$E_i = (\chi_i, \sigma_i) : 0 \to A_i \to B_i \to C_i \to 0 \;\; \textit{for} \;\; i = 1, 2 \tag{30}$$

is given by

$$E_1 \oplus E_2 : 0 \to A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \to 0 \tag{31}$$

Finally we have

Theorem 3.1.16. *The set $Ext_R(C, A)$ is an Abelian group under the operation which assigns to the congruence classes of extensions E_1 and E_2 the congruence class of the extension $E_1 + E_2 = \nabla_A (E_1 \oplus E_2) \Delta_c$, where the right hand side indicates composition of the homomorphisms ∇_A and Δ_C with the extension $E_1 \oplus E_2$ as we have defined such compositions above.*

The identity for this group is the split extension

$$0 : 0 \to A \to A \oplus C \to C \to 0. \tag{32}$$

The inverse of E is $(-1_A)E$.

Theorem 3.1.17. *Composition with homomorphisms is distributive over this sum, i.e:*

$$\alpha(E_1 + E_2) \equiv \alpha E_1 + \alpha E_2, \quad (E_1 + E_2)\gamma \equiv E_1\gamma + E_2\gamma, \tag{33}$$

$$(\alpha_1 + \alpha_2)E \equiv \alpha_1 E + \alpha_2 E, \quad E(\gamma_1 + \gamma_2) \equiv E\gamma_1 + E\gamma_2. \tag{34}$$

This theorem shows that the maps

$$\alpha_* : \text{Ext}(C, A) \to \text{Ext}(C, A') : E \to \alpha E \tag{35}$$

and

$$\gamma^* : \text{Ext}(C, A) \to \text{Ext}(C', A) : E \to E\gamma \tag{36}$$

are group homomorphisms.

We also get the following isomorphisms from these properties

Theorem 3.1.18.

$$Ext(C, A_1 \oplus A_2) \cong Ext(C, A_1) \oplus Ext(C, A_2), \tag{37}$$

$$Ext(C_1 \oplus C_2, A) \cong Ext(C_1, A) \oplus Ext(C_2, A). \tag{38}$$

Example 3.1.19. *For an Abelian group A and the cyclic group C_m we have*

$$Ext_{\mathbb{Z}}(C_m, A) \cong A/mA. \tag{39}$$

3.2. **Obstructions to the Extension of Homomorphisms.** Recall that the functor Hom is not exact. Indeed if $0 \to A \to B \to C \to 0$ is exact, we can only say $0 \to \text{Hom}_R(C, G) \to \text{Hom}_R(B, G) \to \text{Hom}_R(A, G)$, is exact.

The problem stems from the following fact

Lemma 3.2.1. *If A is a submodule of B so that there is an exact sequence*

$$E : 0 \to A \to B \to C \to 0 \tag{40}$$

with $C = B/A$, then a homomorphism $\alpha : A \to G$ can be extended to a homomorphism $\alpha' : B \to G$ iff the extension αE is split.

In this case αE is simply the identity of $\text{Ext}_R(C, G)$. In general, we call $\alpha E \in \text{Ext}_R(C, G)$ the *obstruction* of the homomorphism $\alpha : A \to G$.

Thus we have a group homomorphism

$$E^* : \text{Hom}_R(A, G) \to \text{Ext}_R(C, G) \tag{41}$$

which sends a homomorphism to its obstruction.

This is a connecting homomorphism in the following sequence

Theorem 3.2.2. *The following sequence of Abelian groups is exact for any R-module G*

$$\begin{aligned} 0 \to Hom_R(C, G) \to Hom_R(B, G) \to Hom_R(A, G) \\ \xrightarrow{E^*} Ext_R(C, G) \xrightarrow{\sigma^*} Ext_R(B, G) \xrightarrow{\chi^*} Ext_R(A, G) \end{aligned}$$

given that $0 \to A \to B \to C \to 0$ is exact.

These results have duals which are stated as follows

Lemma 3.2.3. *If $C = B/A$, giving rise to an exact sequence E, then a homomorphism $\gamma : G \to B/A$ can be lifted to a homomorphism $\gamma' : G \to B$ iff the extension $E\gamma$ splits.*

We call $E\gamma \in \text{Ext}_R(G, A)$ the obstruction to lifting γ. Thus we have a homomorphism of groups

$$E_* : \text{Hom}(C, G) \to \text{Ext}(G, A)$$

It is a connecting homomorphism in the following

Theorem 3.2.4. *If* $0 \to A \to B \to C \to 0$ *is exact, then*

$$0 \to Hom_R(G,A) \to Hom_R(G,B) \to Hom_R(G,C)$$
$$\xrightarrow{E_*} Ext_R(G,A) \longrightarrow Ext_R(G,B) \longrightarrow Ext_R(G,C)$$

is exact for any R-module G.

Since an R-module P is projective iff every extension by P splits, then

Theorem 3.2.5. *P is projective iff* $Ext_R(P,G) = 0$ *for all R-modules G.*

Given any two R-modules C and G, we'd like to be able to calculate $\mathrm{Ext}_R(C,G)$. The following theorem allows us to do this.

Theorem 3.2.6. *Given any exact sequence*

$$0 \longrightarrow K \xrightarrow{\chi} P \longrightarrow C \longrightarrow 0$$

with P projective we have

$$Ext_R(C,G) \cong Hom_R(K,G) \, / \, \chi^* \, Hom_R(P,G)$$

independent of the exact sequence chosen.

Since any module C can be represented as the quotient of a free module, we can always use this theorem with P free.

Theorem 3.2.7. *For Abelian groups, the sequences in theorems* (3.2.2) *and* (3.2.4) *remain exact when a zero is added to the right.*

3.3. **The Universal Coefficient Theorem.** Given the homology of a complex K of free Abelian groups it is possible to calculate its cohomology $H^n(K,G)$

Theorem 3.3.1. *If* K_n *are free Abelian groups and G any coefficient group, then for each dimension, the following sequence is exact*

$$0 \longrightarrow Ext(H_{n-1}(K),G) \xrightarrow{\beta} H^n(K,G) \xrightarrow{\alpha} Hom(H_n(K),G) \longrightarrow 0$$

with homomorphisms α, β *natural in K and G. In fact, this sequence is split by a homomorphism natural in G (but not K).*

The map α is defined on cohomomology classes. E.g: let f be an n-cocycle of $\mathrm{Hom}(K,G)$, i.e: a homomorphism $F : K_n \to G$ vanishing on ∂K_{n+1}. It induces a map $f_* : H_n(K) \to G$, thus we define $\alpha(\mathrm{cls}\ f) = f_*$. This is well defined, since if $f = \delta g$ is a coboundary, it vanishes on cycles, thus $(\delta g)_* = 0$, i.e: f_* only depends on the class of f in $H^n(K,G)$.

This theorem only relies on the simple fact that subgroups of free Abelian groups are free. This idea can be extended to free modules over a principal ideal domain. That is, the universal coefficient theorem holds for K a complex of free modules over such a PID, call it D say, and G a D-module.

A simple example is

Corollary 3.3.2. *If K is a complex of vector spaces* K_n *over a field F and V an arbitrary vector space over F, then there is a natural isomorphism*

$$H^n(K,V) \cong Hom(H_n(K),V)$$

This theorem holds by virtue of the fact that $\mathrm{Ext}(H_{n-1}(K),V)$ is trivial. This is because $H_{n-1}(K)$ is projective, since the homology is the quotient of a vector space with a subspace.

If $V = F$ then this simply says that $H^n(K,F)$ is the vector space dual of $H_n(K)$.

Another useful application of the universal coefficient theorem is

Corollary 3.3.3. *If $f : K \to K'$ is a chain transformation between complexes K and K' of free Abelian groups, inducing for each n an isomorphism $f_* : H_n(K) \cong H_n(K')$, then for any coefficient group G, the cohomologies are also isomorphic, by $F^* : H^n(K', G) \cong H^n(K, G)$.*

Proof: In the universal coefficient theorem, α and β are natural in K and so the following diagram commutes

$$\begin{array}{ccccc} 0 \to \operatorname{Ext}(H_{n-1}(K'), G) & \longrightarrow & H^n(K', G) & \longrightarrow & \operatorname{Hom}(H_n(K'), G) \to 0 \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ 0 \to \operatorname{Ext}(H_{n-1}(K), G) & \longrightarrow & H^n(K, G) & \longrightarrow & \operatorname{Hom}(H_n(K), G) \to 0 \end{array}$$

and since the maps $f_n : H_n(K) \to H_n(K')$ are isomorphisms, so are the outside vertical maps of this diagram. Thus by the short five lemma, so is the middle map. $\square$