Gradient flows for the harmonic map energy

Problem Sheet 3

Peter Topping July 2017.

3.1. In Lemma 3.3.1 it is claimed that if (M, g) is a closed Riemannian manifold and $h \in \Gamma(\mathrm{Sym}^2 T^*M)$, then there exist a vector field X on M and a section h_0 of $\mathrm{Sym}^2 T^*M$ that is divergence free, i.e. $\delta h_0 \equiv 0$, giving the L^2-orthogonal decomposition

$$h = h_0 + \mathcal{L}_X g.$$

Prove this, using the Fredholm alternative.
Remark: Note that divergence is defined by $\delta := -\mathrm{tr}\nabla$ for us.
Hint: This is analogous to the decomposition of a vector field H into the sum of a divergence-free vector field H_0 and the gradient of a function f. To prove that we would solve the PDE

$$\Delta f = \mathrm{div} H$$

and set $H_0 = H - \nabla f$. Try to mimic that.

3.2. Suppose that X is a Killing vector field on a closed n-dimensional Riemannian manifold (M, g) with strictly negative Ricci curvature, i.e. a solution to $\mathcal{L}_X g = 0$, (i.e. an infinitesimal isometry). Prove that $X \equiv 0$.
Hint: For such X, try computing the Bochner-type formula (valid for $\mathcal{L}_X g = 0$)

$$\Delta(\frac{1}{2}|X|^2) = |\nabla X|^2 - \mathrm{Ric}(X, X).$$

3.3. Suppose M is a closed oriented surface of genus $\gamma \geq 2$, and suppose g_0 is a hyperbolic metric on M. Suppose that we are given a point in Teichmüller space that is represented by a hyperbolic metric g. Define $u : (M, g_0) \to (M, g)$ to be the harmonic map homotopic to the identity that exists by the Eells-Sampson Theorem 2.2.1 (and is unique in this case, as we discussed). Now we consider the Hopf differential of this harmonic map; it is holomorphic by Lemma 1.6.1, so we have constructed a map that takes g to an element of $\mathcal{H}(M, g_0)$.
Prove that we have constructed a well-defined map from Teichmüller space $\mathcal{T}(M)$ to $\mathcal{H}(M, g_0)$, i.e. the map is independent of the representative g we took.
Beautiful Fact: (M.Wolf [39].) This map is a bijection, i.e. because $\mathcal{H}(M, g_0)$ is a vector space of complex dimension $3(\gamma - 1)$, we have a global chart for $\mathcal{T}(M)$.

3.4. Use the Gauss-Bonnet theorem to compute the area of any pair of pants.

3.5. What is the area of a genus $\gamma \geq 2$ closed hyperbolic surface? If such a surface is decomposed into pairs of pants, how many pairs of pants does one obtain?
Hint: You could use Gauss-Bonnet and the previous question.