3. MODULAR EQUATIONS AND ETA EVALUATIONS

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Abstract. My talk at the recent AustMS annual meeting met with some success [1], immediately initiating a request that I write a summary for the *Gazette*. In response, I give an informal essay surveying the ideas which precipitated my interest in modular equations.

Efforts to evaluate quotients of the Dedekind eta function $\eta(\tau)$ go back 175 years. My interest in the matter is only some two years old and derives from the authors of [4] showing me their paper and its predecessor [11]. Those two papers deal with explicit evaluation of $\eta(\tau)$ at quadratic irrationals and the relation of those evaluations to singular values of L-series.

Such questions fall into the general scope of complex multiplication. That theory tells one that evaluating eta quotients at elements in an imaginary quadratic number field yields values belonging to certain algebraic number fields — essentially 'ring class fields'; see [6]. Such evaluations turn out to be valuable, for example in the work of [6], since they lead to very much 'smaller' generators for those ring class fields than is done by more evident functions such as Klein's j function, and are much more useful computationally.

The point is that evaluating *analytic* functions yields *algebraic* information. All of us know that evaluating the function $q = e^{2\pi i\tau}$ at rational points yields roots of unity and that these generate cyclotomic number fields. Complex multiplication can be viewed as the analogue, for the computational geometry of elliptic curves, of trigonometry and the circle. (A pleasant and readable introduction to the beautiful topic 'complex multiplication' is provided by Cox [3].)

One could choose to identify special values from their decimal expansion on a computer screen or one might proceed, as I have, to find algebraic techniques for eliciting those values directly. The latter has clear advantage even over the most intelligent numerical method.

The eta function is a 24-th root of the discriminant $\Delta(\tau)$ of an elliptic curve. The discriminant function $\Delta(\tau)$, defined for τ in the complex upper half plane, is periodic, with period 1; that is, it is invariant under the transformation $T : \tau \to \tau + 1$. Thus $\Delta(\tau)$ has a Fourier expansion, that is an expansion in powers of $q = e^{2\pi i\tau}$, a so-called q-series expansion; it happens that this q-series has no negative powers of q, even its constant term vanishes, so it is properly a power series in q. Remarkably, the theory of elliptic functions implies that $\Delta(\tau)$ never vanishes for τ in the upper half plane. More, Δ is a *modular form*, meaning that Δ is also transformed in a well known way by the transformation $S : \tau \to -1/\tau$ ([9] gives detailed information on modular forms and functions).

The more familiar absolute modular invariant $j(\tau)$ is the quotient of another modular form of weight 12 by $\Delta(\tau)$. It is a modular *function*, invariant under the action of the full group of fractional linear transformations generated by S and T.

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The eta function, being a 24-th root of Δ, is given by the q-series

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \qquad q = e^{2\pi i \tau}.$$

The actions of S and T on $\eta(\tau)$ multiply it by certain 24-th roots of unity; see [8].

Weber functions are a special case of eta quotients. Specifically, Weber [10, §34] defines his functions as normalized quotients of the eta function:

$$\mathfrak{f}(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \qquad \mathfrak{f}_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \qquad \mathfrak{f}_2(\tau) = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)}.$$

Weber also lists a number of identities which these functions satisfy; these identities proved to be invaluable in my investigations.

First, there are the identities involving the modular transformations

$$\begin{pmatrix} \mathfrak{f} \\ \mathfrak{f}_1 \\ \mathfrak{f}_2 \end{pmatrix} \circ T = \begin{pmatrix} \zeta_{48}^{-1}\,\mathfrak{f}_1 \\ \zeta_{48}^{-1}\,\mathfrak{f} \\ \zeta_{24}\,\mathfrak{f}_2 \end{pmatrix}, \qquad \begin{pmatrix} \mathfrak{f} \\ \mathfrak{f}_1 \\ \mathfrak{f}_2 \end{pmatrix} \circ S = \begin{pmatrix} \mathfrak{f} \\ \mathfrak{f}_2 \\ \mathfrak{f}_1 \end{pmatrix}, \tag{1}$$

where ζ_n denotes the n-th root of unity $e^{2\pi i/n}$.

In addition, the following beautiful identities hold

$$\mathfrak{f}^8 = \mathfrak{f}_1^8 + \mathfrak{f}_2^8, \qquad \mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}, \qquad \mathfrak{f}_1(\tau) = \frac{\sqrt{2}}{\mathfrak{f}_2(\tau/2)}. \tag{2}$$

Even more important, Weber developed an extensive theory of modular equations for his functions — here a modular equation is a polynomial relationship between $\mathfrak{f}(\tau)$ and $\mathfrak{f}(n\tau)$ for some $n \in \mathbb{Z}$ (where, usually, n is a prime p). Such relations are analogues of the well known polynomial relations between $\cos(\tau)$ and $\cos(n\tau)$.

These modular equations have their own elegance, and Weber lists many of them in a number of different flavours (see [10]; also see [2] for the work of others, including Ramanujan, on modular equations).

In particular, Weber noticed that the *Schläfli modular equations* can be used to evaluate his functions at certain points of the form $\tau = \sqrt{-m}$, with $m \in \mathbb{N}$; see [10, §128-132].

The story of Weber's method goes more or less as follows. Write u for $\mathfrak{f}(\tau)$ and v for $\mathfrak{f}(p\tau)$, where p is prime. Then introduce the product $P = uv$ and quotient $Q = v/u$ of these functions. A Schläfli modular equation is a polynomial relationship between functions A and B of the form

$$A = Q^l \pm 1/Q^l \quad \text{and} \quad B = P^k \pm (c/P)^k,$$

for some constant $c \in \mathbb{N}$ and exponents $k, l \in \mathbb{N}$. For example, for the prime $p = 7$, Weber's method sets

$$A = Q^4 + 1/Q^4 \quad \text{and} \quad B = P^3 + (2/P)^3.$$

He then derives a particularly simple and elegant expression, the Schläfli modular equation

$$A = B - 7\,. \tag{3}$$

Weber realised that if one sets $\tau = \sqrt{-p}/p$ then $p\tau = \sqrt{-p} = -1/\tau$. But from (1) $\mathfrak{f}(-1/\tau) = \mathfrak{f}(\tau)$, so that in the case $p = 7$ the modular equation (3) becomes

$$\left(\frac{\mathfrak{f}(\sqrt{-7})}{\mathfrak{f}(\sqrt{-7})}\right)^4 + \left(\frac{\mathfrak{f}(\sqrt{-7})}{\mathfrak{f}(\sqrt{-7})}\right)^4 = \mathfrak{f}(\sqrt{-7})^6 + 8/\mathfrak{f}(\sqrt{-7})^6 - 7\,.$$

Thus if $x = \mathfrak{f}(\sqrt{-7})$, x satisfies the equation

$$x^{12} - 9x^6 + 8 = 0.$$

In fact, $\sqrt{2}$ plainly is a root of this equation and it turns out that $\mathfrak{f}(\sqrt{-7}) = \sqrt{2}$.

One can easily check this by plugging the data into PARI [5], being careful to recall that the function $\eta(x)$ is given by the PARI command `eta(x,1)`, and not by `eta(x)`.

Such magical applications of modular equations are highly entertaining. However, one quickly realises that these equations are useful only for evaluating $\mathfrak{f}(\sqrt{-p})$ for certain primes p. Other of Weber's modular equations can be used to evaluate $\mathfrak{f}_1(\sqrt{-p})$ for some p. However, the magic wears off when one tries to evaluate $\mathfrak{f}_1\big(\frac{1}{4}(1+\sqrt{-47}\,)\big)$.

I made progress with this question only after noticing a nice numerical fact with the help of PARI. It is perhaps not surprising, almost a hundred years before the advent of the personal computer, that Weber's computational powers were not quite up to spotting such a numerical coincidence.

The first step in evaluating $\alpha = \mathfrak{f}_1\big(\frac{1}{4}(1+\sqrt{-47}\,)\big)$ seems to be to restrict oneself to evaluating its absolute value $|\alpha|$.

It turns out, because of a minor coincidence, that it is possible to use Weber's own evaluation of $\mathfrak{f}(\sqrt{-47})$ and some simple modular substitutions, to evaluate the auxiliary expression $\mathfrak{f}_2\big(\frac{1}{4}(1+\sqrt{-47}\,)\big)$. However, subsequent evaluation of $|\alpha|$ requires a little more work.

Set $\tau_1 = \frac{1}{12}(1+\sqrt{-47}\,)$. Then $S\tau_1 = \frac{1}{4}(-1+\sqrt{-47}\,)$. This simple observation yields the two useful results

$$\mathfrak{f}_2\left(\frac{1+\sqrt{-47}}{4}\right) = \mathfrak{f}_1\left(\frac{-1+\sqrt{-47}}{12}\right) \tag{4}$$

and

$$\mathfrak{f}_1\left(\frac{1+\sqrt{-47}}{4}\right) = \mathfrak{f}_2\left(\frac{-1+\sqrt{-47}}{12}\right). \tag{5}$$

But, more than this, it is easy to see from the definition of the eta function that $\mathfrak{f}_1\left((a+\sqrt{-d}\,)/c\right)$ is the complex conjugate of $\mathfrak{f}_1\left((-a+\sqrt{-d}\,)/c\right)$. Thus all four of the expressions in (4) and (5) above can be expressed in terms of τ_1 and $3\tau_1$. All this strongly suggests the use of a modular equation of degree three for Weber functions.

When we plug all the values into the appropriate modular equation, an amazing thing happens. Values are always accompanied by their complex conjugate! In other words, our modular equation only involves the absolute values of the quantities we are studying.

Specifically, let $u = \mathfrak{f}(\tau)$ and $v = \mathfrak{f}(3\tau)$ be defined as above and set

$$u_1 = \mathfrak{f}_1(\tau), \quad v_1 = \mathfrak{f}_1(3\tau), \quad u_2 = \mathfrak{f}_2(\tau), \quad \text{and} \quad v_2 = \mathfrak{f}_2(3\tau).$$

Then the relevant modular equation is, [10, §75],

$$u^2v^2 = u_1^2\,v_1^2 + u_2^2\,v_2^2. \tag{6}$$

If further, we write

$$l = |v|^2, \qquad m = |v_2|^2/\sqrt{2} \quad \text{and} \quad n = |v_1|^2/\sqrt{2},$$

then the modular equation (6) becomes the delightful identity

$$m^5\,n + n^5\,m = 1. \tag{7}$$

Of course the second identity $\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}$ of (2) ensures that $lmn = 1$, but the surprise is the above-mentioned numerical coincidence. It turns out that $l = m+1$.

It's not clear how one might guess this result from first principles. I do not know whether $l = m+1$ is just fortuitous: only a consequence of a law of small numbers, or whether it is a manifestation of a more general pattern. In any case, I have been able to make use of analogous coincidences to complete evaluations in other cases.

The 'assumption' $l = m+1$ makes it possible to solve (7) algebraically. However, once one has somehow 'guessed' a solution of (7) it is straightforward to verify formally that one has indeed guessed correctly. It is now easy to check that indeed $l = m + 1$. Thus we use numerical information to find a root of the equation (7), and we then use the fact that this 'possible root' is indeed a root to verify a piece of information that until then had only been 'known' numerically.

It turns out, moreover, that the root we find is real and must be precisely the root we need to complete our evaluation of $|\alpha|$. The upshot is that $x = |\alpha|^2/\sqrt{2}$ satisfies the equation

$$x^5 + x^4 + x^3 - x^2 - 2x - 1 = 0.$$

I have used variants of this method to obtain a number of other eta evaluations. Moreover, the approach soon led me to wonder about possible generalizations of the Weber functions. Eventually, I could show that these exist and I obtained modular equations for them. This is the work I spoke about at the AustMS meeting in July, 2003. More recently, this work has enabled me to obtain large numbers of new eta evaluations. I expect to report the results I have obtained in the not-too-distant future. In any case, full details are to be recorded in my thesis [7].

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