0. PRELIMINARIES

Introduction

This chapter essentially consists of an introduction to the topic of eta evaluations discussing what is known from the literature. We describe a number of useful results which will be referred to in later chapters.

It also seems useful to have an idealic version of the work of van der Poorten and Chapman and so we discuss this in section two of this chapter.

Some interesting analogies of parts of our topic with the cyclotomic number field case are noted in the final section.

1. The Chowla-Selberg Formula

The Dedekind eta function is defined for τ on the upper half complex plane by the q-series

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{1}$$

where $q = e^{2\pi i \tau}$. Contrast this with the discriminant function for an elliptic curve

$$\Delta(\tau) = (2\pi)^{12}\, q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

which is, upto a constant factor, the twenty-fourth power of $\eta(\tau)$.

The first important result concerning eta evaluations was due to Chowla and Selberg in 1967. In terms of the discriminant function, it is essentially stated in [2] as

Theorem (Chowla-Selberg) 1.0.1.

$$\prod_{(a,b,c)\in H(d)} a^{-6}\Delta(\tau) = (2\pi|d|)^{-6h} \left[\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{(\frac{d}{m})} \right]^{3w}$$

where the product on the left is over a complete set of reduced binary quadratic forms for the fundamental discriminant $d < 0$, with $\tau = \frac{b+\sqrt{d}}{2a}$ a complex root of that quadratic form, and where the right side is a product of gamma functions, whose exponent is given by $(\frac{d}{m})$ the Legendre-Jacobi-Kronecker symbol, h is the class number and w is equal to the number of roots of unity in the quadratic field $\mathbb{Q}(\sqrt{d})$.

Clearly if there is only one reduced binary quadratic form corresponding to the discriminant d, this formula provides an explicit evaluation of a single discriminant function in terms of gamma functions. However this can only happen when the class number is one, which only occurs for finitely many fundamental discriminants

Theorem (Heegner-Stark) 1.0.2. *An imaginary quadratic number field with fundamental discriminant has class number one iff it has discriminant*

$$d = -3, -4, -7, -8, -11, -19, -43, -67, -163.$$

Nevertheless, for other discriminants where the class number is higher, this formula gives us the product of h values $\Delta(\tau)$. If we are then prepared to take the real twenty-fourth root of this expression (the right hand side is real, since all the arguments of the gamma functions are real) and remove some factors of π, we obtain an expression for the product over absolute values of eta functions evaluated at these same roots of reduced binary quadratic forms.

$$\prod_{(a,b,c)\in H(d)} \left|a^{-\frac{1}{4}}\eta(\tau)\right| = (2\pi|d|)^{-\frac{1}{4}} \left[\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right]^{\frac{w}{8}}$$

This is a product of all the values we are interested in. Now in order to evaluate each of the values separately, all we need to find is the absolute value of the quotient of each eta value with the next in the list. Later we will see that such quotients of absolute values of eta functions are in fact units in certain number fields. It is these units that form the basis of most of our research.

2. L-series and the Kronecker Limit Formula

Given a quadratic form

$$Q(x,y) = ax^2 + bxy + cy^2, \quad d = b^2 - 4ac < 0$$

we investigate the Epstein zeta function

$$Z(s) = {\sum_{x,y=-\infty}^{\infty}}' \frac{1}{Q(x,y)^s}$$

where the sum is over all integers $x, y \neq 0$. This series converges for $s > 1$.

The Kronecker limit formula expresses this zeta function in terms of our eta functions as follows

Theorem (Kronecker Limit Formula) 2.0.3.

$$Z(s) = \frac{2\pi}{\sqrt{|d|}}\left(\frac{1}{(s-1)} + 2\gamma - \log|d| - 4\log\left(a^{-1/4}\left|\eta\left(\tau\right)\right|\right)\right) + O(s-1)$$

where $\tau = \frac{b+\sqrt{d}}{2a}$ *is a root of* $Q(x,y)$.

We now choose a character of the form class group for a fundamental discriminant d (or equivalently by class field theory, of the absolute ideal class group of the quadratic number field), and form take a character sum over Epstein zeta functions, where the quadratic forms appearing are a complete set of reduced forms for the discriminant d. We also take the limit of this expression as $s \to 1$. From the Kronecker limit formula we have an expression for each term of this character sum. However the character sum over the constant terms of the limit formula will disappear, since the sum over all the values of a character is zero. Thus we are left with

$$\lim_{s\to 1}\sum_{[Q]}\sum_{x,y=-\infty}^{\infty} \frac{\chi([Q])}{Q(x,y)^s} = \frac{-8\pi}{\sqrt{|d|}}\sum_{[Q]}\chi([Q])\log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right)$$

Now it is well known that a quadratic form belonging to a fundamental discriminant d represents an integer n when there is an ideal of the quadratic number field $K = \mathbb{Q}(\sqrt{d})$ whose norm is n. However there is one representation of each n for each of the roots of unity in K. This means that upto a factor which is the number of these roots of unity, sums over quadratic forms can be expressed as sums over norms of ideals. Moreover since the form class group and the ideal class group are canonically isomorphic then the character value for the class of the ideal is the same as the character value for the corresponding quadratic form. Therefore, the left hand side of this last expression can be replaced as follows

(2)
$$w(d)L(1,\chi) = w(d)\lim_{s\to 1}\sum_{\mathfrak{a}\in\mathcal{O}_K}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \frac{-8\pi}{\sqrt{|d|}}\sum_{[Q]}\chi([Q])\log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right)$$

where the sum on the left is over all integral ideals of the number field $K = \mathbb{Q}(\sqrt{d})$ and $w(d)$ is the number of roots of unity in the number field $K = \mathbb{Q}(\sqrt{d})$. Thus we have an ordinary Hecke L-series to evaluate.

Clearly this formula holds for any character of the absolute ideal class group. One strategy will be to consider all of the expressions obtained from these different characters and "solve" for the individual eta values in terms of the values of the L-series. The first time it was realised that this strategy could be employed and that there were enough L-series to completely break up the Chowla-Selberg formula, was in [16]. Of course it then remains to evaluate the L-series that occur there. However this paper represented proof in principle at least, that a more general form of the Chowla-Selberg formula exists, where there is no restriction to a product over the whole class group, or even over a single genus.

This method of evaluating in terms of L-series can be extended slightly for certain non-fundamental discriminants $D = f^2 d$ with d fundamental as above. This situation corresponds to an order $\mathcal{O}$ of conductor f in $\mathcal{O}_K$ the ring of integers of K. The complete set of reduced forms corresponding to this discriminant D is also a group whose order is given by (see for example [3])

Theorem 2.0.4.
(3)
$$h(\mathcal{O}) = \frac{h(\mathcal{O}_K)f}{[\mathcal{O}_K^* : \mathcal{O}^*]}\prod_{p|f}\left(1-\left(\frac{d_K}{p}\right)\frac{1}{p}\right)$$

where d_K is the discriminant of K, $\mathcal{O}^*$ is the group of units of the order, etc, and the product is over all primes dividing the conductor f of the order $\mathcal{O}$.

In a paper [15] the idea is presented to trivially lift a character of the absolute ideal class group to become a character of $\mathrm{Pic}(\mathcal{O})$ [1]. Then we can take the character sum of Epstein zeta functions whose quadratic forms are a complete set of reduced binary quadratic forms with non-fundamental discriminant D. Using the Kronecker limit formula this process will again yield a character sum of the terms of the limit formula. [15] also shows how to write this expression in terms of L-functions in a manner similar to the above.

In terms of our goal of finding the explicit values of single eta functions however, this technique does not work in general. The reason is that there are not enough characters of the absolute ideal class group, that we lift, to split apart all the eta functions that arise from the limit formula expressions. For there is one eta function per class in $\mathrm{Pic}(\mathcal{O})$. Thus, the only time when this approach will work is if the absolute class group has order the same as that of $\mathrm{Pic}(\mathcal{O})$.

[1] similar to a class group, but defined to be the proper $\mathcal{O}$-ideals modulo principal proper $\mathcal{O}$-ideals. It is also isomorphic as a group to the $\mathcal{O}_K$-ideals prime to the conductor f, modulo the principal such ideals $\alpha\mathcal{O}_K$ where $\alpha \equiv a \pmod{f\mathcal{O}_K}$ for an integer a prime to f, and is also isomorphic to the form class group of reduced binary quadratic forms of non-fundamental discriminant D. See [3] for more details.

From (3) above, we see that the two "class groups" are of the same order only under special conditions. In fact ignoring the cases $d = -3, -4$ where the order of the unit group $\mathcal{O}_K^*$ is not two, the term $[\mathcal{O}_K^* : \mathcal{O}^*]$ becomes equal to one. We then cancel the prime factors p of the conductor f in the formula (3) with the denominators of the terms in the product. The product then becomes a product of terms of the form $(p+1)$ or $(p-1)$. The conductor f has now also disappeared from the expresssion (being absorbed in this way into the product) unless it was not a square-free number.

This argument shows that for the cardinalities of the absolute class group and Pic $\mathcal{O}$ to be the same, the values $p+1$ and $p-1$ that appear can only have the value 1 and the conductor must be square-free. Thus, no primes other than two can occur in the conductor, and even then $\left(\frac{d_K}{p}\right)$ must be 1, i.e: two must split in K. But this means the conductor can only be one or two. The former case gives us just the absolute class group itself leaving only the latter possibility. The expression obtained via the method of [15] in this case is the following

Theorem 2.0.5. *For a non-fundamental discriminant $D = 4d$ with d fundamental and such that 2 splits in $\mathbb{Q}(\sqrt{d})$, we have*

$$(4)\quad w(D)\lim_{s\to 1}\left(\sum_{\substack{\mathfrak{a}\in\mathcal{O}_K\\(\mathfrak{a},2)=1}}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}+\frac{1}{2^{2s}}\sum_{\mathfrak{a}\in\mathcal{O}_K}\frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}\right)$$
$$=\frac{-8\pi}{\sqrt{|4d|}}\sum_{[Q]}\chi([Q])\log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{4d}}{2a}\right)\right|\right)$$

where the sums on the left are over integral ideals prime to two and all integral ideals, respectively, and the sum on the right is over all reduced binary quadratic forms Q of discriminant D.

But the part of the expression on the left involving just the first sum is precisely $w(D)L(1,\chi)$ with the Euler factors for primes above two removed. Recalling that $w(D) = 2$ since $D \neq -3, -4$, we have that the left hand side of (4) is given by

$$(5)\qquad 2\lim_{s\to 1}\left((1-\chi(\mathfrak{p})\,2^{-s})(1-\chi(\mathfrak{p}')\,2^{-s})+\frac{1}{4^s}\right)L(s,\chi)$$

where the prime two splits as $2 = \mathfrak{p}\mathfrak{p}'$.

Combining expressions (4) and (5) for the different characters of Pic($\mathcal{O}$) that are available, we can evaluate the individual absolute values of eta values that appear. Of course we need an expression for the product of all these expressions as before, but this is provided by an extended version of the Chowla-Selberg formula for non-fundamental discriminant found in [5] and [7].

3. Solving L-series

In a series of four papers [19], [20], [21] and [22] certain Epstein zeta functions $Z(s)$ that we have mentioned above are "solved" into quadratic Dirichlet L-series. The authors of these papers use two notations which we introduce here for convenience.

Firstly, for a binary quadratic form with coefficients (a, b, c) the corresponding Epstein zeta function $Z(s)$ is denoted

$$S(a,b,c) = Z(s) = \sum_{x,y=-\infty}^{\infty}{}' \frac{1}{(ax^2+bxy+cy^2)^s}$$

where once again the sum is over all integers $x, y \neq 0$.

Secondly, the notation L_d is introduced for a quadratic Dirichlet L-series, where d is some integer, either positive or negative. They are defined by

$$L_d = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s}$$

where $\left(\frac{d}{n}\right)$ is the Legendre-Jacobi-Kronecker symbol.

These L-series were evaluated by Dirichlet. His result is the following

Theorem (Dirichlet) 3.0.6. *For discriminant $d = b^2 - 4ac$ we have the evaluations*

$$d < 0 \quad L_d(1) = \frac{2h(d)\pi}{w(d)\sqrt{|d|}}$$

$$d > 0 \quad L_d(1) = \frac{2h(d) \ln \epsilon_0}{\sqrt{d}}$$

where ϵ_0 is a fundamental unit in the real quadratic number field, $h(d)$ is the class number of the quadratic number field with discriminant d and $w(d)$ the number of roots of unity that it contains.

The theorem that enables us to ultimately evaluate individual Epstein zeta series $S(a, b, c)$ is the following

Theorem (Kronecker) 3.0.7. *To each decomposition of a negative discriminant $d = d_1 d_2$ into discriminants d_1 and d_2 (whether positive or negative) including the decompostition $d = 1.d$, there is a corresponding (quadratic) genus character χ_{d_1,d_2}. Then we have the formula*

$$\sum_{[(a,b,c)]} \chi_{d_1,d_2}((a, b, c))S(a, b, c) = w(d)L_{d_1}L_{d_2}$$

where the sum is over a complete set of reduced forms for the discriminant d.

Note that L_1 in the preceeding formula is precisely the Riemann zeta function.

The final result that we need to complete evaluations is the following

Theorem 3.0.8.

$$S(a, b, c) = S(a, -b, c).$$

Applying the Kronecker limit formula to this we obtain

Corollary 3.0.9.

$$\left|\eta\left(\frac{-b + \sqrt{d}}{2a}\right)\right| = \left|\eta\left(\frac{b + \sqrt{d}}{2a}\right)\right|.$$

This last corollary is also obvious from the definition (1) of eta itself.

We now give an example of the use of these theorems from [22].

Example 3.0.10. *For the discriminant $d = -39$ we have from theorem 3.0.7*

$$S(1, 1, 10) + S(3, 3, 4) + S(2, 1, 5) + S(2, -1, 5) = 2L_1L_{-39}$$

and

$$S(1, 1, 10) + S(3, 3, 4) - S(2, 1, 5) - S(2, -1, 5) = 2L_{-3}L_{13}.$$

hence

$$2S(2, 1, 5) + 2S(2, -1, 5) = 2L_1L_{-39} - 2L_{-3}L_{13}.$$

But since, from theorem 3.0.9, $S(2, 1, 5) = S(2, -1, 5)$ we must have

$$S(2, 1, 5) = S(2, -1, 5) = \frac{L_1L_{-39} - L_{-3}L_{13}}{2}.$$

Applying the Kronecker limit formula to the left hand side of this, and theorem 3.0.6 to the right hand side, gives us the explicit eta evaluation we are after. Note however that this method does not split apart $S(1,1,10)$ and $S(3,3,4)$.

We will want to know exactly when all the eta functions are split up by these relations. But this can only occur when there is exactly one class per genus. Then there will be as many characters as there are genera and hence classes. This will lead to as many equations as there are eta functions to split, which is the situation we are after. This situation is characterized by the following

Theorem 3.0.11. *For an imaginary quadratic number field, there is precisely one class per genus when the class number is 1 or 2, or when the class number is 4 and the class group is isomorphic to KV, or more generally if the class group is 2-torsion.*

Proof: It is known that there are always the same number of classes in each genus [3]. Hence the proposition is equivalent to there being one class in the principle genus, i.e. in the genus of square classes. Hence the square of every class in the class group is the identity. Therefore the order of each element other than the identity is 2. But this limits us to exactly the possibilities given in the proposition. □

The only other way that we can get the above technique to evaluate eta functions individually is when we are lucky as in the example, and a class and its inverse are the only classes in a particular genus. For this situation we have

Theorem 3.0.12. *There are precisely two classes per genus when the class group is isomorphic to $C_4 \times C_2^s$ where $s \in \mathbb{N} \cup \{0\}$.*

Proof: There are two classes in the principal genus. Call them 1 and g. Clearly $g^2 = 1$. Since every other class h of the class group must have $h^2 = g$ or $h^2 = 1$, then every element of the group has order 1, 2 or 4. Thus we are reduced to $C_4^r \times C_2^s$ with $r, s \in \mathbb{N} \cup \{0\}$. However, if r is anything other than one, the subgroup of squares does not have exactly two elements. Therefore we are left with the possibilities of the theorem. □

Note that by class field theory, these theorems imply that the Hilbert class field of the quadratic number field K we are considering is K itself or, at worst, constructed by successive quadratic extensions of K or of a quartic extension of K.

4. Generalizing the Chowla-Selberg Formula

Based on the ideas above, it was hoped that the Chowla-Selberg formula could be extended to the product of the classes of forms in a single genus. This was accomplished for fundamental discriminant in [18]. Their result is as follows

Theorem (Williams, Zhang) 4.0.13. *Let G be one of the $2^{t(d)}$ genera of fundamental discriminant d then*

$$\prod_{(a,b,c)\in G} a^{-1/4} \left| \eta\left(\frac{b+\sqrt{d}}{2a}\right) \right| = \left(\frac{2\pi}{|d|}\right)^{\frac{-h(d)}{2^{t(d)+2}}} \tag{6}$$

$$\times \left(\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)} \right)^{\frac{w(d)}{2^{t(d)+3}}} \prod_{\substack{d_1.d_2=d \\ d_1>1}} \varepsilon_{d_1}^{\frac{-w(d_1)\chi_{d_1,d_2}(G)h(d_1)h(d/d_1)}{w(d/d_1)2^{t(d)+1}}}$$

where ε_{d_1} is the fundamental unit of the real quadratic number field $\mathbb{Q}(d_1)$.

Of course when there is one class per genus, this formula produces the results that we have already mentioned. Versions of this also exist for non-fundamental discriminant; see [4]. This time, in the formulae, various Euler factors appear due to the exclusion of ideals not prime to the conductor.

5. Elliptic units

Quotients of the discriminant function are simple cases of what are known more generally as elliptic units. Various definitions of elliptic units have surfaced over the years e.g: those of Siegel [12], Ramachandra [8], Robert [10] and more recently Coates, Wiles [1] and Rubin [11].

A general elliptic unit involves a quotient of discriminant functions, Jacobi theta functions and exponential functions. Although the author is not aware of any explicit calculations of elliptic units in the literature which do not use numerical methods, there are various other characterizations of them.

Robert (see [10]) defines certain certain class field invariants $\varphi_{\mathfrak{f}}(C) = \varphi_{\mathfrak{f}}(t, \mathfrak{b})$ where the pair $C = (t, \mathfrak{b})$ uniquely defines a certain ray class. For our purposes, all we need is the following

Definition 5.0.14.

$$\varphi_{(1)}(C) = |(2\pi)^{-12}\mathcal{N}(\mathfrak{b})^6\Delta(\mathfrak{b})|$$

where $\mathfrak{b}$ is any ideal of the absolute class C.

Note that we are here considering the quadratic ideal $\mathfrak{b}$ as a lattice with generators $\{\omega_1, \omega_2\}$ with $\tau = \frac{\omega_1}{\omega_2}$ in the complex upper half plane, so that we can define $\Delta(\mathfrak{b}) = \Delta(\tau)$. Likewise $\mathcal{N}$ is introduced so that, as it turns out, $\varphi_{(1)}$ depends only on the class of $\mathfrak{b}$.

The important theorem is the following (see [10] prop. 3)

Theorem 5.0.15. *For all classes C, C' and C'' containing ideals $\mathfrak{b}, \mathfrak{b}'$ and $\mathfrak{b}''$ respectively, we have*

(i) $\left(\dfrac{\varphi_{(1)}(C)}{\varphi_{(1)}(C')}\right)^{h(d)}$ *is a unit of the Hilbert class field H of $K = \mathbb{Q}(\sqrt{d})$,*

(ii) $\left(\dfrac{\Delta(\mathfrak{b})}{\Delta(\mathfrak{b}')}\right)^{\sigma_{C''}} = \dfrac{\Delta(\mathfrak{b}\mathfrak{b}'')}{\Delta(\mathfrak{b}'\mathfrak{b}'')}$;

where $h(d)$ is the class number of K and $\sigma_{C''}$ is the Galois action given by the Artin map from the ideal class C''.

Now there are two annoying problems with this result. Firstly we do not want the extra exponent $h(d)$ in (i) above. We will shortly see that it is indeed unnecessary. We also want to consider eta quotients rather than these discriminant quotients, which are a twenty-fourth power of an eta quotient. In general we may find that very much lower powers of eta quotients will suffice. In the next chapter we will quote a theorem which tells us precisely what powers to take. They will be much lower than the twenty-fourth powers quoted here.

6. Stark Units

As we have already related our eta quotients to special values of L-functions at $s = 1$, it is natural to look at the Stark conjectures and ascertain whether they have anything to say. These conjectures are related to the values of Artin L-series at $s = 0$ and $s = 1$.

Stark, in his first paper [13] suggests that he began looking at these matters by considering a series not unlike our Epstein zeta function. He defines the quadratic Artin L-series for the binary quadratic form $Q(m,n) = am^2 + bmn + cn^2$ as follows

Definition 6.0.16.

$$L(s,\chi,Q) = \frac{1}{2}\sum_{m,n}{}' \frac{\chi(Q(m,n))}{Q(m,n)^s}$$

where $\chi = \left(\frac{k}{n}\right)$, is the Legendre-Jacobi-Kronecker symbol.

Stark also reduces his L-series to an ideal theoretic version which he describes as follows (see pp. 342)

Lemma 6.0.17.

$$L(s,\chi,\mathcal{F}) = \sum_{\mathfrak{a}\in\mathcal{F}} \chi(\mathfrak{a})N(\mathfrak{a})^{-s}$$

where the sum is over all integral ideals in the single absolute class $\mathcal{F}$ corresponding to the quadratic form $Q(m,n)$.

The effect of Stark's character χ is to split up the ideal class $\mathcal{F}$ of the quadratic number field $K = \mathbb{Q}(\sqrt{d})$, into ring classes of the order with discriminant $D = dk^2$, where χ was a character modulo $|k|$. As Stark remarks, one can get back to ordinary Hecke L-series by taking a character sum of such partial L-series over all of the absolute classes of K.

Now the only evaluation of these L-series that Stark provides is the obvious one using the Kronecker limit formula as we have above. However in a manner similar to Robert, Stark gives other characterizations of these values. The main result of his paper is

Theorem (Stark) 6.0.18. *For $K = \mathbb{Q}(\sqrt{d})$ with $d < -4$*

$$L(1,\chi,\mathcal{F}) = \frac{2\pi}{24|k|\sqrt{|d|}}\log\varepsilon \qquad (7)$$

where ε is a unit in $\mathbb{R}\{H(\sqrt{k})\}$, ie: at worst, a quadratic extension of the (not totally) real subfield of the Hilbert class field H of K.

We note that in the degenerate case where $k = 1$ the unit is actually in this real subfield. From now on, this field will be called the maximal real subfield. It needs to be stressed however that it is not *totally* real.

Notice that this result is an improvement on what we have above. The class number exponent has disappeared, however we are still dealing with quotients of the discriminant function, essentially thanks to the factor of twenty-four in the denominator on the right hand side of (7).

Now the $s = 1$ Stark conjecture is not the most developed. Normally the more explicit conjectures are expressed in terms of special values of L-series at $s = 0$. The device that will allow us to look at our L-series at $s = 0$ is the functional equation. We quote a version of this given in [6] pp. 254.

Functional Equation 6.0.19. *Let $\mathfrak{F}$ be an ideal class of a number field K with discriminant d_K, with $N = [K : \mathbb{Q}]$ and with r_1 real embeddings and r_2 pairs of complex embeddings, then if*

$$A = 2^{-r_2}|d_K|^{1/2}\pi^{-N/2}$$

and

$$\Lambda(s, \mathfrak{F}) = A^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta(s, \mathfrak{F})$$

then

$$\Lambda(s, \mathfrak{F}) = \Lambda(1 - s, \mathfrak{F}') \tag{8}$$

where $\mathfrak{F}'$ is the ideal class of $(\mathfrak{d}\mathfrak{a})^{-1}$ with $\mathfrak{d}$ the different of $K/\mathbb{Q}$ and where $\mathfrak{a}$ was an ideal in $\mathfrak{F}$. Both sides of (8) are analytic except for simple poles at $s = 0$ and $s = 1$.

Taking a character sum of such partial zeta functions in the imaginary quadratic case $K = \mathbb{Q}(\sqrt{d})$, we obtain the following relation between our L-series at $s = 0$ and $s = 1$.

Theorem 6.0.20.

$$L(1, \chi) = \frac{2\pi}{\sqrt{|d|}}\lim_{s\to 0} s^{-1}L(s, \overline{\chi}) = \frac{2\pi}{\sqrt{|d|}}L'(0, \overline{\chi})$$

where $\overline{\chi}$ is the complex conjugate character of χ.

Note that we have here used the fact that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 and that $\Gamma(0) = 1$. This result simplifies (2) to the following elegant form

$$L'(0, \overline{\chi}) = \frac{-4}{w(d)}\sum_{[Q]}\chi([Q])\log\left(a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right|\right)$$

A similar simplification of (4) and (5) can be made, bringing to a close this line of investigation.

Next we note that, in a later paper, Stark made a so called 'first order zero' conjecture about the values of $L'(0, \chi)$ (see [14]). We will see however, that this conjecture excludes the particular case we are interested in and is therefore not of interest to us.

We recall that an ordinary L-series $L(1, \chi)$ has an Euler product expansion into so-called Euler factors. The Dedekind zeta function for a number field K can also be viewed as an L-series with the trivial character χ_0, which takes the value one on all classes of ideals in the number field K.

In order to understand what the first order zero conjecture says, we need to look at a special class of L-functions which will be just like our ordinary L-series, but with certain Euler factors excluded. In particular, if we choose a set S of places of K which also contains all the infinite places of K, then we can make the following definition

Definition 6.0.21.

$$L_S(s, \chi) = \prod_{\mathfrak{p}\notin S}\left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)^{-1}.$$

Here the product runs over all finite places that are not in S (recall that all the infinite places are in S, so we only need Euler factors for the finite places).

Note that in order to obtain such an L-series, we simply multiply our ordinary L-series by a term of the form $(1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s})$ for each finite prime $\mathfrak{p}$ in S. In particular if we are considering a zeta function, where the character is the trivial one, then these factors are of the form $(1 - N\mathfrak{p}^{-s})$.

Now loosely speaking, when $s = 0$ these factors themselves vanish, and so in multiplying our original zeta function by them, we are adding an extra order of zero for each such factor. The story for L-series in general is not that much more difficult. It is dealt with by the following theorem which we have essentially quoted from [9]

Theorem 6.0.22. *Let $r(\chi_0) = Card\ S - 1$ and in general $r(\chi)$ be the number of places $v \in S$ such that the decomposition group D_v of v is contained in the kernel of χ, ie: such that $\chi|_{D_v} = 1$ (this is trivially true for the infinite places in S), then the order of vanishing at $s = 0$ of $L_S(s, \chi)$ is equal to $r(\chi)$.*

The other concept we need in order to understand the first order zero conjecture is that of a partial zeta function. For any ideal class which corresponds by the Artin map to an element σ of the Galois group G of the field extension N/K that we are investigating, we define the partial zeta function for that class by

Definition 6.0.23.

$$\zeta_S(s, \sigma) = \sum_{\substack{(\mathfrak{a}, S)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} N\mathfrak{a}^{-s}$$

where the sum is over all ideals in a certain class prime to the places of S. The class is defined to be all ideals $\mathfrak{a}$ with Artin symbol $\sigma_{\mathfrak{a}}$ equal to a fixed σ.

Clearly our S-zeta functions and L-series can be built up by taking a character sum of such partial zeta functions.

Now the idea of the first order zero conjecture is that for an appropriate set S of excluded primes, the values of the derivatives at $s = 0$ of such partial zeta functions are logarithms of certain units (called Stark units). The exact statement is as follows (see [9])

Conjecture (Stark) 6.0.24. *Assume S contains all infinite places including a distinguished infinite place v which is totally split in our field extension N/K. Also let w be a certain one of those places which divide v. Further assume that Card $S \geq 2$. Let m be the number of roots of unity contained in our extension field N. Then there exists an S-unit ε of N such that*

$$log|\sigma(\varepsilon)|_w = -m\zeta_S'(0, \sigma)$$

Furthermore, $N(\sqrt[m]{\varepsilon})/K$ is an Abelian extension and if Card $S \geq 3$ then ε is actually a unit.

Taking a character sum of expressions like the one in the theorem leads to something very much like the Kronecker limit formula. The question then arises as to why the case that we have been dealing with, involving ordinary zeta functions and L-series, has been excluded. Specifically the theorem requires Card $S \geq 2$, whereas for an ordinary L-series given for an unramified extension of an imaginary quadratic field, we have only a solitary complex archimedean prime in S.

The answer is that the Stark conjecture hopes to relate properties of certain Stark units to the first non-zero coefficient in the Taylor series expansion of the partial zeta functions. However for the ordinary partial zeta functions of an imaginary quadratic number field, these coefficients are all the same constant, essentially given by the analytic class number formula.

Theorem (Analytic Class Number Formula at $s = 0$) 6.0.25. *The partial zeta function of a field K has the first non-zero coefficient in its Taylor series expansion at $s = 0$ given by*

$$\lim_{s \to 0} s^{-r_1 - r_2 + 1} \zeta_K(s, \sigma) = -\frac{R_K}{w}$$

where w is the number of roots of unity in K and R_K is the regulator.

Unfortunately for our humble imaginary quadratic number field, this constant becomes w^{-1}. There is clearly no information about units here. This all means that the Stark conjectures must demand a first order zero for the partial zeta function in order to avoid this situation. This is essentially equivalent, as we have noted, to the condition that Card $S \geq 2$. Thus the Stark conjectures at $s = 0$ do not help us at all with our stated objective of evaluating eta functions.

We finish this subsection by noting that the situation we described in (2) is also like that of L-series of cyclotomic fields. That is, in both cases the terms occurring inside logarithms on the right hand side are not necessarily absolute values of S-units as in the Stark conjectures. Rather *quotients* of these expressions turn out to be units.

We quote the results from the cyclotomic case here for comparison (see [17] pp. 37)

Theorem 6.0.26. *For a Dirichlet character χ of conductor f we have*

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = -\frac{\tau(\chi)}{f} \sum_{a=1}^{f} \overline{\chi}(a) \log |1 - \zeta_f^a|; \quad \textit{where } \zeta_f = e^{2\pi/f},$$

and $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{f} \chi(a) e^{2\pi i a/f}.$$

It is also easy to demonstrate that in most cases, quotients of expressions such as $(1 - \zeta_f^a)$ are units (see for example [17] pp. 2, 12). We will revisit these ideas at a later stage.

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