7. ETA EVALUATIONS USING GENERALIZED WEBER FUNCTIONS

Introduction

This chapter is intended to continue the work which we began in chapter two, of providing actual eta evaluations. As hinted in the previous chapters we are now able to provide a number of new evaluations which depend not on Weber functions but on the modular equations for their generalizations. In particular we will make use of the Schläfli type modular equations which we developed earlier.

As in chapter two we need to be very familiar with the various identities which our functions satisfy. These are recorded in chapter five, at least for the level 3 and 5 functions.

1. Class Number Five Evaluations Using Level Three Functions

We begin by adapting a technique of Weber. We let ω be the root $(-r+\sqrt{-m})/2$ where $m = 4n - r^2$, of the quadratic form $z^2 + rz + n$. Note in particular that

$$\omega/n = -1/(\omega + r). \tag{1}$$

We cannot make use of this in the case of the class number seven disciminінant $d = -71$, since to express $71 = 4n - r^2$ for a prime degree n is not possible (consider the equation modulo 8).

We do have however $131 = 4 \times 53 - 9^2$.

We make use of the Schläfli type modular equation for level three functions and degree $n = 53$. In general this will provide a polynomial relationship between $\mathfrak{g}_1(\tau)$ and $\mathfrak{g}_1(53\tau)$ for general τ.

Firstly we make the transformation $\tau \to \tau+1$ in our modular equation. This simply makes it a polynomial relationship between $\zeta_{12}^{-1}\,\mathfrak{g}_2(\tau)$ and $-\mathfrak{g}_3(53\tau)$. Alternatively, since this will be an identity for all τ, sending $\tau \to \tau/53$ this becomes a relationship between $\zeta_{12}^{-1}\,\mathfrak{g}_2(\tau/53)$ and $-\mathfrak{g}_3(\tau)$.

However for the value $\tau = \omega = (-9+\sqrt{-131})/2$ this will become a relationship involving $\zeta_{12}^{-4}\,\mathfrak{g}_3(\omega)$ and $-\mathfrak{g}_3(\omega)$ or between the values $\zeta_6^{-1}\,\mathfrak{g}_1((1+\sqrt{-131})/2)$ and $\zeta_6^{-2}\,\mathfrak{g}_1((1+\sqrt{-131})/2)$.

The relevant modular equation defines

$$A_\infty = (uv_\infty)^2 + 3^2/(uv_\infty)^2 \quad \text{and} \quad B_\infty = (v_\infty/u)^3 - (u/v_\infty)^3.$$

Thus in our case if we let $x = \mathfrak{g}_1(\omega)$ these functions will reduce to the values

$$A = x^4 + 9/x^4 \quad \text{and} \quad B = 0. \tag{2}$$

The following piece of code calculates the modular equation of degree 53 and stores it in the variable m. It has been adapted from the code used in chapter 6.

```
etaq = prod(n=1,300,1-q^n+O(q^300))
etaq53 = subst(etaq+O(q^6),q,q^53)+O(q^300)
gq = etaq/subst(etaq,q,q^3)
gq53 = etaq53/subst(etaq53,q,q^3)
a = vector(14);b=vector(10)
a[1] = 1;b[1] = 1
a[2] = q^(-9)*(gq*gq53)^2+3^2/(q^(-9)*(gq*gq53)^2)
b[2] = q^(-13)*(gq53/gq)^3-q^(13)*(gq/gq53)^3
for(i=3,14,a[i]=a[i-1]*a[2])
for(i=3,10,b[i]=b[i-1]*b[2])
n = a[14]-b[10]
m = A^13-B^9
div9(y)=for(k=0,100,if(y-k*9==0,return(k)))
l(z)=for(i=0,9,if((z-i*13)%9==0,return(B^i*A^((z-i*13)/9))))
larr(x)=for(p=0,9,if((x-p*13)%9==0,return(b[p+1]*a[div9(x-p*13)+1])))
for(j=0,116,if(polcoeff(n,j-116,q)<>0,m=m-polcoeff(n,j-116,q)*l(116-j
   );n=n-polcoeff(n,j-116,q)*larr(116-j)))
```

In order to find the minimum polynomial for $x = \mathfrak{g}_1(\omega)$ we simply substitute the values given by (2), remove the denominator by multiplying by x^{52} and factorize the resulting polynomial in x. This can be done with the following code fragment

```
A = x^4+9/x^4
B = 0
factor(eval(m))
```

We find that there are only two factors which have degree divisible by 5 and it is not hard to substitute the value $\mathfrak{g}_1(\omega)$ into both and find that the correct minimal polynomial is $x^{20} - 13x^{18} + 38x^{16} + 9x^{14} + 185x^{12} + 352x^{10} + 555x^8 + 81x^6 + 1026x^4 - 1053x^2 + 243$.

The major disadvantage of this approach is that it does not calculate the complex absolute value of the eta quotient but the quotient itself. It is not trivial to determine the minimum equation of one from the other. We would like to calculate the minimal polynomial of $|x|^2/\sqrt{3}$ since it is this value which we conjecture is a real unit in the Hilbert class field. Unfortunately our methods have not led to such an evaluation.

It is clear from the minimal polynomial that x^2 is of degree 10 over $\mathbb{Q}$ and so it must be in the Hilbert class field. This is precisely what was proved using Gee's results in chapter 1.

We move on to the next discriminant of class number five which is $d = -179$. We have that $179 = 4 \times 47 - 3^2$ and so the same technique as above can be applied.

We first make the transformation $\tau \to \tau + 1$ in the modular equation of degree 47 for signature 3 functions. We end up with a polynomial relationship between $\zeta_{12}^{-1}\,\mathfrak{g}_2(\tau)$ and $\zeta_{12}^{-4}\,\mathfrak{g}_3(47\tau)$. Then we replace τ with $\tau/47$.

Now by virtue of the fact that (1) holds and $m = 179$, $n = 47$ and $r = 3$ we see that our modular equation becomes, for $\tau = \omega = (1 + \sqrt{-179})/2$, a polynomial relationship between $\zeta_{12}^{-1}\,\mathfrak{g}_1((1+\sqrt{-179})/2)$ and $\zeta_{12}^{-3}\,\mathfrak{g}_1((1+\sqrt{-179})/2)$.

Now for the modular equation of degree 47 we must have

$$A_\infty = (uv_\infty) + 3/(uv_\infty) \quad \text{and} \quad B_\infty = (v_\infty/u)^6 + (u/v_\infty)^6.$$

But from what we have just written we see that if we define $x = \zeta_6^{-1}\,\mathfrak{g}_1((1 + \sqrt{-179})/2)$ then $A = x^2 + 3/x^2$ whilst $B = 2$.

We modify the code that was used in the last example to obtain the modular equation of degree 47. We make the substitutions just mentioned and then factor the resulting expression to obtain a minimal polynomial for the value x.

There are two factors whose degree is divisible by 5. It is not hard to substitute the value $x = \zeta_6^{-1}\,\mathfrak{g}_1((1+\sqrt{-179})/2)$ into both factors and discover that in fact $x^{20} + 17x^{18} + 170x^{16} + 607x^{14} + 1869x^{12} + 3400x^{10} + 5607x^8 + 5463x^6 + 4590x^4 + 1377x^2 + 243$ is its minimal polynomial.

Once again x^2 has minimal polynomial of degree 10 over $\mathbb{Q}$ and since we already knew that x^2 must be in some extension of the Hilbert class field (since it is a root of a discriminant function quotient) which is degree 10 over $\mathbb{Q}$ we know that it must actually lie in the Hilbert class field. Again this is what we proved in chapter one.

2. A Class Number Five Evaluation Using Level Five Functions

We return to the discriminant $d = -131$ of class number five. One of the values we would like to evaluate is $\mathfrak{h}_1((3+\sqrt{-131})/2)$ where $\mathfrak{h}_1(\tau) = \eta(\tau/5)/\eta(\tau)$ is the level five function defined in chapter five.

The method we use is virtually the same as for the last section. We again use the fact that $131 = 4 \times 53 - 9^2$. We let ω be the root of $z^2 + rz + n$ indicated at the start of section one, where $r = 9$, $m = 131$ and $n = 53$.

Our modular equation of degree 53 for level 5 functions is a polynomial relationship between $\mathfrak{h}_1(\tau)$ and $\mathfrak{h}_1(53\tau)$.

After applying the transformation $\tau \to \tau + 2$ and replacing τ with $\tau/53$ it becomes a polynomial relationship between $\zeta_6^{-1}\,\mathfrak{h}_3(\tau/53)$ and $\zeta_6^{-4}\,\mathfrak{h}_2(\tau)$.

Substituting the value $\tau = \omega = ((1+\sqrt{-131})/2)$ it becomes a relation between $\zeta_6^{-1}\,\mathfrak{h}_1((1+\sqrt{-131})/2)$ and $\zeta_6^{-2}\,\mathfrak{h}_1((1+\sqrt{-131})/2)$.

The modular equation of degree 53 for level 5 functions will have

$$A_\infty = (uv_\infty) + 5/(uv_\infty) \quad \text{and} \quad B_\infty = (v_\infty/u)^3 + (u/v_\infty)^3.$$

Letting $x = \mathfrak{h}_1((1+\sqrt{-131})/2)$ and plugging in the values above, these functions can be expressed $A = -x^2 - 5/x^2$ and $B = -2$.

Following Pari code to required to generate the modular equation of degree 53 for level 5 functions. It is included since it differs in important ways from the code we presented for the degree 53 level 3 case and took the author some time to adapt correctly from the earlier construction. In particular it needs to check that the total order of each of the terms of the modular equation is even.

```
etaq = prod(n=1,500,1-q^n+O(q^500))
etaq53 = subst(etaq+O(q^10),q,q^53)+O(q^500)
gq = etaq/subst(etaq,q,q^5)
gq53 = etaq53/subst(etaq53,q,q^5)
a = vector(53);b=vector(19)
a[1] = 1;b[1] = 1
a[2] = q^(-9)*(gq*gq53)+5/(q^(-9)*(gq*gq53))
b[2] = q^(-26)*(gq53/gq)^3+q^(26)*(gq/gq53)^3
for(i=3,53,a[i]=a[i-1]*a[2])
for(i=3,19,b[i]=b[i-1]*b[2])
n = a[53]-b[19]
m = A^52-B^18
div9(y)=for(k=0,100,if(y-k*9==0,return(k)))
l(z)=for(i=0,18,if((z-i*26)%9==0,if(((z-i*26)/9+i)%2==0,return(B^i*A^
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   ((z-i*26)/9)))))
larr(x)=for(p=0,18,if((x-p*26)%9==0,if(((x-p*26)/9+p)%2==0,return(b[p
   +1]*a[div9(x-p*26)+1]))))
for(j=0,467,if(polcoeff(n,j-467,q)<>0,m=m-polcoeff(n,j-467,q)*l(467-j
   );n=n-polcoeff(n,j-467,q)*larr(467-j)))
```

Making the substitutions above for A and B and factorizing we find there is only one factor whose degree is divisible by 5. Thus we find that $x^{20} - 13x^{18} + 36x^{16} + 561x^{14} + 335x^{12} - 2552x^{10} + 1675x^8 + 14025x^6 + 4500x^4 - 8125x^2 + 3125$ is the minimum polynomial of $x = \mathfrak{h}_1((3 + \sqrt{-131})/2)$.

3. Class Number Seven Evaluations

We wish to deal with the class number 7 case, $d = -71$. Firstly we write $-71 = 71^2 - 72 \times 71$. Thus we can take ω to be the root $(-71+\sqrt{-71})/2$ of $z^2+71z+18\times 71$. We note that $\omega/71 = -1/((\omega + 71)/18)$.

We use the modular equation of degree 71 for level three functions. We can make it into a polynomial in $\mathfrak{g}_1(\tau/71)$ and $\mathfrak{g}_1(\tau)$. Plugging in the value $\tau = \omega$ given above it becomes a relation between $\mathfrak{g}((71 + \sqrt{-71})/36)$ and $\mathfrak{g}_1((-71 + \sqrt{-71})/2)$. But this can be made into a relation between $\zeta_6\,\mathfrak{g}((-1 + \sqrt{-71})/36)$ and $\mathfrak{g}_1((1 + \sqrt{-71})/2)$ or between $\zeta_6\,\mathfrak{g}_1((1 + \sqrt{-71})/2)$ and $\mathfrak{g}_1((1 + \sqrt{-71})/2)$.

Now the modular equation of degree 71 (which is given in chapter six) defines

$$A_\infty = (uv_\infty) + 3/(uv_\infty) \quad \text{and} \quad B_\infty = (v_\infty/u)^6 + (u/v_\infty)^6.$$

Thus if we let $x = \zeta_{12}\,\mathfrak{g}_1((1 + \sqrt{-71})/2)$ then we have that $A = x^2 + 3/x^2$ and $B = 2$ for the particular value of τ we have chosen.

Substituting these into the modular equation of degree 71 and factorizing we find only one factor whose degree is divisible by 7. In fact the minimal polynomial of the value x is $x^{28} + 2x^{24} + 57x^{22} + 177x^{20} + 296x^{18} + 864x^{16} + 2081x^{14} + 2592x^{12} + 2664x^{10} + 4779x^8 + 4617x^6 + 486x^4 + 2187$.

We move on to the next discriminant of class number 7, $d = -151$. We write $-151 = 151^2 - 152 \times 151$. So we can take $\omega = (-151+\sqrt{-151})/2$ a root of $z^2+151z+38\times 151$. We see from the quadratic equation that $\omega/151 = -1/((\omega + 151)/38)$.

We use the modular equation of degree 151 for level five functions. It starts as a polynomial relationship between $\mathfrak{h}_1(\tau)$ and $\mathfrak{h}_1(151\tau)$. After sending $\tau \to \tau - 3$ and then making the change $\tau \to \tau/151$ it becomes a relation between $\mathfrak{h}_3(\tau/151)$ and $\mathfrak{h}_3(\tau)$.

For the specific value $\tau = \omega$ that we have chosen this becomes a relation between $\mathfrak{h}_3((151+\sqrt{-151})/76)$ and $\mathfrak{h}_3((-151+\sqrt{-151})/2)$. This can be changed to a relation between $\mathfrak{h}_5((-1 + \sqrt{-151})/76)$ and $\zeta_6^4\,\mathfrak{h}_1((3 + \sqrt{-151})/2)$ or between $\zeta_6\,\mathfrak{h}_1((3 + \sqrt{-151})/2)$ and $\zeta_6^4\,\mathfrak{h}_1((3 + \sqrt{-151})/2)$.

The modular equation of degree 151 must have

$$A_\infty = (uv_\infty)^3 + 5^3/(uv_\infty)^3 \quad \text{and} \quad B_\infty = (v_\infty/u) + (u/v_\infty).$$

If we take $x = \mathfrak{h}_1((3 + \sqrt{-151})/2)$ then the values of these functions at ω are equal to $A = -x^6 - 5^3/x^6$ and $B = -2$.

We adapt the Pari code that we used for the discriminants -131 and -179 and find the modular equation of degree 151 for signature 5 functions. We then substitute in the given value of A and B and factorize.

There are two factors whose degree is divisible by 7 and we easily find that the mimimum polynomial for the value x above is $x^{28} - 23x^{26} + 191x^{24} + 146x^{22} -$

$4249x^{20}+6199x^{18}+27581x^{16}-100057x^{14}+137905x^{12}+154975x^{10}-531125x^{8}+91250x^{6}+596875x^{4}-359375x^{2}+78125.$

It is unfortunate that our technique does not provide eta evaluations similar to those of chapter two, namely evaluations of the *absolute value* of an eta quotient. Of course we should still be happy to provide eta evaluations as we have, of the square of an eta quotient but an important question is raised by the difference in results between chapter two and the current chapter.

The astute reader will have realised that in chapter two we made use of Weber's modular equations of irrational form. These involve all the Weber functions and not just a single function. However the modular equations which we developed in chapter five and have used here are of Schläfli type and only involve the function $\mathfrak{g}_1$ or $\mathfrak{h}_1$, etc. The question is, what sort of modular equation plays the part of the equation of irrational form for our higher level functions?

In the next section we develop a modular equation which seems to be such a generalization. It involves all the level three functions at once. Even though it does not help us to provide eta evaluations similar to those of chapter two we report on the development of this kind of modular equation since it leads to identities which are most curious indeed and perhaps worthy of further study. This will be a final act in our development of modular equations which has arisen from our study in the evaluation of the Dedekind eta function.

4. A Permanent Identity

It will be necessary to consider modular equations of prime degree p where $p \equiv 1 \pmod 3$.

For our functions $\mathfrak{g}_i$ of level 3 we define functions associated to each $\mathfrak{g}_i$

$$v_\infty^{(i)} = \mathfrak{g}_i(p\tau)$$

$$v_c^{(i)} = \left(\frac{3}{p}\right) \mathfrak{g}_i((\tau + 36mc)/p) \text{ for } -(p-1)/2 \le c \le (p-1)/2$$

where $36mc = ap + 1$ is the smallest natural number congruent to 1 modulo p.

Note: we also extend this defintion to the function $\mathfrak{g}$, where we write simply v_∞ and v_c for the appropriate functions associated with $\mathfrak{g}$.

Also for convenience, we let $u^{(i)} = \mathfrak{g}_i(\tau)$, and also let $u = \mathfrak{g}(\tau)$.

We will be interested in functions $\Psi(\tau)$, built polynomially from the functions $u^{(i)}v_c^{(i)}$, which are invariant under the full modular group; or what is the same thing, invariant under both the modular substitutions S and T. Thus these functions will belong to the modular function field $\mathbb{C}(j)$.

In addition we will demand that these functions $\Psi(\tau)$ have no poles in the complex upper half plane or at $i\infty$. The first of these conditions ensures that Ψ is expressible as a polynomial in j, and the second, in light of the q-expansion for j, ensures that it is in fact a constant. Thus we will have the identity $\Psi(\tau) = C$ for some constant C and all τ in the upper half plane. This identity will be the modular equation we are after.

The condition that there be no poles on the upper half plane is automatic, given that our functions will be built from terms like $u^{(i)}v_c^{(i)}$ which themselves have no such poles.

Now to simplify matters, an argument similar to that in chapter five shows that we only need to construct a function $\Psi_0(\tau)$ built polynomially from the $u^{(i)}v_\infty^{(i)}$, which vanishes at $q = 0$. This then ensures an identity $\Psi_\infty(\tau) = 0$ which will be the modular equation we are after.

We now investigate the action of the modular substitutions on the functions $u^{(i)}$ and $v^{(i)}$. Firstly for the $u^{(i)}$ the relevant data is easily found and summarized in the following table.

	u	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$
$\tau \to \tau + 1$	$\varepsilon_1 u$	$\varepsilon_2 u^{(2)}$	$u^{(3)}$	$u^{(1)}$
$\tau \to -1/\tau$	$u^{(1)}$	u	$u^{(3)}$	$u^{(2)}$

where $\varepsilon_1 = e^{\frac{2\pi i}{12}}$ and $\varepsilon_2 = e^{-\frac{2\pi i}{12}}$.

Now for the v's. Firstly consider the action $\tau \to \tau + 1$.

For $v_\infty : \mathfrak{g}(p(\tau+1)) = e^{\frac{2p\pi i}{12}}\, \mathfrak{g}(p\tau) = e^{\frac{2p\pi i}{12}}\, v_\infty$.

For $i = 2, 3$ we have $v_\infty^{(i)} : \mathfrak{g}_i(p(\tau+1)) = e^{-\frac{2\pi i}{12}\frac{(p-1)}{3}}\, \mathfrak{g}_{i+1}(p\tau) = e^{-\frac{2\pi i}{12}\frac{(p-1)}{3}}\, v_\infty^{(i+1)}$ where we adjust, as we will from now on without comment, $(i+1)$, by three if necessary, so that it is in the range 1..3 and for $i = 1$ we have $v_\infty^{(1)} : \mathfrak{g}_1(p(\tau+1)) = e^{-\frac{2\pi i}{12}\frac{(p+2)}{3}}\, \mathfrak{g}_2(p\tau) = e^{-\frac{2\pi i}{12}\frac{(p+2)}{3}} v_\infty^{(2)}$.

For the other $v_c^{(i)}$ we note that

$$\frac{\tau + 36mc + 1}{p} = \frac{\tau + 36m(c+1) - ap}{p}.$$

Thus

$$v_c(\tau + 1) = e^{-\frac{2a\pi i}{12}}\, v_{c+1} = e^{\frac{2p\pi i}{12}}\, v_{c+1},$$

since $a \equiv -p \pmod{12}$. Note we also adjust the $c + 1$ by p if necessary (which we can do in light of the periodicity of our function) so that it lies in the correct range.

Now noting that $a \equiv 2 \pmod 3$ in light of the fact that $p \equiv 1 \pmod 3$, we also have

$$\begin{aligned}
v_c^{(1)}(\tau+1) &= e^{\frac{2\pi i}{12}\frac{(a-2)}{3}} v_{c+1}^{(2)}(\tau) \\
v_c^{(2)}(\tau+1) &= e^{\frac{2\pi i}{12}\frac{(a+1)}{3}} v_{c+1}^{(3)}(\tau) \\
v_c^{(3)}(\tau+1) &= e^{\frac{2\pi i}{12}\frac{(a+1)}{3}} v_{c+1}^{(1)}(\tau)
\end{aligned}$$

again adjust $c + 1$ wherever necessary.

Turning to $\tau \to -1/\tau$ we have

$$v_\infty : \mathfrak{g}(-p/\tau) = \mathfrak{g}_1(\tau/p) = \left(\frac{3}{p}\right) v_0^{(1)}$$

and vice versa, and

$$v_0 : \left(\frac{3}{p}\right) \mathfrak{g}(-1/p\tau) = \left(\frac{3}{p}\right) \mathfrak{g}_1(p\tau) = \left(\frac{3}{p}\right) v_\infty^{(1)}$$

and vice versa.

Similarly we have

$$v_\infty^{(2)} : \mathfrak{g}_2(-p/\tau) = \mathfrak{g}_3(\tau/p) = \left(\frac{3}{p}\right) v_0^{(3)}$$

and vice versa, and

$$v_0^{(2)} : \left(\frac{3}{p}\right) \mathfrak{g}(-1/p\tau) = \left(\frac{3}{p}\right) \mathfrak{g}_3(p\tau) = \left(\frac{3}{p}\right) v_\infty^{(3)}$$

and vice versa.

Now consider the other $v_c^{(i)}$ and v_c for $c \neq 0, \infty$.

We solve

$$\begin{pmatrix} 1 & 36mc \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & p \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

This gives the following series of equations assuming the first matrix on the right is in $SL_2(\mathbb{Z})$.

$$36mc = \delta, \quad -1 = \delta A + \gamma p, \quad p = \beta, \quad 0 = \beta A + \alpha p, \quad \alpha\delta - \beta\gamma = 1.$$

Clearly the third and fourth equalities imply $\alpha = -A$. Then the final one follows automatically from the others. Thus it only remains for us to satisfy the second of these requirements. We need

$$\gamma = \frac{-36mcA - 1}{p} \in \mathbb{Z}.$$

So we want $36mcA - 1 \equiv 0 \pmod p$. If we additionally impose the condition that $A = 36mc'$, then clearly we need $cc' \equiv -1 \pmod p$, i.e: $c' \equiv -1/c \pmod p$ as we might have expected.

Now, referring to the linear tranformations of the functions $\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2$ and $\mathfrak{g}_3$ as given in chapter five, we obtain the following.

$$v_c : v_c(-1/\tau) = (-1)^{\frac{\beta-1}{2}} \left(\frac{\beta}{3}\right) v^{(1)}_{-1/c}(\tau) = \left(\frac{3}{p}\right) v^{(1)}_{-1/c}(\tau)$$

since $\beta = p \equiv 1 \pmod 3$.

$$v_c^{(1)} : v_c^{(1)}(-1/\tau) = (-1)^{\frac{\gamma-1}{2}} \left(\frac{\gamma}{3}\right) v_{-1/c}(\tau) = \left(\frac{3}{p}\right) v_{-1/c}(\tau)$$

since $\gamma p \equiv -1 \pmod{36}$.

$$v_c^{(2)} : v_c^{(2)}(-1/\tau) = e^{\frac{2\pi i}{3}} \left(\frac{3}{p}\right) e^{-\frac{2\pi i}{9}[4p^2-1]} v^{(3)}_{-1/c}(\tau).$$

$$v_c^{(3)} : v_c^{(3)}(-1/\tau) = e^{\frac{2\pi i}{3}} \left(\frac{3}{p}\right) e^{-\frac{2\pi i}{9}[4\gamma^2-1]} v^{(2)}_{-1/c}(\tau) = e^{\frac{2\pi i}{3}} \left(\frac{3}{p}\right) e^{-\frac{2\pi i}{9}[4p-1]} v^{(2)}_{-1/c}(\tau)$$

since if $p \equiv 1,\ 4,\ 7 \pmod 9$ then $\gamma^2 \equiv 1,\ 4,\ 7 \pmod 9$ respectively.

As with our previous modular equations, we don't particularly care how the various functions are permuted. We only care about the various roots of unity that are induced by the two transformations. These are summarized in the following tables.

	uv_∞	uv_0	uv_c	$uv_\infty^{(1)}$	$uv_0^{(1)}$	$uv_c^{(1)}$
S	ε_1	ε_1	ε_1	ε_2	ε_3	ε_3
T	ε_6	ε_6	ε_6	ε_6	ε_6	ε_6

	$uv_\infty^{(2)}$	$uv_0^{(2)}$	$uv_c^{(2)}$	$uv_\infty^{(3)}$	$uv_0^{(3)}$	$uv_c^{(3)}$
S	ε_4	ε_5	ε_5	ε_4	ε_5	ε_5
T	ε_6	ε_6	ε_6	ε_6	ε_6	ε_6

where $\varepsilon_1 = e^{\frac{2(p+1)\pi i}{12}}$, $\varepsilon_2 = e^{-\frac{2(p+5)\pi i}{12}}$, $\varepsilon_3 = e^{\frac{2\pi i}{12}\frac{a-5}{3}}$, $\varepsilon_4 = e^{-\frac{2\pi i}{12}\frac{p-1}{3}}$, $\varepsilon_5 = e^{\frac{2\pi i}{12}\frac{a+1}{3}}$ and $\varepsilon_6 = \left(\frac{3}{p}\right)$.

These roots of unity clearly depend on p modulo 36. There are six cases $p \equiv 1, 7, 13, 19, 25, 31 \pmod{36}$. However in all cases it is convenient to introduce

$$A_c = (uv_c)^3 - (u^{(1)}v_c^{(1)})^3 - (u^{(2)}v_c^{(2)})^3 + (u^{(3)}v_c^{(3)})^3$$

$$B_c = 1/(uv_c)^3 - 1/(u^{(1)}v_c^{(1)})^3 - 1/(u^{(2)}v_c^{(2)})^3 + 1/(u^{(3)}v_c^{(3)})^3$$

We then find a polynomial $F(x, y)$ so that the q-series of $F_\infty = F(A_\infty, B_\infty)$ disappears upto and including the constant term. The $F_c = F(A_c, B_c)$ are then permuted upto sign by our transformations. In the cases $p \equiv 7, 19, 31 \pmod{36}$ they are permuted exactly. In the remaining cases, they are permuted always with a factor of -1. However in all cases the q-series of the F_c can be transformed into each other upto sign if the total degrees of the terms of F are either all odd or all even.

We look at the q-series for some small primes and give the corresponding modular equations:

For $p = 7$:

$$A_\infty(q^3) = 0 + O(q)$$

For $p = 13$:

$$A_\infty(q^6-) = 0 + O(q)$$

For $p = 19$:

$$A_\infty(q^3) = 0 + O(q)$$

For $p = 31$:

$$A_\infty(q^3) = 0 + O(q)$$

As can be seen these are all the same. In fact numerically we have tested the modular equations of this form for many values of the degree p. Not only do we obtain $A = 0$ for p a prime satisfying the specified congruence but in fact the relation appears to hold for all natural numbers (in fact the identity appears to be even more general than that).

We do not prove this identity here since the proof must surely involve either determining special arithmetic properties of the q-series involved or the application of some heavy analysis. Both of these would be beyond the scope of this thesis, not being directly relevant to eta evaluations.

Nevertheless we conjecture that this identity holds for all degrees n where n is a natural number.

At first glance this identity might seem to follow from the identity involving sixth powers of the functions $\mathfrak{g}_i$ which is reported in the second section of chapter five. However closer inspection reveals that this is merely the special case of our new identity where $n = 1$.

In fact this new identity seems to be some kind of *permanent identity* (see Fine [1] §41 for examples of permanent identities). There are many open questions related to this sort of identity not the least of which is whether they truly are the correct generalization of the modular equations of irrational form given by Weber for his functions.

We have not looked for similar identities for our level 5, 7 or 13 functions and there are clearly a multitude of future research directions which have opened up to us and which might be pursued in the future. Due to constraints of time and space we cannot pursue them all here but it will be interesting to see these questions investigated further in the future.

References

[1] Nathan J. Fine *Basic hypergeometric series and applications* Mathematical Surveys and Monographs, 27. Amer. Math. Soc., Providence, 1988.