4. A GUIDE TO WEBER'S METHOD

Introduction

The purpose of this chapter is to detail Weber's development of Schläfli modular equations and modular equations of irrational kind.

This has necessitated translation of extensive portions of Weber's Lehrbuch [2] from German and a concise rearrangement and summary of the material.

It is not our intention to claim any great originality here. Nor has it been the intention to give exhaustive proofs and commentary. Rather, we have opted to fill in gaps where they are perceived to exist in Weber and to give the reader information which they may not so easily find in modern treatments of the topic.

In particular some of the proofs and results are new. In particular the treatment of the theorems of section 1.6 is new. Weber takes a long excursus at this point through the theory of the sigma function and seems unaware that such a direct path exists. We have added extra intermediate results here in order to make the theory go through.

Towards the end of the chapter we bring together many of Weber's comments from throughout his book in order to explain how he justified his modular equations. It was not clear from reading Weber that he had actually done this anywhere, however we found that many of the details are scattered throughout his book somewhere, even if they are only touched on briefly.

One of our main aims has been to save the reader time. All the unnecessary material is pared away, holes are filled in and difficult parts of the argument are hopefully elucidated. However it is still Weber's method, not the modern one. In that sense this chapter can be viewed as a 'guide to Weber' which may profitably be read alongside his book. Above all we were keen to present a summary which did not span the 200 pages (pages 60-274 of [2]) of Weber's own treatment.

1. Theta Functions

Our first task will be to describe Weber's presentation of the theory of theta functions. Although theta functions had been discovered well before Weber's time, he follows an unusual route through their development. However this extended process appears to have paid dividends in that it may have been what inspired his understanding of modular equations; which is also our aim in synthesizing his study.

1.1. T-Functions.

We begin by looking at a type of function more general than a theta function, i.e. that which Weber, in [2] §17, calls a *T-function.*

Definition 1.1.1. *An entire function $T(u)$ with two "quasi-periods", ω_1 and ω_2, is called a T-*function*, i.e.*

$$T(u+\omega_1) = \exp(-\pi i(a_1(2u+\omega_1)+b_1))\,T(u) \tag{1}$$

$$T(u+\omega_2) = \exp(-\pi i(a_2(2u+\omega_2)+b_2))\,T(u) \tag{2}$$

with a_i, b_i some (usually rational) constants. The exponential factors on the right hand side are called period factors*.*

If $T(u)$ vanishes at a point $u_0 \in \mathbb{C}$, then it clearly vanishes at all congruent points modulo the lattice $[\omega_1, \omega_2]$. Thus there is a concept of a fundamental parallelogram for a T-function, defined by its lattice of quasi-periods, much as there is for an elliptic function.

We say that $T(u)$ is of m-th order if it vanishes at exactly m points in a fundamental parallelogram. Clearly multiplying two T-functions, of orders m and m' respectively, gives a T-function of order $m + m'$, counting multiplicities. The new period factors are also easily seen to be the products of the corresponding original period factors.

The first interesting question that presents itself is that of finding the space of all T-functions of 0-th order. Weber shows

Theorem 1.1.2. *An entire function $L(u)$ is a T-function of 0-th order iff it is of the form*

$$L(u) = C \exp(-\pi i(\lambda u^2 + \mu u)) \tag{3}$$

where C, λ and μ are arbitrary constants (it will sometimes be useful to consider the zero function to be a T-function of order 0).

By what we have just said, a T-function of order m, when multiplied by such a function $L(u)$, remains a T-function of order m and vanishes at exactly the same points.

By integrating around a period parallelogram for an m-th order T-function $T(u)$ one finds

Theorem 1.1.3. *A T-function with quasi-periods given by (1) and (2) has order*

$$m = a_2\omega_1 - a_1\omega_2.$$

Similarly by integrating the logarithmic derivative of $T(u)$ we obtain

Theorem 1.1.4. *If the (not necessarily distinct) zeroes of $T(u)$ within its period parallelogram are $\alpha_1, \alpha_2, \ldots, \alpha_m$ and the lattice of quasi-periods is $L = [\omega_1, \omega_2]$, then*

$$\sum \alpha_i \equiv \frac{1}{2}(b_1\omega_2 - b_2\omega_1) + \frac{m}{2}(\omega_1 + \omega_2) \ \ (mod\, L). \tag{4}$$

The quantity on the right is called the *character* of the T-function.

The values b_1 and b_2 are usually given rational values g_1 and g_2 and clearly from the definition only need to be defined modulo 2. The symbol (g_1, g_2), given modulo 2, is then called the *characteristic* of the T-function. It is often subscripted, $T_{(g_1,g_2)}(u)$, for convenience.

The product of two arbitrary T-functions with respective characters γ_1 and γ_2 and characteristics (g_1, g_2) and (g_1', g_2') is a T-function with character $\gamma_1 + \gamma_2$, and characteristic $(g_1 + g_1', g_2 + g_2')$ (taken modulo 2).

Later we look at theta functions with characteristics. One needs to take some care however, as modern accounts of theta functions, such as [2], define period factors with the quantity $-2\pi i$ replacing $-\pi i$ in the expressions which correspond to (1) and (2). This leads to characteristics defined modulo $\mathbb{Z}$.

1.2. **Similar T-functions.** In [2] §18, Weber calls two T-functions with the same quasi-periods, ω_1, ω_2, the same orders and same characters *similar* T-functions. Requiring that a set of T-functions be similar imposes restrictions on the period factors that appear. This information can be used to show

Theorem 1.2.1. *Given two similar T-functions $T_1(u)$ and $T_2(u)$ there exists a T-function $L(u)$ of 0-th order such that*

$$T_2(u) = L(u) \cdot T_1(u).$$

A useful corollary is that if similar T-functions share $m-1$ zeroes in the period parallelogram then they also share the m-th. This also follows from the relation (4) for the character of a T-function.

We are now able to prove the following theorem which will be very important in the sequel.

Theorem 1.2.2. *Between any $m+1$ similar T-functions $T, T_1, T_2, \ldots, T_m$ of order m there exists a relation*

$$LT = L_1T_1 + L_2T_2 + \cdots + L_mT_m \tag{5}$$

with 'coefficients' $L, L_1, \ldots, L_m$ which are T-functions of 0-th order, not all zero.

Proof: We assume that we don't already have such a relation purely between the m functions T_i of the right hand side. We will show that a relation then exists between the $m+1$ functions as stated.

According to (3), each of the L_i on the right hand side has two internal parameters, λ_i and μ_i, and one external parameter, C_i, say. It is possible to choose the internal parameters of the L_i so that each term L_iT_i is a T-function with the same period factors as T.

Setting the right hand side of (5) to be zero at each of the zeroes of T we end up with m equations in the m unknowns $C_1, C_2, \ldots, C_m$. Thus we can set the C_i so that the right hand side is zero precisely when T is. Then the right hand side is a T-function similar to T. The result now follows by application of Theorem (1.2.1). □

We now wish to show that we can explicitly construct a T-function of given order m and lattice $L = [\omega_1, \omega_2]$ that has any m given zeroes $\alpha_1, \alpha_2, \ldots, \alpha_m$ modulo L.

This is indeed possible if we can explicitly construct but a single T-function of first order with lattice L and a specified zero. For, let such a $t(u)$ be given with character γ. Since $t(u)$ is of first order, we see from (4) that γ is simply the single zero of $t(u)$ modulo L. Define the functions

$$t_i(u) = t(u - \alpha_i + \gamma),\ i = 1, 2, \ldots, m$$

where the α_i are the zeroes that we require our T-function of order m to have. Each of the t_i has exactly one zero mod L, namely the respective α_i. Then the function

$$T(u) = t_1(u)\, t_2(u) \ldots t_m(u)$$

is a T-function of m-th order with the given zeroes $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Again we can ensure that $T(u)$ has any given period factors by multiplying by an appropriate 0-th order T-function $L(u)$. Thus our assertion holds and it is indeed enough to construct a T-function of order one with specified zero.

We note as an aside that any elliptic function with lattice L can be constructed as the quotient of two appropriate T-functions.

Weber begins his construction of a first order T-function with lattice L in [2] starting at §19.

We have already noted that an arbitrary T-function can be adjusted by multiplying it by a 0-th order T-function $L(u)$. In this way we obtain a T-function which we have termed 'similar', but this new T-function may have different period factors.

Because of the restrictions that exist on the parameters a_1, a_2, b_1 and b_2 of the period factors of classes of similar T-functions this process may not allow us to eliminate the period factors altogether, however Weber proves the following

Theorem 1.2.3. *The most general T-functions of first order can be constructed from a single first order T-function with parameters $a_1 = 0$, $b_1 = g_1$ and $b_2 = g_2$, where (g_1, g_2) is the characteristic taken modulo 2.*

We find by theorem (1.1.3) that $a_2 = 1/\omega_1$ is fixed by this choice. Thus the quasi-period relations (1) and (2) for such a function $t(u)$ become

$$t(u + \omega_1) = \exp(-\pi i g_1)\; t(u) \tag{6}$$

$$t(u + \omega_2) = \exp(-\pi i(2u + \omega_2)/\omega_1)\; \exp(-\pi i g_2)\; t(u). \tag{7}$$

There is only one function (up to a constant independent of u) with these properties. For if there were another the quotient of the two, being entire and bounded in the complex plane, must be constant by Liouville's theorem.

Since the function t is dependent on u, ω_1 and ω_2 we denote it by $t(u, \omega_1, \omega_2)$. We may also subscript the characteristics (g_1, g_2) if these are not clear from the context.

We now only require the existence of a t-function which satisfies the conditions of theorem (1.2.3) to complete our proof that T-functions of all orders exist. It is by way of the existence of certain theta functions that we will be able to accomplish this.

1.3. **Theta Functions With Characteristics.** If h is an arbitrary parameter the function $t(hu, h\omega_1, h\omega_2)$ satisfies the same relations (6) and (7) as $t(u, \omega_1, \omega_2)$. Thus it is the same function except for a constant factor dependent only on the lattice.

Setting $h = 1/\omega_1, v = u/\omega_1$ and $\omega = \omega_2/\omega_1$ we therefore obtain a function

$$\theta(v, \omega) = t(v, 1, \omega) = C(\omega_1, \omega_2) \cdot t(u, \omega_1, \omega_2) \tag{8}$$

for some constant C dependent only on the lattice.

We order ω_1 and ω_2 in such a way that ω is in the complex upper half plane. The resulting function $\theta(v, \omega)$ is then called a *theta function with characteristics* (g_1, g_2).

From the properties of the T-function from which it is derived and its definition (8), we have the following (as per §20 of [2])

Definition 1.3.1. *A theta function with characteristics (g_1, g_2) is an entire function of v of first order which obeys the following relations*

$$\theta(v + 1) = \exp(-\pi i g_1)\; \theta(v) \tag{9}$$

$$\theta(v + \omega) = \exp(-\pi i(2v + \omega))\; \exp(-\pi i g_2)\; \theta(v), \tag{10}$$

i.e. it has quasi-periods 1 *and* ω.

We sometimes subscript the characteristics, $\theta_{g_1,g_2}(v,\omega)$, if these are not clear from the context.

The theta function is clearly a special case of the t-function (ie: T-function of first order). The general t-function can be recovered

$$t(u) = C\exp(-\pi i(\lambda u^2 + \mu u))\,\theta\left(\frac{u}{\omega_1}, \frac{\omega_2}{\omega_1}\right)$$

where C, λ, μ are constants independent of ω_1 and ω_2.

Jacobi is honoured by having the θ-functions with integral characteristics named after him. He constructed these explicitly and in fact it is not difficult to construct theta functions with arbitrary rational characteristics. For an elegant modern treatment of this see [1]. We will not reproduce this existence proof here.

Just as we can have T-functions of arbitrary order, we can do the same for theta functions. We define

Definition 1.3.2. *A Θ-function of order m is an entire function satisfying the following conditions*

$$\Theta(v+1) = \exp(-\pi i g_1)\,\Theta(v)$$
$$\Theta(v+\omega) = \exp(-m\pi i(2v+\omega))\,\exp(-\pi i g_2)\,\Theta(v).$$

Again we emphasize the characteristics by writing $\Theta_{g_1,g_2}(v,\omega)$.

These Θ-functions are all special cases of the T-function, and one can recover the general T-function via

$$T(u) = C\exp(-\pi i(\lambda u^2 + \mu u))\,\Theta\left(\frac{u}{\omega_1}, \frac{\omega_2}{\omega_1}\right).$$

We note that the definition of order for a theta function agrees with that for a T-function, by theorem (1.1.3).

We call Θ-functions with the same quasi-periods, same orders and same characteristics, *similar*. This agrees with the definition of similar T-functions, by theorem (1.1.4). In fact, theorem (1.1.4) on the character of a T-function becomes the following when applied to a Θ-function.

Theorem 1.3.3. *If $\Theta_{(g_1,g_2)}(v)$ is of m-th order with quasi-periods given by the lattice $L = [1,\omega]$ and has zeroes $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the period parallelogram, then*

$$\sum_{i=1}^{m} \alpha_i = m\frac{1+\omega}{2} + \frac{g_1\omega - g_2}{2} \quad (mod\ L).$$

An application of theorem (1.2.2) to Θ-functions then yields the so-called fundamental theorem of Θ-functions:

Theorem 1.3.4. *Any $m+1$ similar Θ-functions of order m are linearly dependent, with constant coefficients.*

1.4. **Jacobi Theta Functions.** In [2] §21, Weber now looks at what happens when the characteristics of θ-functions are chosen to be integral. There are four possibilites

$$(g_1, g_2) = (0,0), (0,1), (1,0) \text{ or } (1,1)$$

From their various properties we recognize these θ-functions to be the Jacobi θ-functions. The following is helpful when making a comparison with other literature. In the language of Jacobi

$$\theta_{00}(v) = \theta_3(v), \quad \theta_{01}(v) = \theta_4(v), \quad \theta_{10}(v) = \theta_2(v), \quad \theta_{11}(v) = \theta_1(v)$$

The first three of these functions are even functions, whilst the last is odd.

Each of them has precisely one zero in the fundamental parallelogram with sides 1 and ω

$$\theta_{11}(0) = 0, \quad \theta_{01}(\omega/2) = 0, \quad \theta_{10}(1/2) = 0, \quad \theta_{00}((1+\omega)/2) = 0 \tag{11}$$

It is natural to investigate theta functions of higher order built from these four functions of order one. Recall that multiplying theta functions with given orders and characteristics adds the orders and adds the characteristics modulo 2. Thus the squares of Jacobi θ-functions are all Θ_{00} functions of 2nd order.

By Theorem (1.3.4) there must be a linear relation between any three squares of Jacobi theta functions. These relations turn out to be

$$\theta_{01}^2\,\theta_{10}^2(v) = \theta_{10}^2\,\theta_{01}^2(v) - \theta_{00}^2\,\theta_{11}^2(v)$$
$$\theta_{01}^2\,\theta_{00}^2(v) = \theta_{00}^2\,\theta_{01}^2(v) - \theta_{10}^2\,\theta_{11}^2(v).$$

The coefficients in these expressions involve the so-called *theta constants*, which are the values of the θ-functions evaluated at $v = 0$, i.e. $\theta_{00} = \theta_{00}(0)$, etc. They are constants with respect to v but still depend on ω.

From these identities one can also derive the following identity:

$$\theta_{00}^4 = \theta_{01}^4 + \theta_{10}^4. \tag{12}$$

We do not offer proofs of these identities here, as they are found in many standard texts (see for example [1]).

1.5. **Higher Order Theta Functions.** We can continue to build up general Θ-functions of higher order with integral characteristics. But by the fundamental theorem of Θ-functions, arbitrary Θ-functions will have expressions in terms of bases of Θ-functions which we will choose in some convenient way.

We let, for example, Θ_0 and Θ_1 be any two different squares of Jacobi θ-functions. Because of the location of the zeroes of Jacobi theta functions (11), monomials of degree k composed of these two functions turn out to be linearly independent, and are clearly always even and of order $2k$. In fact they generate the space of all even Θ-functions of order $2k$.

Also, by a similar argument, given an arbitrary Θ-function with order m and specified characteristics, one can express it as the sum of monomials in the four Jacobi θ-functions, whose common degree is m and which all have the specified characteristics.

Now any θ-function appearing twice in such a monomial can be replaced by an expression in terms of the two theta squares Θ_0 and Θ_1. The result is that each monomial is broken into monomials of Θ_0 and Θ_1, possibly multiplied by some left over θ-functions, each appearing exactly once. But each monomial has been replaced with a new expression which must have the same order and characteristics as the original one. Clearly the only way this can happen is if the left over θ-functions are the same for each of the original monomials. Taking these out as a common factor, the expression that remains is a homogeneous polynomial in Θ_0 and Θ_1.

Weber provides the following table of such expressions starting from an arbitrary Θ-function of order m, denoted $\Theta = \Theta^{(m)}(v)$. We include his table here, with $F^{(k)}(\Theta_0, \Theta_1)$ denoting the resulting homogeneous polynomial in Θ_0 and Θ_1 of degree k with coefficients independent of v, which results from the process we have just mentioned

I. For m even and Θ even

$$\Theta_{00}^{(m)}(v) = F^{\left(\frac{m}{2}\right)}(\Theta_0, \Theta_1) \tag{13}$$

$$\Theta_{01}^{(m)}(v) = \theta_{00}(v)\,\theta_{01}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{14}$$

$$\Theta_{10}^{(m)}(v) = \theta_{00}(v)\,\theta_{10}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{15}$$

$$\Theta_{11}^{(m)}(v) = \theta_{10}(v)\,\theta_{01}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{16}$$

II. For m even and Θ odd

$$\Theta_{00}^{(m)}(v) = \theta_{00}(v)\,\theta_{10}(v)\,\theta_{01}(v)\,\theta_{11}(v)\,F^{\left(\frac{m-4}{2}\right)}(\Theta_0, \Theta_1) \tag{17}$$

$$\Theta_{01}^{(m)}(v) = \theta_{10}(v)\,\theta_{11}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{18}$$

$$\Theta_{10}^{(m)}(v) = \theta_{01}(v)\,\theta_{11}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{19}$$

$$\Theta_{11}^{(m)}(v) = \theta_{00}(v)\,\theta_{11}(v)\,F^{\left(\frac{m}{2}-1\right)}(\Theta_0, \Theta_1) \tag{20}$$

III. For m odd and Θ even

$$\Theta_{00}^{(m)}(v) = \theta_{00}(v)\,F^{\left(\frac{m-1}{2}\right)}(\Theta_0, \Theta_1) \tag{21}$$

$$\Theta_{01}^{(m)}(v) = \theta_{01}(v)\,F^{\left(\frac{m-1}{2}\right)}(\Theta_0, \Theta_1) \tag{22}$$

$$\Theta_{10}^{(m)}(v) = \theta_{10}(v)\,F^{\left(\frac{m-1}{2}\right)}(\Theta_0, \Theta_1) \tag{23}$$

$$\Theta_{11}^{(m)}(v) = \theta_{00}(v)\,\theta_{01}(v)\,\theta_{10}(v)\,F^{\left(\frac{m-3}{2}\right)}(\Theta_0, \Theta_1) \tag{24}$$

IV. For m odd and Θ odd

$$\Theta_{00}^{(m)}(v) = \theta_{01}(v)\,\theta_{10}(v)\,\theta_{11}(v)\,F^{\left(\frac{m-3}{2}\right)}(\Theta_0, \Theta_1) \tag{25}$$

$$\Theta_{01}^{(m)}(v) = \theta_{00}(v)\,\theta_{10}(v)\,\theta_{11}(v)\,F^{\left(\frac{m-3}{2}\right)}(\Theta_0, \Theta_1) \tag{26}$$

$$\Theta_{10}^{(m)}(v) = \theta_{00}(v)\,\theta_{01}(v)\,\theta_{11}(v)\,F^{\left(\frac{m-3}{2}\right)}(\Theta_0, \Theta_1) \tag{27}$$

$$\Theta_{11}^{(m)}(v) = \theta_{11}(v)\,F^{\left(\frac{m-1}{2}\right)}(\Theta_0, \Theta_1) \tag{28}$$

1.6. **Transformations Of Theta Functions.** We are now in a position to begin discussing transformations of theta functions (§27 of [2]). Weber notes that this investigation proves to be easier if transformations of T-functions are discussed first.

By a transformation of a T-function we mean a transformation of the modulus $[\omega_1', \omega_2']$ to another, $[\omega_1, \omega_2]$, by setting

$$\omega_1 = d\omega_1' - b\omega_2'$$
$$\omega_2 = -c\omega_1' + a\omega_2'$$

where a, b, c, d are integers, and $ad-bc = n$ is the determinant of the transformation. This transformation has the effect of changing to a new quasi-period parallelogram which is scaled in area by the factor n. We still have a T-function but the order of the new T-function is larger.

Thus our original T-function in ω_1', ω_2', $T'(u, \omega_1', \omega_2')$ say, can simultaneously be viewed as a T-function in ω_1 and ω_2, which we can write as $T(u, \omega_1, \omega_2) = T'(u, \omega_1', \omega_2')$.

If m and m' are the orders of T and T', then $m = m'n$ and if (g_1', g_2') and (g_1, g_2) are the corresponding characteristics then

$$(g_1, g_2) = (dg_1' - bg_2' - m'bd, -cg_1' + ag_2' - m'ac). \tag{29}$$

We now make the appropriate restrictions to turn our new transformed T-function T into a Θ-function. It is actually possible to pick the various parameters of T in such a way that T becomes a Θ-function of order m with no further scaling

$$T(u, \omega_1, \omega_2) = \Theta^{(m)}(u, \omega) \tag{30}$$

where $\omega = \omega_2/\omega_1$.

Also, if we carefully follow the details through we find that these restictions then imply

$$\omega_1' = \frac{a + b\omega}{n} \tag{31}$$

$$\omega_2' = \frac{c + d\omega}{n}. \tag{32}$$

However the original T-function, T', is then not quite a Θ-function, but

Lemma 1.6.1.

$$\Theta^{(m')}_{g_1',g_2'}(v, \omega') = \exp\left(\frac{\pi i m' b u^2}{\omega_1'}\right) T'(u, \omega_1', \omega_2') \tag{33}$$

is a Θ-function of order m' with arguments $v = u/\omega_1'$, $\omega' = \omega_2'/\omega_1'$ and characteristics (g_1', g_2').

Finally the fact that our two T-functions are equal induces a relation between the Θ-functions that we have just built from them. Thus the statement $T' = T$, with reference to (30), (31), (32) and (33) becomes

Theorem 1.6.2.

$$\exp\left(-\frac{\pi i m' n b u^2}{a + b\omega}\right) \Theta^{(m')}_{g_1',g_2'}\left(\frac{nu}{a + b\omega}, \frac{c + d\omega}{a + b\omega}\right) = \Theta^{(m'n)}_{g_1,g_2}(u, \omega)$$

with the characteristics being related by (29).

What this theorem shows is that if we apply a transformation to a Θ-function, we essentially get another Θ-function, but of different (known) order and characteristics. This result is called the *transformation principle* of the general Θ-function.

A simple rearrangement of this result will be useful later. The result is used by Weber, but derived in a completely different way via the theory of sigma functions.

We relabel

$$u' = \frac{nu}{a + b\omega} \quad \text{and} \quad \omega' = \frac{c + d\omega}{a + b\omega}.$$

The last result becomes

Theorem 1.6.3.

$$\exp(-\pi i m' b u u')\, \Theta^{(m')}_{g_1',g_2'}(u', \omega') = \Theta^{(m'n)}_{g_1,g_2}(u, \omega)$$

with the characteristics being related by (29).

We apply this result to the Jacobi θ-functions with characteristics $(0,0), (0,1)$ and $(1,0)$, i.e. the even Jacobi theta functions. For each of these, the order $m' = 1$. We will transform them using a linear transformation, i.e. with determinant $n = 1$.

Lemma 1.6.4. *In the previous theorem, if $m' = 1$ and the determinant of the transformation $n = ad - bc = 1$ then the characteristics $(0,0), (0,1)$ and $(1,0)$ are permuted by the transformation.*

Proof: To show that these characteristics are permuted we need to show that the transformation is always non-degenerate and that it is never possible to end up with both characteristics odd.

Firstly we note that at least one of a, b, c, and d must be even. The argument is the same for each case, so suppose a is even. Then both b and c must be odd.

Thus modulo 2 we have that $(g_1, g_2) \equiv (d(g_1'+1)+g_2', g_1')$. Regardless of what d is modulo 2, this is clearly non-degenerate. Also the only way to have both the resulting characteristics odd is for both of the original characteristics to be odd, and our assertion follows. □

We call any Θ-function with one of the three listed characteristics, even.

Any even Θ-function of order one is, by (21) to (23), related by some constant factor to a Jacobi theta function with the same characteristics. This includes the Θ-function on the right hand side of theorem (1.6.3). We can calculate this constant by setting $u = 0$. In this case we also have $u' = 0$ and so theorem (1.6.3) can be rewritten

Theorem 1.6.5.

$$\exp(-\pi i b u u') \, \frac{\theta_{g_1', g_2'}(u', \omega')}{\theta_{g_1', g_2'}(0, \omega')} = \frac{\theta_{g_1, g_2}(u, \omega)}{\theta_{g_1, g_2}}$$

for characteristics (g_1', g_2') equal to one of $(0,0), (0,1)$ or $(1,0)$ and (g_1, g_2) being related by (29) and thus also being one of the three listed possibilities.

Since the three possible 'even' characteristics are permuted by a linear transformation, we have the following result

Theorem 1.6.6.

$$\frac{\theta_{00}(u,\omega)\,\theta_{01}(u,\omega)\,\theta_{10}(u,\omega)}{\theta_{00}\,\theta_{01}\,\theta_{10}} = \exp(-3\pi i b u u')\,\frac{\theta_{00}(u',\omega')\,\theta_{01}(u',\omega')\,\theta_{10}(u',\omega')}{\theta_{00}(0,\omega')\,\theta_{01}(0,\omega')\,\theta_{10}(0,\omega')} \tag{34}$$

which is Weber's §39 (8) in [2].

A similar argument to that which is used for theorem (1.6.5) gives the following additional result which Weber establishes at the start of §39 of [2]

Theorem 1.6.7.

$$\frac{\exp(\pi i b u u')}{a + b\omega}\,\frac{\theta_{11}(u,\omega)}{\theta_{11}'} = \frac{\theta_{11}(u',\omega')}{\theta_{11}'(0,\omega')}$$

Note here that the dashed theta functions denote the derivative with repsect to their first argument.

We now quote without proof, a number of well known results for the Jacobi θ-functions.

Theorem 1.6.8.

$$\theta_{11}(u,\omega+1) = e^{\frac{\pi i}{4}}\,\theta_{11}(u) \tag{35}$$

$$\theta_{10}(u,\omega+1) = e^{\frac{\pi i}{4}}\,\theta_{10}(u) \tag{36}$$

$$\theta_{01}(u,\omega+1) = \theta_{00}(u) \tag{37}$$

$$\theta_{00}(u,\omega+1) = \theta_{01}(u) \tag{38}$$

$$e^{-\frac{\pi i u^2}{\omega}}\,\theta_{11}\left(\frac{u}{\omega}, -\frac{1}{\omega}\right) = -i\sqrt{-i\omega}\,\theta_{11}(u) \tag{39}$$

$$e^{-\frac{\pi i u^2}{\omega}}\,\theta_{01}\left(\frac{u}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\,\theta_{10}(u) \tag{40}$$

$$e^{-\frac{\pi i u^2}{\omega}}\,\theta_{10}\left(\frac{u}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\,\theta_{01}(u) \tag{41}$$

$$e^{-\frac{\pi i u^2}{\omega}}\,\theta_{00}\left(\frac{u}{\omega}, -\frac{1}{\omega}\right) = \sqrt{-i\omega}\,\theta_{00}(u) \tag{42}$$

Theorem 1.6.9. *Again letting a dash represent the derivative with respect to the first argument*

$$\theta'_{11} = \pi\,\theta_{00}\,\theta_{10}\,\theta_{01}$$

We also have the following representations of the θ-functions in terms of the Dedekind eta function and the three Weber functions (certain eta quotients to be defined later).

Theorem 1.6.10.

$$\theta'_{11} = 2\pi\eta(\omega)^3 \tag{43}$$

$$\theta_{00} = \eta(\omega)\,\mathfrak{f}(\omega)^2 \tag{44}$$

$$\theta_{01} = \eta(\omega)\,\mathfrak{f}_1(\omega)^2 \tag{45}$$

$$\theta_{10} = \eta(\omega)\,\mathfrak{f}_2(\omega)^2 \tag{46}$$

The equations (35) to (42) can be used to work out any *linear transformation* of the θ-functions, i.e. any transformation having determinant one. However particularly useful results regard transformations of higher order.

Up to a linear transformation, any higher order transformation can be represented by one of the two transformations

$$\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \text{ or } \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}. \tag{47}$$

These are called the *fundamental transformations* of order n.

We note that applying one of these fundamental transformations, but of specifically even order, in the case of integral characteristics will by (29) change the characteristics (g_1, g_2) to $(0, g_2)$ or $(g_1, 0)$ respectively, whereas for odd order the characteristics remain unchanged. These situations are fundamentally different and have to be treated separately.

1.7. **Second Order Transformations.** Starting at §32 of [2] Weber discusses the fundamental transformations of second order, i.e. with $n = 2$ in (47).

From Theorem (1.6.2) we have the following equations

$$\theta_{g_1,g_2}\left(u, \frac{\omega}{2}\right) = \Theta_{g_1,0}(u,\omega)$$

$$\theta_{g_1,g_2}(2u, 2\omega) = \Theta_{0,g_2}(u,\omega)$$

where the right hand sides are Θ-functions of 2nd order. We can get the form of their expression in terms of Jacobi θ-functions from (13) to (28). Finally by putting $u = 0$ we can evaluate the constants.

The results are the so-called *Gauss transformations*

Theorem 1.7.1.

$$\theta_{10}\left(0, \frac{\omega}{2}\right)\,\theta_{11}\left(u, \frac{\omega}{2}\right) = 2\,\theta_{01}(u,\omega)\,\theta_{11}(u,\omega)$$

$$\theta_{10}\left(0, \frac{\omega}{2}\right)\,\theta_{10}\left(u, \frac{\omega}{2}\right) = 2\,\theta_{00}(u,\omega)\,\theta_{10}(u,\omega)$$

and the *Landen Transformations*

Theorem 1.7.2.

$$\theta_{01}(0, 2\omega)\,\theta_{11}(2u, 2\omega) = \theta_{10}(u,\omega)\,\theta_{11}(u,\omega)$$

$$\theta_{01}(0, 2\omega)\,\theta_{01}(2u, 2\omega) = \theta_{00}(u,\omega)\,\theta_{01}(u,\omega).$$

One is now able to derive some identities which are important for odd order transformations.

Firstly replace ω by 2ω in the second Gauss transformation and and swap the sides of the resulting equation. Now multiply this expression with the second Landen transformation. We obtain

$$2\,\theta_{00}(u,2\omega)\;\theta_{10}(u,2\omega)\,\theta_{01}(0,2\omega)\,\theta_{01}(2u,2\omega)=\theta_{10}\,\theta_{10}(u)\,\theta_{00}(u)\,\theta_{01}(u) \tag{48}$$

Now if we set $u=0$ and replace ω with 2ω in the second Gauss transformation and set $u=0$ in the second Landen transformation we obtain respectively

$$2\,\theta_{00}(0,2\omega)\,\theta_{10}(0,2\omega)=\theta_{10}^2$$

$$\theta_{01}(0,2\omega)^2=\theta_{00}\,\theta_{01}$$

Now if we divide (48) by both of these relations, we obtain

$$\frac{\theta_{00}(u,2\omega)\,\theta_{10}(u,2\omega)\,\theta_{01}(2u,2\omega)}{\theta_{00}(0,2\omega)\,\theta_{10}(0,2\omega)\,\theta_{01}(0,2\omega)}=\frac{\theta_{00}(u)\,\theta_{10}(u)\,\theta_{01}(u)}{\theta_{00}\,\theta_{10}\,\theta_{01}}. \tag{49}$$

This expression looks elegant except for the last factor in the numerator of the left hand side. To make a balanced expression requires a little more work.

Firstly notice that if we multiply expressions of the form $\theta_{01}\left(\frac{v}{n}\right)$, for an odd integer n, over a complete residue system mod n, the resulting expression is the same as that obtained by multiplying over the same residue system but with v replaced with $2v$ throughout.

We also notice that for any of the theta functions in this expression, replacing v by $n-v$ does not change the expression, since θ_{01} is even and depends only on its argument modulo 1.

The net effect is that we only need to take the product over the first $\frac{n-1}{2}$ terms, i.e.

$$\prod_{v=1}^{\frac{n-1}{2}}\theta_{01}\left(\frac{v}{n}\right)=\prod_{v=1}^{\frac{n-1}{2}}\theta_{01}\left(\frac{2v}{n}\right) \tag{50}$$

Similar results hold for θ_{00} and θ_{10}, except we need to take into account the fact that $\theta_{10}(v+1)=-\theta_{10}(v)$.

This technique allows us to overcome the inelegance of the factor on the left hand side of (49) by simply taking the product of that equation over such a residue system and using (50).

We have thus obtained the following important result

Theorem 1.7.3. *The following expression is invariant if ω is replaced by 2ω*

$$\frac{\prod_{v=1}^{\frac{n-1}{2}}\theta_{00}\left(\frac{v}{n}\right)\theta_{10}\left(\frac{v}{n}\right)\theta_{01}\left(\frac{v}{n}\right)}{\theta_{00}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}}$$

Clearly by repeated application of this theorem, we see that the expression remains unchanged by replacing ω by $2\omega,4\omega,8\omega,\ldots..$ But in the limit, i.e. as $2^n\omega\to i\infty$, this value must always be the same regardless of the starting value ω. Thus the function is seen to be constant for all values of ω in the complex upper half plane.

At $\omega=i\infty$ our θ-functions are known to have the values

$$\theta_{00}(v)=1,\;\theta_{01}(v)=1,\;\frac{\theta_{10}(v)}{\theta_{10}}=\cos\pi v.$$

Using the trig identity

$$2^{\frac{n-1}{2}} \prod_{v=1}^{\frac{n-1}{2}} \cos\frac{v\pi}{n} = 1$$

we end up with the identity

$$2^{\frac{n-1}{2}} \prod_{v=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{v}{n}\right)\theta_{10}\left(\frac{v}{n}\right)\theta_{01}\left(\frac{v}{n}\right) = \theta_{00}^{\frac{n-1}{2}}\theta_{10}^{\frac{n-1}{2}}\theta_{01}^{\frac{n-1}{2}}. \tag{51}$$

Making use of (50) and its analogues for the other theta functions, we find that

$$2^{\frac{n-1}{2}} \prod_{v=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{2v}{n}\right)\theta_{10}\left(\frac{2v}{n}\right)\theta_{01}\left(\frac{2v}{n}\right) = (-1)^{\frac{n^2-1}{8}}\theta_{00}^{\frac{n-1}{2}}\theta_{10}^{\frac{n-1}{2}}\theta_{01}^{\frac{n-1}{2}}. \tag{52}$$

1.8. **Transformations of Odd Order.** Weber begins his treatment of transformations of odd order in §33 of [2].

From theorem (1.6.3) we know that $\theta_{11}(nu, n\omega)$ is a Θ_{11}-function of order n, since $b = 0$ in a fundamental transformation of order n. It is also clear from the definition of this function and (11) that its zeroes are

$$u = \frac{\nu + \mu n\omega}{n} = \frac{\nu}{n} + \mu\omega \text{ for } \mu,\, nu \in \mathbb{Z}.$$

One gets all the zeroes that are incongruent modulo the lattice for a fixed integer μ by taking ν over a complete residue system modulo n. We use the residue system

$$0, \pm 1, \pm 2, \ldots, \pm\frac{n-1}{2}.$$

Since we know that $\theta_{11}(u)$ has a single zero at $u = 0$, we can express our transformed theta function in terms of this simpler θ-function (up to a constant C, independent of u). A similar argument applies to the other θ-functions and we obtain

Theorem 1.8.1.

$$C\,\theta_{11}(nu, n\omega) = \theta_{11}(u) \prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n} + u\right)\theta_{11}\left(\frac{v}{n} - u\right). \tag{53}$$

$$C\,\theta_{10}(nu, n\omega) = \theta_{10}(u) \prod_{v=1}^{\frac{n-1}{2}} \theta_{10}\left(\frac{v}{n} + u\right)\theta_{10}\left(\frac{v}{n} - u\right). \tag{54}$$

$$C\,\theta_{01}(nu, n\omega) = \theta_{01}(u) \prod_{v=1}^{\frac{n-1}{2}} \theta_{01}\left(\frac{v}{n} + u\right)\theta_{01}\left(\frac{v}{n} - u\right). \tag{55}$$

$$C\,\theta_{00}(nu, n\omega) = \theta_{00}(u) \prod_{v=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{v}{n} + u\right)\theta_{00}\left(\frac{v}{n} - u\right). \tag{56}$$

Proof: Since the theta function on the left and the expression on the right hand side are both entire with exactly the same zeroes, their quotient must be a constant in each case. □

Firstly rewrite each of these expressions by dividing by the first factor of the right hand side. Thus we have a product on the right hand side and a quotient on the left in each case. Now take the limit as $u \to 0$ in the first expression. We end up with

$$Cn\,\frac{\theta'_{11}(0,n\omega)}{\theta'_{11}(0,\omega)} = \prod_{v=1}^{n-1} \theta_{11}\left(\frac{v}{n}\right). \tag{57}$$

Also set $u = 0$ in the other three equations and obtain three further expressions.

Rewriting the left hand side of (57) by means of Theorem (1.6.9) we can now use these other three expressions to obtain, after taking the square root

$$C\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right) = \sqrt{n}\prod_{v=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{v}{n}\right)\theta_{01}\left(\frac{v}{n}\right)\theta_{10}\left(\frac{v}{n}\right).$$

Alternatively by applying (51) to this expression we obtain

$$C\,2^{\frac{n-1}{2}}\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right) = \sqrt{n}\,\theta_{00}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}. \tag{58}$$

It remains to let $\omega \to \infty$ in the various expressions and evaluate the constants. We find always that $C = 1$. Now by multiplying our four equations (53) to (56) by the equation (58) we obtain finally the relations

$$\begin{aligned}\sqrt{n}\,\theta_{11}(nu,nw)\,\theta_{00}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}\\ = 2^{\frac{n-1}{2}}\,\theta_{11}(u)\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right)\theta_{11}\left(\frac{v}{n}+u\right)\theta_{11}\left(\frac{v}{n}-u\right)\end{aligned} \tag{59}$$

$$\begin{aligned}\sqrt{n}\,\theta_{10}(nu,nw)\,\theta_{00}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}\\ = 2^{\frac{n-1}{2}}\theta_{10}(u)\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right)\theta_{10}\left(\frac{v}{n}+u\right)\theta_{10}\left(\frac{v}{n}-u\right)\end{aligned} \tag{60}$$

$$\begin{aligned}\sqrt{n}\,\theta_{01}(nu,nw)\,\theta_{00}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}\\ = 2^{\frac{n-1}{2}}\theta_{01}(u)\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right)\theta_{01}\left(\frac{v}{n}+u\right)\theta_{01}\left(\frac{v}{n}-u\right)\end{aligned} \tag{61}$$

$$\begin{aligned}\sqrt{n}\,\theta_{00}(nu,nw)\,\theta_{00}^{\frac{n-1}{2}}\,\theta_{10}^{\frac{n-1}{2}}\,\theta_{01}^{\frac{n-1}{2}}\\ = 2^{\frac{n-1}{2}}\theta_{00}(u)\prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right)\theta_{00}\left(\frac{v}{n}+u\right)\theta_{00}\left(\frac{v}{n}-u\right)\end{aligned} \tag{62}$$

The importance of these expression is seen once we apply the relations of Theorem (1.6.10). This is the program of [2] §34.

In particular one differentiates (59), sets $u = 0$, takes the cube root and obtains

$$\sqrt{n}\,\eta(n\omega)\,\eta(\omega)^{\frac{n-3}{2}} = \prod_{v=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{v}{n}\right). \tag{63}$$

Also one sets $u = 0$ in (60) to (62), takes the square root and obtains

$$\sqrt{n}\,\mathfrak{f}(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \mathfrak{f}(\omega)\prod_{v=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{v}{n}\right). \tag{64}$$

$$\sqrt{n}\,\mathfrak{f}_1(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \mathfrak{f}_1(\omega)\prod_{v=1}^{\frac{n-1}{2}} \theta_{01}\left(\frac{v}{n}\right). \tag{65}$$

$$\sqrt{n}\,\mathfrak{f}_2(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \mathfrak{f}_2(\omega)\prod_{v=1}^{\frac{n-1}{2}} \theta_{10}\left(\frac{v}{n}\right). \tag{66}$$

We also note for later reference that from the definition of the θ-functions

$$\theta_{g_1,g_2}\left(\frac{n-v}{n}\right) = (-1)^{g_1(g_2+1)}\,\theta_{g_1,g_2}\left(\frac{v}{n}\right).$$

In the case where n is odd, then when v is odd, $n-v$ is even, i.e. of the form $2h$. Thus, taking care to mind factors of -1, we obtain from (64) to (66)

$$\mathfrak{f}(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \mathfrak{f}(\omega)\prod_{h=1}^{\frac{n-1}{2}} \theta_{00}\left(\frac{2h}{n}\right) \tag{67}$$

$$\mathfrak{f}_1(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \mathfrak{f}_1(\omega)\prod_{h=1}^{\frac{n-1}{2}} \theta_{01}\left(\frac{2h}{n}\right) \tag{68}$$

$$\mathfrak{f}_2(n\omega)\,\eta(\omega)^{\frac{n-1}{2}} = \left(\frac{2}{n}\right)\mathfrak{f}_2(\omega)\prod_{h=1}^{\frac{n-1}{2}} \theta_{10}\left(\frac{2h}{n}\right) \tag{69}$$

Here $\left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}$ is the Kronecker symbol.

Also from (63) we obtain

$$\sqrt{n}\,\eta(n\omega)\,\eta(\omega)^{\frac{n-3}{2}} = \prod_{h=1}^{\frac{n-1}{2}} \theta_{11}\left(\frac{2h}{n}\right). \tag{70}$$

2. Transformations of the Eta Function

2.1. **Linear Transformation of the Eta Function.** In [2] §38, linear transformations applied to the Dedekind eta function are discussed. We start from the well known transformations

Theorem 2.1.1.

$$\eta(\omega \pm 1) = e^{\pm\frac{\pi i}{12}}\,\eta(\omega) \tag{71}$$

$$\eta\left(-\frac{1}{\omega}\right) = \sqrt{-i\omega}\,\eta(\omega). \tag{72}$$

For integers α, β, γ, δ with $\alpha\delta - \beta\gamma = 1$ we define with Weber

Definition 2.1.2.

$$\frac{\eta\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right)}{\eta(\omega)} = E\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \omega\Bigg).$$

Note that Weber writes his matrices differently to what is now standard. We will only follow Weber's conventions when writing his $E(\sigma;\omega)$ function or fractional linear tranformations. However whenever we write matrices, representing fractional linear transformations, on their own, we will rearrange the entries to match the modern conventions. Thus the matrix corresponding to the transformation used in definition (2.1.2) is written in the modern notation as

$$\sigma = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}.$$

One notices that to find the value of E for a composite linear transformation, one simply multiplies the corresponding values of E together. That is, if σ_1 and σ_2 are two different linear transfomations, then if

$$\omega'' = \sigma_2\,\omega' = \sigma_2\,\sigma_1\,\omega \;\text{ then }\; E(\sigma_2\,\sigma_1;\omega) = E(\sigma_2;\omega')\,E(\sigma_1;\omega). \tag{73}$$

Note also that for the fundamental transformations

$$\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we have from (71) and (72)

$$\mathrm{E}\begin{pmatrix} \alpha\pm\beta & \beta \\ \gamma\pm\delta & \delta \end{pmatrix} ;\, \omega\Bigg) = e^{\pm\frac{\pi i}{12}}\,\mathrm{E}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ;\, \omega\pm 1\Bigg)$$

$$\mathrm{E}\begin{pmatrix} -\beta & \alpha \\ -\delta & \gamma \end{pmatrix} ;\, \omega\Bigg) = \sqrt{-i\omega}\,\mathrm{E}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ;\, -\frac{1}{\omega}\Bigg)$$

Now in view of (73) and the fact that all linear transformations are built from the fundamental ones, we can therefore use these last relations to obtain all values of E for a linear transformation. In particular, if we can come up with an expression for E which obeys all of the laws that we have just developed, then because of its uniqueness, this expression must be the correct one. Thus it is only a matter of (tedious) calculation to check that the following gives the values of E for any linear transformation.

Theorem 2.1.3. *Since $\alpha\delta-\beta\gamma = 1$, either α or β is odd, and can be made positive by a fundamental transformation.*

Then for α odd and positive:

$$E\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ;\, \omega\Bigg) = \left(\frac{\beta}{\alpha}\right) i^{\frac{\alpha-1}{2}}\, e^{\frac{\pi i}{12}[\alpha(\gamma-\beta)-(\alpha^2-1)\beta\delta]}\sqrt{(\alpha+\beta\omega)}; \tag{74}$$

and for β odd and positive:

$$E\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ;\, \omega\Bigg) = \left(\frac{\alpha}{\beta}\right) i^{\frac{1-\beta}{2}}\, e^{\frac{\pi i}{12}[\beta(\alpha+\delta)-(\beta^2-1)\alpha\gamma]}\sqrt{-i(\alpha+\beta\omega)}. \tag{75}$$

We can see that the eta function transforms with a 24^{th} root of unity. In fact if we let

$$\mathrm{E}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ;\, \omega\Bigg) = \varepsilon\,\sqrt{(\alpha+\beta\omega)} \tag{76}$$

then ε is a 24^{th} root of unity whose 12^{th} power satisfies

$$\varepsilon^{12} = (-1)^{\alpha\beta+\gamma\delta+\beta\gamma}.$$

2.2. **Linear Transformation of the Weber Functions.** The three Weber functions are defined as follows. They appear in §34 of [2].

Definition 2.2.1.

$$\mathfrak{f}(\omega) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\omega+1}{2}\right)}{\eta(\omega)} \tag{77}$$

$$\mathfrak{f}_1(\omega) = \frac{\eta\left(\frac{\omega}{2}\right)}{\eta(\omega)} \tag{78}$$

$$\mathfrak{f}_2(\omega) = \sqrt{2}\,\frac{\eta\left(2\omega\right)}{\eta(\omega)} \tag{79}$$

The following are well known identities which Weber proves in §34.

Theorem 2.2.2. *For any ω in the complex upper half plane,*

$$\mathfrak{f}(\omega)^8 = \mathfrak{f}_1(\omega)^8 + \mathfrak{f}_2(\omega)^8 \tag{80}$$

$$\mathfrak{f}(\omega)\,\mathfrak{f}_1(\omega)\,\mathfrak{f}_2(\omega) = \sqrt{2}. \tag{81}$$

Theorem 2.2.3. *The Weber functions are transformed in the following way by the fundamental transformations,*

$$\begin{aligned}
\mathfrak{f}(\omega+1) &= \exp\left(-\frac{\pi i}{24}\right)\mathfrak{f}_1(\omega),\\
\mathfrak{f}_1(\omega+1) &= \exp\left(-\frac{\pi i}{24}\right)\mathfrak{f}(\omega),\\
\mathfrak{f}_2(\omega+1) &= \exp\left(\frac{\pi i}{12}\right)\mathfrak{f}_2(\omega),\\
\mathfrak{f}\left(-\frac{1}{\omega}\right) &= \mathfrak{f}(\omega),\\
\mathfrak{f}_1\left(-\frac{1}{\omega}\right) &= \mathfrak{f}_2(\omega),\\
\mathfrak{f}_2\left(-\frac{1}{\omega}\right) &= \mathfrak{f}_1(\omega),
\end{aligned}$$

The identities of this theorem can be derived from the fundamental transformations of the eta function.

We now wish to look at general linear transformations of the Weber functions. As per [2] §40, we start from the definition (79). We note that

$$2\,\frac{\gamma+\delta\omega}{\alpha+\beta\omega} = \frac{2\gamma+\delta\cdot 2\omega}{\alpha+\frac{1}{2}\beta\cdot 2\omega}.$$

Therefore, so long as $\beta \equiv 0 \pmod 2$, we have that

$$\mathfrak{f}_2\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right) = \frac{E\begin{pmatrix} \alpha & \frac{1}{2}\beta \\ 2\gamma & \delta \end{pmatrix}; 2\omega}{E\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \omega}\mathfrak{f}_2(\omega).$$

Now since β is even, α must be odd, and we can use (74) to determine

$$\frac{E\begin{pmatrix} \alpha & \frac{1}{2}\beta \\ 2\gamma & \delta \end{pmatrix}; 2\omega}{E\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \omega} = \left(\frac{2}{\alpha}\right)\exp\left(\frac{\pi i}{12}\left[\alpha\gamma + \frac{\alpha\beta}{2} + (\alpha^2-1)\frac{\beta\delta}{2}\right]\right),$$

so long as α is positive.

We note that $\frac{1}{24} = \frac{3}{8} - \frac{1}{3}$ and that

$$(82) \qquad \alpha\gamma + \frac{\alpha\beta}{2} + (\alpha^2 - 1)\frac{\beta\delta}{2} \equiv \frac{\alpha(2\gamma + \beta)}{2} \pmod 8 \text{ and}$$

$$(83) \qquad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \equiv \alpha(\gamma - \beta) - (\alpha^2 - 1)\beta\delta \pmod 3.$$

Thus we have

Theorem 2.2.4. *If* $\beta \equiv 0 \ (mod\, 2)$ *and* $\alpha > 0$ *then*

$$\mathfrak{f}_2\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = \left(\frac{2}{\alpha}\right) \varrho \ \exp\left(\frac{3\pi i}{8}\left[\alpha(2\gamma + \beta)\right]\right) \mathfrak{f}_2(\omega),$$

where $\varrho = \exp\left(-\frac{2\pi i}{3}[\alpha(\gamma - \beta) - (\alpha^2 - 1)\beta\delta]\right)$.

Notice that if $\alpha < 0$ then we can replace $\alpha, \beta, \gamma, \delta$ with $-\alpha, -\beta, -\gamma, -\delta$. This does not change the transformation, but α becomes positive. Now for odd, positive α, $\left(\frac{2}{\alpha}\right)$ only depends on α modulo 8. But the value of this symbol is $+1$ for $\alpha \equiv \pm 1 \pmod 8$ and -1 for $\alpha \equiv \pm 3 \pmod 8$. Thus if we define $\left(\frac{2}{\alpha}\right) = \left(\frac{2}{-\alpha}\right)$ for odd integers α it is clear that the sign of α becomes irrelevant in the theorem above.

In order to work out the linear transformations of the other Weber functions we successively apply various transformations to expressions we have already obtained in order to obtain new ones.

Firstly, in the previous equation we replace ω with $-\frac{1}{\omega}$ and then replace $\alpha, \beta, \gamma, \delta$ with $\beta, -\alpha, \delta, -\gamma$ respectively, throughout. The quantity ϱ remains unchanged by this and we obtain

Theorem 2.2.5. *If* $\alpha \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}_2\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = \left(\frac{2}{\beta}\right) \varrho \ \exp\left(\frac{3\pi i}{8}\left[\beta(2\delta - \alpha)\right]\right) \mathfrak{f}_1(\omega).$$

Now replace ω by $\omega + 3$ and γ, α with $\gamma - 3\delta, \alpha - 3\beta$ respectively throughout, and we get

Theorem 2.2.6. *If* $\alpha - \beta \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}_2\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = -\varrho \ \exp\left(\frac{3\pi i}{8}\left[\beta(2\delta - \alpha)\right]\right) \mathfrak{f}(\omega).$$

In the equations of theorems (2.2.4) to (2.2.6) we now set

$$\mathfrak{f}_2\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = \mathfrak{f}_1\left(-\frac{\alpha + \beta\omega}{\gamma + \delta\omega}\right)$$

and replace $\alpha, \beta, \gamma, \delta$ with $-\gamma, -\delta, \alpha, \beta$ throughout. We obtain

Theorem 2.2.7. *If* $\delta \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}_1\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = \left(\frac{2}{-\gamma}\right) \varrho \ \exp\left(-\frac{3\pi i}{8}\left[\gamma(2\alpha - \delta)\right]\right) \mathfrak{f}_2(\omega).$$

Theorem 2.2.8. *If* $\gamma \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}_1\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = \left(\frac{2}{-\delta}\right) \varrho \ \exp\left(-\frac{3\pi i}{8}\left[\delta(2\beta + \gamma)\right]\right) \mathfrak{f}_1(\omega).$$

Theorem 2.2.9. *If* $\gamma - \delta \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}_1\left(\frac{\gamma + \delta\omega}{\alpha + \beta\omega}\right) = -\varrho \ \exp\left(-\frac{3\pi i}{8}\left[\delta(2\beta + \gamma)\right]\right) \mathfrak{f}(\omega).$$

Now in theorems (2.2.9) and (2.2.6) we replace ω by $\frac{-\gamma+\alpha\omega}{\delta-\beta\omega}$ and then $\alpha, \beta, \gamma, \delta$ with $\delta, -\beta, -\gamma, \alpha$ throughout. We obtain respectively

Theorem 2.2.10. *If* $\alpha - \gamma \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right) = -\varrho \ \exp\left(-\frac{3\pi i}{8}\left[\alpha(2\beta+\gamma)\right]\right)\mathfrak{f}_1(\omega).$$

Theorem 2.2.11. *If* $\beta - \delta \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right) = -\varrho \ \exp\left(\frac{3\pi i}{8}\left[\beta(2\alpha-\delta)\right]\right)\mathfrak{f}_2(\omega).$$

Finally in theorem (2.2.10) we replace ω by $\omega + 9$ and γ, α with $\gamma - 9\delta, \alpha - 9\beta$ throughout and we obtain

Theorem 2.2.12. *If* $\alpha + \beta + \gamma - \delta \equiv 0 \ (mod\, 2)$ *then*

$$\mathfrak{f}\left(\frac{\gamma+\delta\omega}{\alpha+\beta\omega}\right) = \left(\frac{2}{\alpha-\beta}\right)\varrho \ \exp\left(-\frac{3\pi i}{8}\left[(\alpha-\beta)(\alpha+\beta+\gamma-\delta)\right]\right)\mathfrak{f}(\omega).$$

3. Elliptic Functions

3.1. **Jacobi Elliptic Functions.** In [2] §42 Weber begins to deal with elliptic functions. In particular we find the theory of the Jacobi functions sn v, cn v and dn v. These functions arise from inverting elliptic integrals such as

$$v(x) = \int_0^x \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}}\, dy.$$

These inverse functions turn out to be elliptic and it can be shown that they can be represented as quotients of Jacobi θ-functions as follows

(84) $$\frac{\theta_{11}(u)}{\theta_{01}(u)} = \sqrt{k}\ \mathrm{sn}\ v$$

(85) $$\frac{\theta_{10}(u)}{\theta_{01}(u)} = \sqrt{\frac{k}{k'}}\ \mathrm{cn}\ v$$

(86) $$\frac{\theta_{00}(u)}{\theta_{01}(u)} = \sqrt{\frac{1}{k'}}\ \mathrm{dn}\ v$$

(87) $$k^2 = \frac{\theta_{10}^4}{\theta_{00}^4}$$

with $v = \pi\, \theta_{00}^2(u)$ (notice that Weber scales the arguments of his elliptic functions by an extra factor of π that does not appear in the modern definition). The quantity k is that which appears in the elliptic integral above, and is called the *modulus* of the elliptic function. The quantity k' is given by the equation $k^2 + {k'}^2 = 1$.

We will not explore the relationships between all these functions any further, as very good accounts of the theory exist in standard texts; and besides, these few notions will be all we require to proceed.

We begin with the well known addition theorems for the Jacobi elliptic functions. An extensive range of such identities are quoted in [2] §44.

Theorem 3.1.1. *The Jacobi elliptic functions satisfy the relations*

$$sn\,(u+v) = \frac{sn\,u\ cn\,v\ dn\,v + sn\,v\ cn\,u\ dn\,u}{1-k^2\ sn^2\ u\ sn^2\ v} \tag{88}$$

$$cn\,(u+v) = \frac{cn\,u\ cn\,v - sn\,u\ sn\,v\ dn\,u\ dn\,v}{1-k^2\ sn^2\ u\ sn^2\ v} \tag{89}$$

$$dn\,(u+v) = \frac{dn\,u\ dn\,v - k^2\ sn\,u\ sn\,v\ cn\,u\ cn\,v}{1-k^2\ sn^2\ u\ sn^2\ v} \tag{90}$$

From these results expressions can be built for double angle formulae, triple angle formulae, etc. However the results quickly get messy. Weber starting in §57 of [2] obtains a neater expression by using the previous theory of θ-functions and the relations (84) to (86).

One starts with $\theta_{11}(nu), \theta_{10}(nu), \theta_{01}(nu)$ and $\theta_{00}(nu)$ which are clearly Θ-functions of order n^2. From (13) to (28) they can be expressed as follows.

For even n

$$\theta_{11}(nu) = \theta_{00}(u)\,\theta_{01}(u\,)\theta_{10}(u)\,\theta_{11}(u)\,F^{\frac{n^2-4}{2}}[\theta_{11}^2(u), \theta_{01}^2(u)]$$

$$\theta_{g_1,g_2}(nu) = F^{\frac{n^2}{4}}[\theta_{11}^2(u), \theta_{01}^2(u)]$$

with $(g_1, g_2) = (1,0), (0,1),$ or $(0,0)$.

For odd n

$$\theta_{g_1,g_2}(nu) = F^{\frac{n^2-1}{2}}[\theta_{11}^2(u), \theta_{01}^2(u)].$$

When we divide these formulae through by $\theta_{01}(u)^{n^2}$ and make use of (84) to (86), we obtain

Theorem 3.1.2. *Letting $x = sn\,v, y = cn\,v, z = dn\,v$ we have*

I. For odd n:

$$\frac{\theta_{01}^{n^2-1}\theta_{00}}{\theta_{10}}\,\frac{\theta_{11}(nu)}{\theta_{01}(u)^{n^2}} = x\ A^{\frac{n^2-1}{2}}(x^2) \tag{91}$$

$$\frac{\theta_{01}^{n^2}}{\theta_{10}}\,\frac{\theta_{10}(nu)}{\theta_{01}(u)^{n^2}} = yB^{\frac{n^2-1}{2}}(x^2) \tag{92}$$

$$\frac{\theta_{01}^{n^2}}{\theta_{00}}\,\frac{\theta_{00}(nu)}{\theta_{01}(u)^{n^2}} = zC^{\frac{n^2-1}{2}}(x^2) \tag{93}$$

$$\theta_{01}^{n^2-1}\,\frac{\theta_{01}(nu)}{\theta_{01}(u)^{n^2}} = D^{\frac{n^2-1}{2}}(x^2) \tag{94}$$

II. For even n:

$$\frac{\theta_{01}^{n^2}\theta_{00}}{\theta_{10}}\,\frac{\theta_{11}(nu)}{\theta_{01}(u)^{n^2}} = xyz\ A^{\frac{n^2}{2}-2}(x^2) \tag{95}$$

$$\frac{\theta_{01}^{n^2}}{\theta_{10}}\,\frac{\theta_{10}(nu)}{\theta_{01}(u)^{n^2}} = B^{\frac{n^2}{2}}(x^2) \tag{96}$$

$$\frac{\theta_{01}^{n^2}}{\theta_{00}}\,\frac{\theta_{00}(nu)}{\theta_{01}(u)^{n^2}} = C^{\frac{n^2}{2}}(x^2) \tag{97}$$

$$\theta_{01}^{n^2-1}\,\frac{\theta_{01}(nu)}{\theta_{01}(u)^{n^2}} = D^{\frac{n^2}{2}}(x^2) \tag{98}$$

where the degrees of the polynomials A, B, C, D are as indicated.

Now in each case, if one respectively divides the first three formulae by the last in each group, then one obtains, remarkably

Theorem 3.1.3. *For odd n:*

$$sn\ nv = \frac{x\,A(x^2)}{D(x^2)} \tag{99}$$

$$cn\ nv = \frac{y\,B(x^2)}{D(x^2)} \tag{100}$$

$$dn\ nv = \frac{z\,C(x^2)}{D(x^2)} \tag{101}$$

and for even n:

$$sn\ nv = \frac{xyz\,A(x^2)}{D(x^2)} \tag{102}$$

$$cn\ nv = \frac{B(x^2)}{D(x^2)} \tag{103}$$

$$dn\ nv = \frac{C(x^2)}{D(x^2)} \tag{104}$$

where the degrees of the polynomials are given by the previous theorem and where the coefficients are polynomial in k^2 of degree less than that of x^2.

These are the forms of the formulae that one would obtain by successive application of the addition theorems. For example with the standard addition formulae one finds easily enough

$$\begin{aligned}
\text{sn } 2v &= \frac{2 \text{ sn } v \text{ cn } v \text{ dn } v}{1 - k^2 \text{ sn}^4 v} \\
\text{cn } 2v &= \frac{\text{cn}^2 v - \text{sn}^2 v \text{ dn}^2 v}{1 - k^2 \text{ sn}^4 v} \\
\text{dn } 2v &= \frac{\text{dn}^2 v - k^2 \text{ sn}^2 v \text{ cn}^2 v}{1 - k^2 \text{ sn}^4 v}.
\end{aligned}$$

From these one can derive the new forms given above by repeated use of the well known identities

$$1 = \text{sn}^2 v + \text{cn}^2 v = \text{dn}^2 v + k^2 \text{ sn}^2 v. \tag{105}$$

In fact in §57 of Weber's book he goes further and derives recurrence relations describing how to construct the polynomials A, B, C and D for multiplication of the elliptic functions by n from the same functions for lower values of n.

For example if we write A_n for the polynomial A which appears in the equation for multiplication by n (102), etc, then Weber shows that

$$A_{2n} = 2A_n B_n C_n D_n.$$

We will not list all his recurrence relations here since their actual statements do not feature in the sequel.

Nevertheless the existence of these recurrence relations is extremely important to Weber's argument. The reason for this is that all of the recurrence relations involve only the polynomials A, B, C and D and the values k^2, x^2, y^2 and z^2 where $x = \text{sn } v$, $y = \text{cn } v$ and $z = \text{dn } v$. Therefore by making use of the relations (105) is is possible to show that the polynomials A, B, C and D can all be expressed as polynomials in x with coefficients in $\mathbb{Q}(k^2)$.

This important fact is used much later to prove that the coefficients in various modular equations are actually in $\mathbb{Q}(k^2)$ rather than $\mathbb{C}(k^2)$.

3.2. **Division Equations.** We define two quantities called *complete elliptic integrals.*

$$K = \int_0^1 \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}} dy$$

and

$$K' = \int_0^{1/k} \frac{1}{\sqrt{(1-y^2)(1-k^2y^2)}} dy.$$

We find that various combinations of these values are periods of the elliptic functions $\mathrm{sn}\ v, \mathrm{cn}\ v$ and $\mathrm{dn}\ v$. In fact we have

Theorem 3.2.1. *The following results hold, from which we can obtain fundamental periods of the elliptic functions*

$$\mathrm{sn}\ (u+2K) = -\mathrm{sn}\ u \tag{106}$$
$$\mathrm{cn}\ (u+2K) = -\mathrm{cn}\ u \tag{107}$$
$$\mathrm{dn}\ (u+2K) = +\mathrm{dn}\ u \tag{108}$$
$$\mathrm{sn}\ (u+2iK') = +\mathrm{sn}\ u \tag{109}$$
$$\mathrm{cn}\ (u+2iK') = -\mathrm{cn}\ u \tag{110}$$
$$\mathrm{dn}\ (u+2iK') = -\mathrm{dn}\ u \tag{111}$$

In particular, from this theorem, it can be seen that $4\mu K + 4\mu' iK'$ turns out to be a (not necessarily fundamental) period for each of the elliptic functions for all integers μ, μ'. However this fact implies that the equations of Theorem (3.1.3) are also satisfied by the values

$$v' = v + 4\mu K + 4\mu' iK'.$$

This is Weber's starting point in [2] §60.

Now for an odd n, since the equations of Theorem (3.1.3) are identities for all v, they must be true for the values $\frac{v'}{n}$. In particular all the values

$$x_{\mu,\mu'} = \mathrm{sn}\ \left(\frac{v}{n} + \frac{4\mu K + 4\mu' iK'}{n}\right) \tag{112}$$

$$y_{\mu,\mu'} = \mathrm{cn}\ \left(\frac{v}{n} + \frac{4\mu K + 4\mu' iK'}{n}\right) \tag{113}$$

$$z_{\mu,\mu'} = \mathrm{dn}\ \left(\frac{v}{n} + \frac{4\mu K + 4\mu' iK'}{n}\right) \tag{114}$$

satisfy the equations

$$D(x^2)\,\mathrm{sn}\ v - xA(x^2) = 0 \tag{115}$$

$$D(x^2)\,\mathrm{cn}\ v - yB(x^2) = 0 \tag{116}$$

$$D(x^2)\,\mathrm{dn}\ v - zC(x^2) = 0. \tag{117}$$

The first of these equations is called the general division equation. It clearly has degree n^2 in x and thus has n^2 roots. Given such a root, the second and third equations gives a unique y and z corresponding to that root x.

Also we note that if μ and μ' traverse complete residue systems modulo n, all the n^2 roots (112) are different. For, since n is odd, no two of the arguments can differ by a period of sn v. Similar reasoning applies to the values (113) and (114). So these values must be all the roots of the equations (115) to (117).

We focus now, as per [2] §61, on the case where $v = 0$ in the above, so that we are considering the division points of a period, i.e.

$$x_{\mu,\mu'} = \text{sn}\left(\frac{4\mu K + 4\mu' iK'}{n}\right).$$

These n^2 values are now the roots of

$$xA(x^2) = 0. \tag{118}$$

Recall that this is a polynomial equation and the coefficients are in $\mathbb{Q}(k^2)$.

We wish to investigate the Galois group of this equation. We follow [2] §63 and following.

We take our base field to be that of the coefficients of the polynomial A, i.e. $\mathbb{Q}(k^2)$.

It is quite clear that the addition theorem (88) is an identity for all u and v

$$\text{sn}\,(u + v) = f(\text{sn}\,u, \text{sn}\,v) \tag{119}$$

with f some rational expression in sn u and sn v with coefficients in $\mathbb{Q}(k^2)$. Similarly, the multiplication of an argument by a positive integer (obtained by repeated addition of that argument) can be expressed as an identity

$$\text{sn}\,mu = \varphi_m(\text{sn}\,u). \tag{120}$$

Applying these expressions to the roots $x_{\mu,\mu'}$ we have

$$x_{\mu+\nu,\mu'+\nu'} = f(x_{\mu,\mu'}, x_{\nu,\nu'})$$

and

$$x_{m\mu,m\mu'} = \varphi_m(x_{\mu,\mu'}).$$

Let G be the Galois group of the polynomial (118) above. Suppose that σ is a permutation belonging to G. We will consider its action on the roots $x_{1,0}$ and $x_{0,1}$. Suppose it takes these roots respectively to

$$x_{d,-c} \text{ and } x_{-b,a}. \tag{121}$$

From (120)

$$x_{\mu,0} = \varphi_\mu(x_{1,0})$$

and applying σ gives

$$\sigma(x_{\mu,0}) = \varphi(x_{d,-c}) = x_{d\mu,-c\mu}.$$

Similarly

$$\sigma(x_{0,\mu'}) = \varphi(x_{-b,a}) = x_{-b\mu',a\mu'}.$$

Finally, from (119) we have

$$x_{\mu,\mu'} = f(x_{\mu,0}, x_{0,\mu'})$$

so that the action of σ yields

$$x_{\nu,\nu'} = \sigma(x_{\mu,\mu'}) = f(\sigma(x_{\mu,0}), \sigma(x_{0,\mu'})) = x_{d\mu-b\mu',-c\mu+a\mu'}, \tag{122}$$

totally fixing the action of σ on a given root $x_{\mu,\mu'}$.

We note that if $m = ad - bc$ is the determinant of this action of σ then its greatest common divisor with the division value n, $d = \gcd(m, n)$, must be one, otherwise the action of σ on the indices is degenerate modulo d so that two roots are taken to the same place under σ. But then σ would not be a permutation of the roots as we have supposed.

So long as $\gcd(m, n) = 1$, the different values of a, b, c, d give actual permutations of the roots. A simple combinatorial argument shows that there are

$$n\,\varphi(n)^2\,\psi(n) \tag{123}$$

different such permutations, where

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

$$\psi(n) = n \prod_{p|n} (1 + \frac{1}{p}).$$

We also see from (122) that if $\gcd(m, n) = 1$ then $\gcd(\mu, \mu', n) = \gcd(\nu, \nu', n)$, i.e. the greatest common divisor of the indices of a root, and n, is invariant under σ.

Hence the action of G on the roots is intransitive, with a different system of intransitivity for each divisor of n. This shows us that the division equation is reducible over $\mathbb{Q}(k^2)$.

Definition 3.2.2. *We call the factor of the division equation containing all the roots $x_{\mu,\mu'}$ with gcd$(\mu, \mu', n) = 1$, denoted*

$$\Phi_n(x) = 0,$$

the characteristic division equation*.*

Note again that the coefficients of this equation are in $\mathbb{Q}(k^2)$.

It is not too hard to show that there are precisely $\varphi(n)\psi(n)$ number pairs (μ, μ') such that $\gcd(\mu, \mu', n) = 1$. Thus this equation has degree $\varphi(n)\psi(n)$.

Note that $x_{1,0}$ and $x_{0,1}$ are amongst the roots of the characteristic division equation. Any permutation of G will change at least one of these roots, so G is the Galois group of this equation also. It is also now clear that starting from one of these roots e.g. $x_{1,0}$, there will be a permutation σ of G that will take this root to any given root of the characteristic division equation. Thus the action of G is transitive and so it follows that the characteristic division equation is irreducible over $\mathbb{Q}(k^2)$.

Now if we consider any one of the roots of our characteristic division equation

$$x_{\mu,\mu'} = \operatorname{sn} \Omega_1$$

then it is clear that the roots

$$\operatorname{sn} h\Omega_1$$

are also roots of the equation, for all values of h prime to n. There are $\varphi(n)$ such roots, and we call these the *first series* of roots.

Again taking any root sn Ω_2 that we have not dealt with thus far we can, by the same method, obtain a second series of $\varphi(n)$ roots, none of which can coincide with those of the first series, and so on. There are then $\psi(n)$ such series each containing $\varphi(n)$ roots of the characteristic division equation.

From the construction of the series we see that two roots are in the same series if and only if there exists an h prime to n such that

$$(\nu, \nu') = (h\mu, h\mu') \pmod{n}.$$

Thus

Theorem 3.2.3. *Two roots $x_{\mu,\mu'} = x_{\nu,\nu'}$ are in the same series if and only if*

$$\mu\nu' - \nu\mu' \equiv 0 \ (mod\, n).$$

But now if we take two roots $x_{\mu 1,\mu 1'}$ and $x_{\mu 2,\mu 2'}$ in the same series, and apply the same permutation σ to these roots, then after expanding out the expression obtained from (122), we realize that the permuted roots are also in the same series as each other. This means that series are permuted with other series, intact, and so the action of G is imprimitive. We can therefore find $\varphi(n)$ conjugate subgroups, G_i of G, that permute the elements of the i-th series only amongst themselves.

We notice that any element of the group (which we will call G_1) that permutes the series containing $x_{1,0}$ to itself, must take $x_{1,0}$ to another root of that series. The only permutations that do this (in the notation of (121)) are of the form

$$\sigma_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

This information gives us quite a good handle on the Galois group of the characteristic division equation and on that equation itself.

4. Transformation Equations

4.1. **The Group Theory of Transformation Equations.** We are now in a position to begin building so-called transformation equations. These are an intermediate step on the way to constructing modular equations; our ultimate goal. Weber discusses these in [2] §65 and following.

Recall that we now have $\psi(n)$ disjoint series each containing $\varphi(n)$ distinct roots of the characteristic division equation.

Firstly we consider a fixed rational function $\xi(z_1, z_2, \ldots, z_m)$, in $m = \varphi(n)$ arguments, which remains invariant under any permutation of those arguments, e.g. a symmetric polynomial in $\varphi(n)$ indeterminates.

Let $\xi_1, \xi_2, \ldots, \xi_{\psi(n)}$ be the values that it takes for each of the $\psi(n)$ series, i.e., for each i, ξ_i is the value that ξ takes when the roots of the i-th series are substituted into it. We will only be interested in the case where each of the ξ_i is distinct.

The fields $F_i = \mathbb{Q}(k^2, \xi_i)$ are then distinct conjugate subfields of the splitting field L of the characteristic division equation. These are clearly the fixed fields of the groups G_i of the previous section.

Weber now notes that these groups have a 'greatest common divisor' G_0. It turns out to be the subgroup consisting, in the notation of the previous section, of all the transformations of the form

$$\sigma_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

where a is an arbitrary integer modulo n which is coprime to n. Thus the order of G_0 is $\varphi(n)$.

The group G_0 clearly fixes all the ξ_i and thus fixes the field they generate over $\mathbb{Q}(k^2)$. Also since G_0 is normal in G its fixed field is a Galois closure of the conjugate fields F_i in L. This field F_0 has degree $\psi(n)$ over K.

It is clear that a group of transformations in G exist which permute the values ξ_i. In fact the ξ_i are roots of an irreducible polynomial equation of degree $\psi(n)$ over $K = \mathbb{Q}(k^2)$. Since the ξ_i are all in F_0 which is Galois over K, the splitting field for this polynomial equation is in fact F_0. We call such an equation a *transformation* *equation.* It is important to note that the coefficients of the transformation equation are in $\mathbb{Q}(k^2)$.

Now the simplest class of functions we can choose for ξ are the products

$$\prod_{h=1}^{n-1} \Phi(\text{sn } h\Omega)$$

where Φ is an arbitrary rational function.

We introduce some terminology at this point for the cases where Φ is an even or odd function. If $\Phi(x)$ is even, then

$$\Phi(\text{sn } h\Omega) = \Phi(\text{sn } (n-h)\Omega).$$

It is then natural to take

$$\Pi(\Omega) = \prod_{h=1}^{\frac{n-1}{2}} \Phi(\text{sn } h\Omega).$$

When m is an integer prime to n, then clearly $\Pi(m\Omega) = \Pi(\Omega)$. The values $\Pi(\Omega)$, for the values of Ω which we picked above to create the various series, are then, by the argument above, the roots of a transformation equation. We call such an equation a *modular equation.*

If however $\Phi(x)$ is odd, then only in certain cases (e.g. when n is a square) is the function $\Pi(\Omega)$ a root of a transformation equation. However in the other cases, $\Pi(\Omega)^2$ is a root of a transformation equation. Such transformation equations are called *multiplikator equations* in the literature.

4.2. **Constructing Transformation Equations.** We are particularly interested from this point onwards, in finding transformation equations (and ultimately modular equations) that involve known functions, such as the Weber functions, etc.

Firstly we shall find a transformation equation whose roots can be expressed in terms of products of theta functions. This process is dealt with in [2] starting at §67. Later we will rewrite this equation in terms of Weber functions.

Recall that we have been dealing with division of periods of elliptic functions by an integer n. To begin with we change our notation to make this a little easier to deal with and to facilitate the change to theta functions. We set

$$\Omega = \frac{4\mu K + 4\mu' iK'}{n}$$

and then as might be expected

$$\omega = \frac{iK'}{K},$$

the corresponding ratio of periods. Then we can define

$$\varpi = \frac{\mu + \mu'\omega}{n}$$

so that

$$\Omega = 4K\varpi.$$

We define some values P_{00}, P_{10}, P_{01} by the following theta product expressions

$$P_{00} = \exp\left(\frac{\pi i}{6}\mu'(\mu + \mu'\omega)\frac{n^2-1}{n}\right)\theta_{00}^{-(\frac{n-1}{2})}\prod_{h=1}^{\frac{n-1}{2}}\theta_{00}(2h\varpi) \tag{124}$$

$$P_{10} = \exp\left(\frac{\pi i}{6}\mu'(\mu + \mu'\omega)\frac{n^2-1}{n}\right)\theta_{10}^{-(\frac{n-1}{2})}\prod_{h=1}^{\frac{n-1}{2}}\theta_{10}(2h\varpi) \tag{125}$$

$$P_{01} = \exp\left(\frac{\pi i}{6}\mu'(\mu+\mu'\omega)\frac{n^2-1}{n}\right)\theta_{01}^{-(\frac{n-1}{2})}\prod_{h=1}^{\frac{n-1}{2}}\theta_{01}(2h\varpi) \tag{126}$$

where the multipliers at the front of these expressions are chosen for later convenience.

If we make the following substitutions in the identity of Theorem (1.6.6)

$$u' = \frac{2h}{n} \text{ and } u = \frac{2h(\mu+\mu'\omega)}{n}$$

and take the product over $h = 1, 2, \ldots, (n-1)/2$, then (52) becomes

$$\begin{aligned}(127)\qquad & 2^{\frac{n-1}{2}}\prod_{h=1}^{\frac{n-1}{2}}\theta_{00}\left(\frac{2h(\mu+\mu'\omega)}{n}\right)\theta_{01}\left(\frac{2h(\mu+\mu'\omega)}{n}\right)\theta_{10}\left(\frac{2h(\mu+\mu'\omega)}{n}\right)\\ &= (-1)^{\frac{n^2-1}{8}}\exp\left(-\frac{\pi i}{2}\mu'(\mu+\mu'\omega)\frac{n^2-1}{n}\right)\theta_{00}^{\frac{n-1}{2}}\theta_{10}^{\frac{n-1}{2}}\theta_{01}^{\frac{n-1}{2}}\end{aligned}$$

which in the notation introduced above, says

$$(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}P_{00}P_{10}P_{01} = 1. \tag{128}$$

We also define the function

$$P_{11} = \exp\left(\frac{\pi i}{6}\mu'(\mu+\mu'\omega)\frac{n^2-1}{n}\right)\eta(\omega)^{-(\frac{n-1}{2})}\prod_{h=1}^{\frac{n-1}{2}}\theta_{11}(2h\varpi). \tag{129}$$

Consider now the quotients

$$\frac{P_{00}^2}{P_{01}P_{10}},\ \frac{P_{10}^2}{P_{01}P_{00}},\ \frac{P_{01}^2}{P_{00}P_{10}},\ \frac{P_{11}^3}{P_{01}P_{10}P_{00}}.$$

We recall the definition of the Jacobi elliptic functions in terms of the theta quotients (84) to (86). The above quotients can then be expressed in terms of the elliptic functions.

However we can also write these quotients differently by applying the identity (128) to their denominators. In this way we now have two different expressions for each of the quotients above. Equating, we have

$$(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}P_{00}^3 = \prod_{h=1}^{\frac{n-1}{2}}\frac{\mathrm{dn}^2 h\Omega}{\mathrm{cn}\ h\Omega} \tag{130}$$

$$(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}P_{10}^3 = \prod_{h=1}^{\frac{n-1}{2}}\frac{\mathrm{cn}^2 h\Omega}{\mathrm{dn}\ h\Omega} \tag{131}$$

$$(-1)^{\frac{n^2-1}{8}}2^{\frac{n-1}{2}}P_{01}^3 = \prod_{h=1}^{\frac{n-1}{2}}\frac{1}{\mathrm{cn}\ h\Omega\ \mathrm{dn}\ h\Omega} \tag{132}$$

$$(-1)^{\frac{n^2-1}{8}}P_{11}^3 = \prod_{h=1}^{\frac{n-1}{2}}\frac{\mathrm{sn}^3 h\Omega}{\mathrm{cn}\ h\Omega\ \mathrm{dn}\ h\Omega} \tag{133}$$

These expressions are roots of transformation equations since the expressions on the right are symmetric functions of roots of a series. Thus we have precisely what is required for

Theorem 4.2.1. *P_{00}^3, P_{10}^3 and P_{01}^3 are roots of modular equations, and P_{11}^6 is the root of a multiplikator equation.*

We still have a little work to do in order to show that P_{00}, P_{01} and P_{10} themselves are roots of transformation equations. This is accomplished as follows.

In equation (94), setting $u = 2h\varpi$ we obtain

$$D(\mathrm{sn}^2 h\Omega) = \frac{\theta_{01}^{n^2}\,\theta_{01}(2h(\mu+\mu'\omega))}{\theta_{01}\,\theta_{01}(2h\varpi)^{n^2}} \tag{134}$$

$$= \frac{\theta_{01}^{n^2}}{\theta_{01}(2h\varpi)^{n^2}}\exp(4\pi i\omega h^2\mu'^2), \tag{135}$$

upon application of the quasi-period relations (9) and (10).

Now taking the product over $h = 1, 2, \ldots, \frac{n-1}{2}$ and rearranging, we obtain

$$P_{01}^{n^2} = \prod_{h=1}^{\frac{n-1}{2}} \frac{1}{D(\mathrm{sn}^2 h\Omega)} \tag{136}$$

so that $P_{01}^{n^2}$ is the root of a transformation equation.

Also, dividing equation (131) by (130) above, we have

$$\frac{P_{10}}{P_{00}} = \prod_{h=1}^{\frac{n-1}{2}} \frac{\mathrm{cn}\, h\Omega}{\mathrm{dn}\, h\Omega} \tag{137}$$

and dividing equation (132) by (130) we have

$$\frac{P_{01}}{P_{00}} = \prod_{h=1}^{\frac{n-1}{2}} \frac{1}{\mathrm{dn}\, h\Omega}. \tag{138}$$

Now if 3 does not divide n then at least 3 divides n^2-1. Thus by taking a power of the expression (132) above, we can get an expression for $P_{01}^{n^2-1}$. This we can divide by (136), giving us an expression for P_{01}. Thus P_{01} is the root of a transformation equation. Now it is clear from (138) and (137) that we can say the same for P_{00} and thus P_{10} in turn.

We can summarise as follows

Theorem 4.2.2. *If n is not divisible by 3, then P_{01}, P_{00} and P_{10} are the roots of a transformation equation.*

4.3. **Weber Functions as Roots of Transformation Equations.** We now wish to turn our expressions for the values P_{00}, P_{01}, P_{10} and P_{11} into ones involving the Weber functions.

We use the expressions (67) to (70) for this purpose. In particular if we denote the P-values for the series containing $x_{1,0}$ by P^0, we have

$$P_{00}^0 = \frac{\mathfrak{f}(n\omega)}{\mathfrak{f}(\omega)^n} \tag{139}$$

$$P_{01}^0 = \frac{\mathfrak{f}_1(n\omega)}{\mathfrak{f}_1(\omega)^n} \tag{140}$$

$$P_{10}^0 = \left(\frac{2}{n}\right)\frac{\mathfrak{f}_2(n\omega)}{\mathfrak{f}_2(\omega)^n} \tag{141}$$

$$P_{11}^0 = \sqrt{n}\,\frac{\eta(n\omega)}{\eta(\omega)}. \tag{142}$$

The other roots of the respective transformation equations, corresponding to the P-values of the other series are, as we indicated in an earlier section, obtained from these values by linear transformations applied to the indices μ and $\mu' = 0$ of the roots $x_{\mu,0}$ of the first series that we have used here. We will describe what these transformations do to the values P_{00}^0, P_{01}^0, etc. Thus we will obtain expressions for all the other roots P^i of the respective transformation equations.

Firstly however, we prove a simple matrix result related to progressing from the first series to another by a linear transformation. We wish to show that an arbitrary transformation

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of any order h can be broken down into a product

$$\sigma = \sigma_1\sigma_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

of a linear transformation σ_1 and another transformation σ_2 of order h, i.e. where $ad = h$, with third entry zero.

Clearly if such a composition exists $\sigma_1\sigma_2 = \sigma$ then $\sigma_2 = \sigma_1^{-1}\sigma$. So we are looking for a linear transformation σ_1^{-1} of σ that turns the third entry of the matrix to zero.

We shall find explicitly a composite linear transformation that does this, composed of

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now the action of S on σ is to replace the first row by the sum of both the rows. The action of T is to switch the rows and then multiply the new first row by -1.

We build up the following table of composite transformations

$$TT = \text{ change sign of top and bottom rows} \tag{143}$$

$$(S)^m = \text{ add bottom row to top row } m \text{ times} \tag{144}$$

$$(STS)^m = \text{ add top row to bottom row } m \text{ times} \tag{145}$$

$$T^3(S)^mT = \text{ subtract top row from bottom row } m \text{ times} \tag{146}$$

$$T^3(STS)^mT = \text{ subtract bottom row from top row } m \text{ times.} \tag{147}$$

From these combinations it is clear that it is possible to reduce one of the values α or γ to zero. Finally, applying T again if necessary, we can always reduce our matrix to the given form as required, proving the result.

Now as we noted, using a linear transformation to change the series that we are considering from the first to any other, gives us expressions for the other roots of the transformation equations, P^i.

In order to find these expressions, the general ideal will be to make a linear transformation of ω in expressions (139) to (142). We need to examine the expressions (124) to (126) and (129). In the P-values for the first series, i.e. with $\mu = 1$ and $\mu' = 0$, the value of ϖ is $\frac{1}{n}$, which is independent of ω. Thus a transformation, $\omega' = \frac{\gamma+\delta\omega}{\alpha+\beta\omega}$, will only change the second argument ω, of the theta functions that appear. We will investigate the effect of this transformation on these theta functions.

We start with the expression (1.6.7). Using the identities (43) and (76) we obtain

$$\exp(-\pi i\beta uu')\,\theta_{11}\,(u',\omega') = \epsilon^3\sqrt{\alpha+\beta\omega}\,\theta_{11}(u,\omega).$$

Next we make a simplifying assumption. We suppose our transformation is actually of the form

$$\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{16}. \tag{148}$$

This is compatible with the condition that $\alpha\delta-\beta\gamma=1$, but we leave it till later to show that such transformations actually allow us to obtain all the other series' of roots.

Theorem (1.6.5) now tells us, upon application of (29) and theorem (1.6.10), that

$$\exp(-\pi ibuu')\,\frac{\theta_{g_1,g_2}\,(u',\omega')}{\eta(\omega')\,\mathrm{F}^2(\omega')} = \frac{\theta_{g_1,g_2}(u,\omega)}{\eta(\omega)\,\mathrm{F}^2(\omega)}$$

for characteristics (g_1,g_2) equal to any one of $(0,0),(0,1)$ or $(1,0)$ and where

$$F(\omega) = \begin{cases} \mathfrak{f}(\omega), & \text{if } (g_1,g_2)=(0,0) \\ \mathfrak{f}_1(\omega), & \text{if } (g_1,g_2)=(0,1) \\ \mathfrak{f}_2(\omega), & \text{if } (g_1,g_2)=(1,0) \end{cases}$$

We are now able to use the transformation formulae of the Weber functions and the eta function to obtain in all cases

Theorem 4.3.1. *For a transformation of the form* (148) *we have*

$$\exp(-\pi i\beta uu')\,\theta_{g_1,g_2}\,(u',\omega') = \varepsilon^3\sqrt{\alpha+\beta\omega}\,\theta_{g_1,g_2}(u,\omega),$$

where (g_1,g_2) *are any integral characteristics and* ε *is some root of unity.*

Substituting $u=0$ and thence $u'=0$ into this expression and then dividing this original expression through by the new expression, we obtain

Theorem 4.3.2. *For a transformation of the form* (148) *we have*

$$\frac{\theta_{g_1,g_2}\,(u',\omega')}{\theta_{g_1,g_2}(0,\omega')} = \exp(\pi i\beta uu')\,\frac{\theta_{g_1,g_2}(u,\omega)}{\theta_{g_1,g_2}(0,\omega)},$$

where (g_1,g_2) *can be any of the characteristics* $(0,0),(0,1)$ *or* $(1,0)$.

Also, we have

Theorem 4.3.3. *For a transformation of the form* (148) *we have*

$$\frac{\theta_{11}\,(u',\omega')}{\theta'_{11}(0,\omega')} = \frac{\exp(\pi i\beta uu')}{\alpha+\beta\omega}\,\frac{\theta_{11}(u,\omega)}{\theta'_{11}(0,\omega)}.$$

This last expression can be rewritten using (43) and (76)

Theorem 4.3.4. *For a transformation of the form* (148) *we have*

$$\frac{\theta_{11}\,(u',\omega')}{\eta(\omega')} = \exp(\pi i\beta uu')\,\varepsilon^2\,\frac{\theta_{11}(u,\omega)}{\eta(\omega)}.$$

where ε *is the 24-th root of unity given by theorem* (2.1.3).

Now we are ready to apply the transformation $\omega \to \omega'$ to our roots P^0, expressions for which are on the one hand given by (124) to (126) and (129) with $\mu = 1, \mu' = 0$ and on the other hand by (139) to (142).

Clearly for $\mu = 1, \mu' = 0$ the value of ϖ is $\frac{1}{n}$. Now suppose that we choose our transformation such that

$$\alpha \equiv \mu \text{ and } \beta \equiv \mu' \pmod{n},$$

for some new values of μ and μ'. Note that this choice can always be made, as it is compatible with (148). By (4.3.2) and (4.3.4) the expressions for the roots, upon transformation, become

$$\exp\left(\pi i \sum_{h=1}^{\frac{n-1}{2}} \frac{(2h)^2\mu'(\mu+\mu'\omega)}{n^2}\right) \theta_{g_1,g_2}^{-(\frac{n-1}{2})} \prod_{h=1}^{\frac{n-1}{2}} \theta_{g_1,g_2}(2h\varpi)$$

for $(g_1, g_2) = (0,0), (0,1)$ or $(1,0)$ and

$$\exp\left(\pi i \sum_{h=1}^{\frac{n-1}{2}} \frac{(2h)^2\mu'(\mu+\mu'\omega)}{n^2}\right) \epsilon^{n-1}\eta(\omega)^{-\frac{n-1}{2}} \prod_{h=1}^{\frac{n-1}{2}} \theta_{11}(2h\varpi)$$

But simplifying these expressions we obtain (at least up to some specific root of unity) the expressions for the other roots P^i of the transformation equations given by e:P00 to (126) and (129) with general parameters μ and μ'.

In fact we have shown that all the roots of the transformation equations are given by

Theorem 4.3.5.

(149) $$P_{00}^i = \frac{\mathfrak{f}(n\omega')}{\mathfrak{f}(\omega')^n}$$

(150) $$P_{01}^i = \frac{\mathfrak{f}_1(n\omega')}{\mathfrak{f}_1(\omega')^n}$$

(151) $$P_{10}^i = \left(\frac{2}{n}\right) \frac{\mathfrak{f}_2(n\omega')}{\mathfrak{f}_2(\omega')^n}$$

(152) $$P_{11}^i = \varepsilon^{1-n}\sqrt{n}\,\frac{\eta(n\omega')}{\eta(\omega')}$$

for $i = 0, \ldots, \psi(n) - 1$ and where ε is given by theorem (2.1.3).

We now turn to the expression $n\omega'$ on the right hand side of these equations. In matrix notation, this expression when considered as a transformation of ω has the form

(153) $$\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} n\delta & n\gamma \\ \beta & \alpha \end{pmatrix},$$

i.e. the composition of a fundamental transformation of order n and a linear transformation.

As before, we can use the composite linear transformations of (143) to (147) to arrange a zero in the third entry of this transformation, i.e.

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Now left multiplying by the inverse of the linear transformation on the left hand side, we obtain

$$\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \tag{154}$$

writing our original transformation as the product of a linear transformation and one of order n in the form required by the above discussion.

Now consider what happens when we make the following replacements in this equation

$$\alpha' \to \alpha' + h\beta', \gamma' \to \gamma' + h\delta', b \to b - dh.$$

Our linear transformation remains linear, whilst b has changed by a multiple of d. Thus, since n is odd, we can ensure that b is divisible by a given power of 2. Also if n is not divisible by three, we can similarly demand that b be divisible by 3.

From (154) we have that $n\delta = \alpha' a$. However since the transformation $\omega \to \omega'$ has matrix satisfying (148) and since $ad = n$ then $\alpha' \equiv d \pmod{16}$. In fact by similar arguments, if we demand that $b \equiv 0 \pmod{16}$ we can show that

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \equiv \begin{pmatrix} d & 0 \\ 0 & d^3 \end{pmatrix} \pmod{16}. \tag{155}$$

So in the numerators of our expressions (149) to (152), $n\omega'$ has become

$$\frac{\beta' + \alpha' \frac{a\omega+b}{d}}{\delta' + \gamma' \frac{a\omega+b}{d}}$$

with $\alpha', \beta', \gamma', \delta'$ satisfying the congruences given by the previous equation.

Now we can apply the general theory of linear transformations of the Weber functions which we have detailed in an earlier section. Evaluating the various roots of unity that appear, with reference to (155), we have

$$P^i_{00} = \varrho \, \frac{\mathfrak{f}(\frac{a\omega+b}{d})}{\mathfrak{f}(\omega)^n} \tag{156}$$

$$P^i_{01} = \varrho \left(\frac{2}{d}\right) \frac{\mathfrak{f}_1(\frac{a\omega+b}{d})}{\mathfrak{f}_1(\omega)^n} \tag{157}$$

$$P^i_{10} = \varrho \left(\frac{2}{a}\right) \frac{\mathfrak{f}_2(\frac{a\omega+b}{d})}{\mathfrak{f}_2(\omega)^n}, \tag{158}$$

where

$$\varrho = \exp\left(-\frac{2\pi i}{3}[\delta'(\beta' - \gamma') - (\delta'^2 - 1)\alpha'\gamma']\right) \times \exp\left(\frac{2n\pi i}{3}[\alpha(\gamma - \beta) - (d^2 - 1)\beta\delta]\right). \tag{159}$$

By similar methods, for the eta function, if we require that $b \equiv 0 \pmod 8$, we obtain

$$P^i_{11} = \left(\frac{\beta}{\alpha}\right)\left(\frac{\gamma'}{\delta'}\right) \varrho \, i^{\frac{d-1}{2}} \sqrt{a} \, \frac{\eta(\frac{a\omega+b}{d})}{\eta(\omega)}.$$

In all the cases above, if 3 does not divide n and b is set to be divisible by 3, then $\varrho = 1$, simplifying matters considerably.

5. Modular Equations for Weber Functions

5.1. **The Function $\mathfrak{f}^{24}$.** In Weber's day the function k^2 was considered fundamental. All modular functions were related to this rather than to Klein's fundamental function $j(\omega)$ as they are today.

From (87) and the transformation formulas (35) to (42) it is easy to see that k^2 is invariant under the transformations

$$\omega \to \omega + 2 \text{ and } \omega \to \omega/(1+2\omega).$$

Clearly all the functions in $\mathbb{Q}(k^2)$ are likewise invariant under these transformations. We note since ${k'}^2 = 1 - k^2$, that ${k'}^2$ is in the field $\mathbb{Q}(k^2)$. However from (87), (12), (1.6.10) and (81) we find in turn that

$$k^2 {k'}^2 = \frac{\theta_{01}^4\, \theta_{10}^4}{\theta_{00}^8} = \frac{\mathfrak{f}_1^8\, \mathfrak{f}_2^8}{\mathfrak{f}^{16}} = \frac{2^4}{\mathfrak{f}^{24}}.$$

Thus we see that $\mathfrak{f}^{24}$ in fact belongs to $\mathbb{Q}(k^2)$ and in particular that they are functions invariant under the same group.

More is true of the function $\mathfrak{f}^{24}$. From its definition in terms of eta functions it is easy to see that it is invariant under

$$\omega \to \omega + 2 \text{ and } \omega \to -1/\omega. \tag{160}$$

In fact we shall prove more

Theorem 5.1.1. *The modular functions for the group G generated by the transformations* (160) *form a field. Furthermore each function in this field is rational in the function $\mathfrak{f}^{24}$.*

Proof: The fact that the modular functions for this group form a field is trivial.

We construct a fundamental domain for the action of the group G on the extended complex upper half plane. We identify edges of this domain which map to each other under elements of G. This domain can in fact be made into a compact Riemann surface R. The function $\mathfrak{f}^{24}$ can then be thought of as a meromorphic function $\mathfrak{f}^{24}(z)$ on this Riemann surface.

The Weber function $\mathfrak{f}$ is a quotient of eta functions and the eta function has no zeroes or poles in the complex upper half plane. In fact $\mathfrak{f}$ takes the value zero exactly once on the compact Riemann surface R. Thus $\mathfrak{f}^{24}(z)$ is a mapping from this Riemann surface to the Riemann sphere S of valence one.

Now let g be any other function for the same group G. It is also a meromorphic function on R. Now by pulling back along $\mathfrak{f}^{24}(z)$ this function can be made into a meromorphic function on the Riemann sphere S. Thus it is a rational function of the Riemann sphere with independent variable $c = \mathfrak{f}^{24}(z)$, i.e. g is rationally expressible in terms of $\mathfrak{f}^{24}$. □

As we noted above $\mathfrak{f}^{24} \in K = \mathbb{Q}(k^2)$. In particular we see from (156) that if $kn \equiv 0 \pmod{24}$ for some $k \in \mathbb{N}$, then the values $\mathfrak{f}_{a,b,d} = \varrho^k\, \mathfrak{f}(\frac{a\omega+b}{d})^k$ are roots of a transformation equation over K which in particular has coefficients in $\mathbb{Q}(k^2)$.

Now suppose that we have constructed, out of the functions $\mathfrak{f}_{a,b,d}$ and $\mathfrak{f}^{24}$, a series of functions $\Phi_{a,b,d}$ which are permuted in exactly the same way as the $\mathfrak{f}_{a,b,d}$ by the transformations (160). We will also assume that the $\Phi_{a,b,d}$ are expressed rationally in terms of their constituent functions with rational coeffients.

We construct the function

$$\Psi[\mathfrak{f}^{24}, x] = \prod_{a,b,d} (x - \mathfrak{f}_{a,b,d}) \sum_{a,b,d} \frac{\Phi_{a,b,d}}{(x - \mathfrak{f}_{a,b,d})}.$$

This function is clearly a polynomial in x of degree at most $\psi(n)-1$ and invariant under the transformations (160), i.e. it is an element of $\mathbb{C}(\mathfrak{f}^{24})[x]$ by the theorem above. But because of its construction which is symmetric in the roots of a transformation equation it also has its coefficients in $\mathbb{Q}(k^2)$. Thus as a polynomial in x it must have coefficients in $\mathbb{C}(\mathfrak{f}^{24})\bigcap\mathbb{Q}(k^2)=\mathbb{Q}(\mathfrak{f}^{24})$.

Let us also denote the product on the right hand side by $P[\mathfrak{f}^{24},x]$ since it also clearly has coefficients in $\mathbb{C}(\mathfrak{f}^{24})$ and by a similar argument to the above these coefficients must also be in $\mathbb{Q}(\mathfrak{f}^{24})$.

Now substituting a particular $\mathfrak{f}_{a,b,d}$ into the formula we obtain

$$\Phi_{a,b,d}=\frac{\Psi[\mathfrak{f}^{24},\mathfrak{f}_{a,b,d}]}{P'[\mathfrak{f}^{24},\mathfrak{f}_{a,b,d}]}$$

where P' denotes the derivative of P with respect to x.

We note in particular that this implies that $\Phi_{a,b,d}$ is in the field $\mathbb{Q}(\mathfrak{f}^{24},\mathfrak{f}_{a,b,d})$.

Now form the polynomial

$$F(x)=\prod_{a,b,d}(x-\Phi_{a,b,d}).$$

It is clear that this is a polynomial in x whose coefficients, being symmetric in the $\Phi_{a,b,d}$, and thus in the $\mathfrak{f}_{a,b,d}$ must belong to $\mathbb{Q}(\mathfrak{f}^{24})$.

Now consider what happens when each of the $\Phi_{a,b,d}$ is chosen so that it is finite for all ω in the complex upper half plane. Each of the coefficients of $F(x)$, considered as a function of ω, will then also have this property. Consider the denominator of such a function as a polynomial in $\mathfrak{f}^{24}$. For reasons which we noted above, $\mathfrak{f}^{24}(\omega)$ attains every finite complex value, for values τ in the complex upper half plane, except the value 0. Thus the denominator of our coefficient turns out to be a pure power of $\mathfrak{f}^{24}$. This means that in fact the rational expression must be a polynomial in both $\mathfrak{f}^{24}$ and $\frac{1}{\mathfrak{f}^{24}}$.

Now we have the theorem

Theorem 5.1.2. *The transformation*

$$\tau\to\frac{\tau-1}{\tau+1} \tag{161}$$

takes $\mathfrak{f}(\omega)$ *to* $\sqrt{2}/\mathfrak{f}(\omega)$.

Proof: Weber's proof goes as follows. From the definitions we note

$$\mathfrak{f}_1(2\omega)\mathfrak{f}_2(\omega)=\sqrt{2}.$$

But by theorem (2.2.3) this can be written

$$\mathfrak{f}(1-1/\omega)\mathfrak{f}(2\omega-1)=\sqrt{2}.$$

Replacing $2\omega-1$ with ω throughout gives us the stated result. □

Suppose now that we also choose the functions $\Phi_{a,b,d}$ to be permuted amongst each other by the transformation $\tau\to\frac{\tau-1}{\tau+1}$. In this case, the coefficients of $F(x)$ must be in

$$\mathbb{Q}(\mathfrak{f}^{24})\left[\mathfrak{f}^{24}+\frac{2^{12}}{\mathfrak{f}^{24}}\right].$$

Now suppose that the $\Phi_{a,b,d}$ remain finite also at $q=0$, i.e. the q-series of the $\Phi_{a,b,d}$ have no negative powers. This would imply that the highest power of $\mathfrak{f}^{24}+\frac{2^{12}}{\mathfrak{f}^{24}}$ appearing in each of the coefficients of $F(x)$ must be zero, since any higher power of it would contribute a negative power of q to the q-series of $\Phi_{a,b,d}$.

In other words, the coefficients of $F(x)$ must be constant. The same could then be said of each of the values $\Phi_{a,b,d}$. What is more, since each of these functions is a modular transformation of each of the others, all the values $\Phi_{a,b,d}$ would all be equal to the same constant C.

If all of the conditions above are met by the functions $\Phi_{a,b,d}$ then we have the equations $\Phi_{a,b,d} = C$. These are the modular equations for Weber function $\mathfrak{f}$ that we are after.

5.2. **Schläfli Modular Equations.** We now come at last to the derivation of some actual modular equations obtained via these methods. The first of these are the *Schläfli modular equations*. Weber deals with these in §73 of [2]. We follow his account.

Let us abbreviate as follows:

$$u = \mathfrak{f}(\omega), \quad v = \mathfrak{f}\left(\frac{a + b\omega}{d}\right)$$
$$u_1 = \mathfrak{f}_1(\omega), \quad v_1 = \left(\frac{2}{d}\right)\mathfrak{f}_1\left(\frac{a + b\omega}{d}\right)$$
$$u = \mathfrak{f}_2(\omega), \quad v = \left(\frac{2}{a}\right)\mathfrak{f}_2\left(\frac{a + b\omega}{d}\right).$$

We need to investigate the effect of the transformations (160) and (161) on at least the functions u and v. We do slightly more and investigate the effect of the first set of transformations on all the Weber functions, since the results will be of use later.

For the first two transformations, we use the theorems (2.2.4) to (2.2.12) to evaluate the roots of unity that appear upon transforming the various functions. This can be summarised in the following table.

(162)

	u	u_1	u_2	v	v_1	v_2
$-\frac{1}{\tau}$	u	u_2	u_1	ϱv	ϱv_2	ϱv_1
$\tau + 1$	$e^{-\frac{\pi i}{24}} u_1$	$e^{-\frac{\pi i}{24}} u$	$e^{\frac{\pi i}{12}} u_2$	$\sigma e^{-\frac{n\pi i}{24}} v_1$	$\sigma e^{-\frac{n\pi i}{24}} v$	$\sigma e^{\frac{n\pi i}{12}} v_2$
$\tau + 2$	$e^{-\frac{\pi i}{12}} u$	$e^{-\frac{\pi i}{12}} u_1$	$e^{\frac{\pi i}{6}} u_2$	$\sigma^2 e^{-\frac{n\pi i}{12}} v$	$\sigma^2 e^{-\frac{n\pi i}{12}} v_1$	$\sigma^2 e^{\frac{n\pi i}{6}} v_2$

where $\varrho = \exp\left(-\frac{2\pi i}{3}[\alpha(\gamma - \beta) + (\alpha^2 - 1)\beta\delta]\right)$ and $\sigma = \left(\frac{2}{a}\right)\exp((n - \lambda)\pi i/24)$ are cubed roots of unity which come from the linear transformations that result from the general transformation theory which is discussed in the last section. These have the value 1 if n is not divisible by 3.

Here λ is a constant which arises when applying the transformation $\omega \to \omega + 1$, i.e. we solve

$$\begin{pmatrix} b & a \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 & a \\ 0 & d \end{pmatrix},$$

transforming the argument $(b\omega + a)/d$ into another suitable argument of the same kind with a linear transformation applied to it.

We now consider just u and v. We want to know what happens to them under the transformation $\tau \to \frac{\tau - 1}{\tau + 1}$. We solve

$$\begin{pmatrix} b & a \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} b' & a' \\ 0 & d' \end{pmatrix},$$

Again using theorems (2.2.4) to (2.2.12) to evaluate the roots of unity resulting from the linear transformation with determinant $\alpha\delta - \beta\gamma = 1$, one obtains the following table

	u	v
$\frac{\tau-1}{\tau+1}$	$\frac{\sqrt{2}}{u}$	$\varrho\left(\frac{2}{n}\right)\frac{\sqrt{2}}{v}$

where $\varrho = \exp\left(-\frac{2\pi i}{3}[\alpha(\gamma-\beta)+(\alpha^2-1)\beta\delta]\right)$. We repeat that if n is not divisible by 3 then $\varrho = 1$.

From the tables we have constructed, a careful examination of the various roots of unity which result leads us to consider various natural combinations of u and v. In particular we consider the action of the permutations on the functions

$$A = \left(\frac{u}{v}\right)^r + \left(\frac{v}{u}\right)^s$$
$$B = (uv)^s + \left(\frac{2}{n}\right)^{r+s} \frac{2^s}{(uv)^s}$$

where we choose the natural numbers r and s so that

$$(n-1)r \equiv 0,\ (n+1)s \equiv 0 \pmod{12}.$$

We note that because of these conditions, if n is divisble by 3 then both r and s are also, and in fact we will have

$$\left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}.$$

Our table now looks as follows

	A	B
$-\frac{1}{\tau}$	A	B
$\tau+2$	$(-1)^{\frac{(n-1)r}{12}}A$	$(-1)^{\frac{(n+1)s}{12}}B$
$\frac{\tau-1}{\tau+1}$	$\left(\frac{2}{n}\right)^r A$	$\left(\frac{2}{n}\right)^r B$

Note that at worst now we only have sign changes occuring under our transformations. It is natural to take

$$\Phi_{a,b,d} = \sum M_{h,k} A^h B^k,$$

for some carefully chosen constants $M_{h,k}$.

We will pick the exponents (h,k) in such a way that all terms simultaneously experience the same sign changes under all permutations from the table above. This will ensure that either $\Phi_{a,b,d}$ or at the worst $\Phi^2_{a,b,d}$ satisfy the condition that we specified above, namely that they be permuted by the transformations (160) and (161). The extra complication of the possible permutation of the $\Phi_{a,b,d}$ only up to sign turns out to present not problem. The former argument goes through as before, but with the $\Phi_{a,b,d}$ replaced with their squares. If the squares of the values are constant then so are the values themselves.

In order to satisfy the other condition that the functions remain finite at $q = 0$ we need to pick our coefficients $M_{h,k}$ carefully so that this is the case. It turns out that this is easiest to accomplish when n is a prime.

The way to go about doing all this is to examine the q-expansions of the functions A and B. Upon doing this, it becomes clear that it will simplify things greatly if we also require

$$\frac{(n-1)r}{12} \equiv \frac{(n+1)s}{12} \pmod{2}.$$

So for example, for the first seven primes we choose

$$n=3,\ A = \left(\tfrac{u}{v}\right)^6 + \left(\tfrac{v}{u}\right)^6,\ B = (uv)^3 - \tfrac{8}{(uv)^3}$$
$$n=5,\ A = \left(\tfrac{u}{v}\right)^3 + \left(\tfrac{v}{u}\right)^3,\ B = (uv)^2 - \tfrac{4}{(uv)^2}$$
$$n=7,\ A = \left(\tfrac{u}{v}\right)^4 + \left(\tfrac{v}{u}\right)^4,\ B = (uv)^3 + \tfrac{8}{(uv)^3}$$

$n = 11,\ A = \left(\frac{u}{v}\right)^6 + \left(\frac{v}{u}\right)^6,\ B =\ uv - \frac{2}{uv}$
$n = 13,\ A = \frac{u}{v} + \frac{v}{u},\ B =\ (uv)^6 - \frac{64}{(uv)^6}$
$n = 17,\ A = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3,\ B =\ (uv)^4 + \frac{16}{(uv)^4}$
$n = 19,\ A = \left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2,\ B =\ (uv)^3 - \frac{8}{(uv)^3}.$

One obtains the q-expansions of the functions as far as is necessary, as follows

$$
\begin{aligned}
n = 3,\ & A = q^{-\frac{1}{2}}(1 - 5q \ldots),\\
& B = q^{-\frac{1}{2}}(1 - 5q \ldots),\\
n = 5,\ & A = q^{-\frac{1}{2}}(1 - 2q \ldots),\\
& B = q^{-\frac{1}{2}}(1 - 2q \ldots),\\
n = 7,\ & A = q^{-1}(1 - 4q \ldots),\\
& B = q^{-1}(1 + 3q \ldots),\\
n = 11,\ & A = q^{-\frac{5}{2}}(1 - 6q + 21q^2 \ldots),\\
& B = q^{-\frac{1}{2}}(1 - q + 2q^2 \ldots),\\
n = 13,\ & A = q^{-\frac{1}{2}}(1 + 2q^2 - 2q^3 \ldots),\\
& B = q^{-\frac{7}{2}}(1 + 6q + 15q^2 + 26q^3 \ldots),\\
n = 17,\ & A = q^{-2}(1 - 3q + 6q^2 - 13q^3 + 25q^4 - 39q^5 + 76q^6 \ldots),\\
& B = q^{-3}(1 + 4q + 6q^2 + 8q^3 + 11q^4 + 28q^5 + 54q^6 \ldots),\\
n = 19,\ & A = q^{-\frac{3}{2}}(1 - 2q + 3q2 - 5q^3 + 11q^4 - 13q^5 + 24q^6 - 28q^7 \ldots),\\
& B = q^{-\frac{5}{2}}(1 + 3q + 3q^2 + 4q^3 + 9q^4 + 4q^5 + 39q^6 - 27q^7 \ldots).
\end{aligned}
$$

One eliminates all the negative powers by taking the following relations. These are the modular equations that we are after.

$$
\begin{aligned}
& n = 3,\ A - B = 0\\
& n = 5,\ A - B = 0\\
& n = 7,\ A - B + 7 = 0\\
& n = 11,\ A - B^5 + B^3 + 2B = 0\\
& n = 13,\ A^7 + 6A^5 + A^3 - 20A - B = 0\\
& n = 17,\ A^3 - B^2 + 17AB - 34A^2 + 34B + 116A + 440 = 0\\
& n = 19,\ A^5 - B^3 + 19AB^2 - 95A^2B + 109A^3 + 128B - 128A = 0.
\end{aligned}
$$

Of course similar equations relating u_1 with v_1 and u_2 with v_2 can be obtained simply by replacing τ with $\tau + 1$ in the one case and then τ with $-\frac{1}{\tau}$ in the other. Both sets of modular equations have the same form so we quote only one. Define

$n = 3,\ A_1 = \left(\frac{u_1}{v_1}\right)^6 - \left(\frac{v_1}{u_1}\right)^6,\ B_1 =\ (u_1v_1)^3 + \frac{8}{(u_1v_1)^3}$
$n = 5,\ A_1 = \left(\frac{u_1}{v_1}\right)^3 - \left(\frac{v_1}{u_1}\right)^3,\ B_1 =\ (u_1v_1)^2 + \frac{4}{(u_1v_1)^2}$
$n = 7,\ A_1 = \left(\frac{u_1}{v_1}\right)^4 + \left(\frac{v_1}{u_1}\right)^4,\ B_1 =\ (u_1v_1)^3 + \frac{8}{(u_1v_1)^3}$

$n = 11,\ A_1 = \left(\frac{u_1}{v_1}\right)^6 - \left(\frac{v_1}{u_1}\right)^6,\ B_1 =\ u_1v_1 + \frac{2}{u_1v_1}$

$n = 13,\ A_1 = \frac{u_1}{v_1} - \frac{v_1}{u_1},\ B_1 =\ (u_1v_1)^6 + \frac{64}{(u_1v_1)^6}$

$n = 17,\ A_1 = \left(\frac{u_1}{v_1}\right)^3 + \left(\frac{v_1}{u_1}\right)^3,\ B_1 =\ (u_1v_1)^4 + \frac{16}{(u_1v_1)^4}$

$n = 19,\ A_1 = \left(\frac{u_1}{v_1}\right)^2 - \left(\frac{v_1}{u_1}\right)^2,\ B_1 =\ (u_1v_1)^3 + \frac{8}{(u_1v_1)^3}.$

For these we obtain the modular equations

$n = 3,\ A_1 + B_1 = 0$

$n = 5,\ A_1 + B_1 = 0$

$n = 7,\ A_1 - B_1 - 7 = 0$

$n = 11,\ A_1 + B_1^5 + B_1^3 - 2B_1 = 0$

$n = 13,\ A_1^7 - 6A_1^5 + A_1^3 + 20A_1 + B_1 = 0$

$n = 17,\ A_1^3 - B_1^2 - 17A_1B_1 - 34A_1^2 - 34B_1 + 116A_1 + 440 = 0$

$n = 19,\ A_1^5 + B_1^3 - 19A_1B_1^2 + 95A_1^2B_1 - 109A_1^3 + 128B_1 - 128A_1 = 0.$

5.3. **The 'Irrational' Modular Equation.** A second form of modular equation can be obtained, called the *modular equation of irrational form.*

Using the same definitions for u, v, u_1, v_1, u_2 and v_2 we consider equations containing the products uv, u_1v_1 and u_2v_2 all at the same time. Clearly from the relations $\mathfrak{f}\,\mathfrak{f}_1\,\mathfrak{f}_2 = \sqrt{2}$ and $\mathfrak{f}_1(2\omega)\mathfrak{f}_2(\omega) = \sqrt{2}$ we can always eliminate all but one of the Weber functions from any such modular equation.

Furthermore we might write such an expression entirely in terms of the function k^2. This is the more familiar way of writing modular equations that Jacobi would have been familiar with. However when this is done a radical sign appears which is where this form of modular equation gets its name.

Of course we need to investigate the effect of transformations on these product functions uv, u_1v_1 and u_2v_2. This time we use the two transformations

$$\omega \to \omega + 1 \ \text{ and } \ \omega \to -1/\omega \tag{163}$$

which generate the entire modular group. To do this we make use of the table (162). We are able to construct the following table for our product functions

ω	uv	u_1v_1	u_2v_2
$-1/\omega$	ϱuv	ϱu_2v_2	ϱu_1v_1
$\omega+1$	$e^{-\frac{(n+1)\pi i}{24}}\sigma u_1v_1$	$e^{-\frac{(n+1)\pi i}{24}}\sigma uv$	$e^{\frac{(n+1)\pi i}{12}}\sigma u_2v_2$

where ϱ and σ have the same meanings as earlier.

As can be seen from the table the roots of unity depend on the value of $n+1$. It turns out that different functions are required depending on the value of n modulo 8. We distinguish three separate cases:

I. $n \equiv 7 \pmod 8$

We set

$$2A = uv + (-1)^{\frac{n+1}{8}}(u_1v_1 + u_2v_2)$$
$$B = \frac{2}{u_1v_1} + \frac{2}{u_2v_2} + (-1)^{\frac{n+1}{8}}\frac{2}{uv}.$$

We obtain the following table of transformations

ω	A	B
$-1/\omega$	ϱA	$\varrho^2 B$
$\omega+1$	$e^{\frac{\pi i(n+1)}{12}}\sigma A$	$e^{-\frac{\pi i(n+1)}{12}}\sigma^2 B$

Now a product of the form $A^h B^k$ will induce factors ϱ^{h-k} and $e^{\frac{(h-k)(n+1)\pi i}{12}}\sigma^{h-k}$ under the two transformations (163). Note that when $n \equiv -1 \pmod 3$ then because we already have $n \equiv 7 \pmod 8$, these roots of unity are equal to 1.

We define

$$\Phi_{a,b,d} = \sum M_{h,k} A^h B^k$$

for various indices and constants h, k and M.

In particular when $n \equiv 0, 1 \pmod 3$ we choose the h and k so that the differences $h-k$ in each term leave the same residues modulo 3.

These choices ensure that the $\Phi_{a,b,d}$ are permuted up to some known constants. The fact that the $\Phi_{a,b,d}$ remain finite for all ω in the complex upper half plane means that the polynomial $F(x) = \prod(x - \Phi_{a,b,d})$ has coefficients which are polynomial in the function $j(\omega)$ which is a function for the group generated by the transformations (163).

Once again, picking the constants $M_{h,k}$ so that the q-series for the $\Phi_{a,b,d}$ do not have negate powers of q ensures that the coefficients of $F(x)$ are constants, the same then being true of the $\Phi_{a,b,d}$.

Computing q-series and picking the coefficients $M_{h,k}$ so that this is so, we obtain the following modular equations

$n = 7,\ A = 0$

$n = 23,\ A = 1$

$n = 31,\ (A^2 - B)^2 - A = 0$

$n = 47,\ (A^2 - A - B) = 2$

$n = 71,\ (A^3 - 4A^2 + 2A - B) = 1.$

Now throughout our explicit calculation of modular equations of various kinds, we have been somewhat silent on one matter so far. We have said that the q-series of the $\Phi_{a,b,d}$ should all have no negative powers of q. However, the q-series we have quoted (where we have done so) have been only those for $\Phi_{1,0,n}$. It turns out of course that the q-series of the other Φ values are related to that of $\Phi_{1,0,n}$ so that what is true of the one is true of them all. We leave it to the reader to check in detail that this is the case.

However, this ceases to be the case in one interesting situation. It turns out that we can construct a modular equation for the composite degree $n = 15$. In this case, we must actually check that the q-series for both $\Phi_{1,0,15}$ and $\Phi_{3,0,5}$ have the required property. This being done we are able to generate the following modular equation

$n = 15,\ A^3 - AB + 1 = 0.$

II. $n \equiv 3 \pmod 8$

We set

$$4A = u^2v^2 - u_1^2v_1^2 - u_2^2v_2^2$$

$$B = \frac{4}{u_1^2 v_1^2} + \frac{4}{u_2^2 v_2^2} - \frac{4}{u^2 v^2}.$$

We obtain the following transformation table

ω	A	B
$-1/\omega$	$\varrho^2 A$	ϱB
$\omega + 1$	$e^{\frac{\pi i(n+1)}{6}} \sigma^2 A$	$e^{-\frac{\pi i(n+1)}{6}} \sigma B$

Here we obtain the modular equations

$n = 3,\ A = 0$

$n = 11,\ A = 1$

$n = 19,\ A^5 - 7A^2 - B = 0.$

III. $n \equiv 1 \pmod 4$

We set

$$8A = u^4 v^4 - u_1^4 v_1^4 - u_2^4 v_2^4$$
$$B = \frac{4}{u_1^4 v_1^4} + \frac{4}{u_2^4 v_2^4} - \frac{4}{u^4 v^4}.$$

We have the following transformation table

ω	A	B
$-1/\omega$	ϱA	$\varrho^2 B$
$\omega + 1$	$e^{\frac{\pi i(n+1)}{3}} \sigma A$	$e^{-\frac{\pi i(n+1)}{3}} \sigma^2 B$

Here we obtain the single modular equation

$n = 5,\ A = 1.$

Weber goes on to derive other modular equations, including many with composite degrees. However these become progressively more complicated in their expression and lie beyond the scope of this document to describe. What we have described here will be sufficient for our purposes.

References

[1] David Mumford, *Tata Lectures on Theta I* Prog. in Math. Birkhäuser, 1983.
[2] Heinrich Weber, *Lehrbuch der Algebra*, Dritter Band, Third Edition. Chelsea N.Y.

E-mail address: wb_hart@maths.mq.edu.au