MA3A6 WEEK 7 ASSIGNMENT : DUE MONDAY 4PM WEEK 7

BILL HART

1. Find an arbitrary $\mathbb{Q}$-basis for $K = \mathbb{Q}(\zeta_5)$, where ζ_5 is a primitive 5-th root of unity and compute the discriminant of the $\mathbb{Q}$-basis you have found.

A primitive 5-th root of unity is of degree 4. The obvious $\mathbb{Q}$-basis to choose is $\{1, \zeta_5, \ldots, \zeta^3\}$ (with 4 elements).

This is a power basis and thus the discriminant is the square of a Vandermonde determinant

$$\Delta[1, \zeta, \zeta^2, \zeta^3] = \prod_{1 \leq i < j \leq 4} (\zeta^i - \zeta^j).$$

We pull out factors of ζ and get

$\zeta^{10}(1-\zeta)(1-\zeta^2)(1-\zeta^3)(1-\zeta)(1-\zeta^2)(1-\zeta) = (1-\zeta)^3(1-\zeta^2)^2(1-\zeta^3).$

The discriminant is the square of this value. After a tedious calculation, one finds that the discriminant is 125.

Note that as our $\mathbb{Q}$-basis is in fact an integral basis, this is the discriminant of the field.

2. Prove that 7 is not a prime element in the ring $\mathbb{Z}[\sqrt{-5}]$.

All prime elements are irreducible, thus to show 7 is not prime, it is enough to show it is reducible. But 7 is in fact irreducible in this ring. Thus we must take a different tack.

To show that 7 is not prime we only need to show that there exist $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ such that $7|\alpha\beta$ but 7 doesn't divide α or β. To show this, simply note that $14 = 7 \times 2 = (3+\sqrt{-5})(3-\sqrt{-5})$. To show that 7 does not divide either $(3+\sqrt{-5})$ or $(3+\sqrt{-5})$ it is enough to consider norms (the norm of 7 is 49, but the norm of the other two elements is 7).

3. Prove that if $11|\mathcal{N}(\alpha)$, for $\alpha \in \mathbb{Z}[\sqrt{-5}]$, then $11|\alpha$, and use this fact to prove that 11 is prime in $\mathbb{Z}[\sqrt{-5}]$.

Let $\alpha = a + b\sqrt{-5}$. Note $\mathcal{N}(\alpha) = a^2 + 5b^2$. Let us suppose 11 divides $a^2 + 5b^2$. We go through the possibilities for a^2 and $5b^2$ modulo 11. If a is not 0 modulo 11 then a^2 has one of the values $1, 4, 9, 5, 3$ modulo 11. If b is not 0 modulo 11 then $5b^2$ must have one of the values $5, 9, 1, 3, 4$ modulo 11. We easily see that the only possibility if for both a and b to be 0 modulo 11. In other words $\alpha = 11(a' + b'\sqrt{-5})$ and so α is divisible by 11.

To prove 11 is prime in the ring we need to show that if 11 divides $\alpha\beta$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ then if 11 does not divide α, it divides β. But taking norms we have that 11^2 divides $\mathcal{N}(\alpha)\mathcal{N}(\beta)$. But if 11 does not divide α then by what we have proved above, 11 cannot divide $\mathcal{N}(\alpha)$. But this implies that 11 divides $\mathcal{N}(\beta)$ and thus by what we proved above, 11 divides β. Thus 11 is prime in $\mathbb{Z}[\sqrt{-5}]$.

E-mail address: `hart_wb@yahoo.com`