Gradient flows for the harmonic map energy

Problem Sheet 1

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1.1. **Isothermal coordinates.** (Have a go at this if you have never seen isothermal coordinates before.) Given a Riemannian metric g on a surface M, and a point $p \in M$, we would like to find local coordinates x, y near p such that $g = \rho^2(dx^2 + dy^2)$ for some locally-defined positive scalar function ρ. In other words, we would like coordinates so that g has no $dx \otimes dy$ term when written in these coordinates. Our starting point for this is that we will *assume* that we can find a harmonic function x defined in a neighbourhood of p which vanishes at p and whose gradient at p is nonzero. (In fact, we can find such a function with prescribed gradient according to either the book [1] p.228, section 5.4, Theorem 1, or [14]

(a) Recalling that we can write the Laplacian of a function in terms of the Hodge star operator and the exterior derivative as $\Delta f = \pm * d * df$, prove that we can find a function y defined near p such that $dy = *dx$, which is then harmonic - the conjugate harmonic function.

(b) Prove that after possibly restricting to a smaller neighbourhood of p, the functions x, y are isothermal coordinates. (For another take on isothermal coordinates, see the book of Hélein.)

Remark: We see that isothermal coordinates are a special case of harmonic coordinates.

1.2. Prove that a C^1 map u from a surface into a Riemannian manifold satisfies $A(u) \leq E(u, g)$ (if the energy is finite) with equality if and only if u is weakly conformal.

1.3. (*Potentially very useful exercise for what's coming.*) Let T^2 be a two-dimensional torus equipped with an arbitrary metric (or arbitrary conformal structure). What is the infimum energy of a degree one map from T^2 to the standard round unit 2-sphere? Can this infimum be realised?

1.4. Suppose (M, g) is a Riemannian surface, where $g = \rho^2(dx^2 + dy^2)$, and we take the local complex coordinate $z := x + iy$. Prove that $\nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z} = 0$, while $\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \frac{2\rho_z}{\rho} \frac{\partial}{\partial z}$.
Show (as claimed in text) that $\partial dz = dz \otimes \nabla_{\frac{\partial}{\partial z}} dz = -\frac{2\rho_z}{\rho} dz^2$ and $\bar{\partial} dz = 0$.

1.5. Suppose the real part of a quadratic differential φdz^2 is zero. Prove that the whole quadratic differential is zero. (This is nothing to do with the quadratic differential being holomorphic or not!)

1.6. Compute the first variation formulae given in the lectures (Theorems 1.5.1 and 1.5.2).

1.7. Prove that any harmonic map from a surface has holomorphic Hopf differential.
Hint: Consider the first variation formula for $E(u, g)$ and plug in a variation that is nothing more than a modification by diffeomorphisms (and therefore does not change the energy).
Comment: This is an example of a symmetry giving a conservation law, cf. Noether's theorem.

1.8. Prove that any harmonic map from S^2 to any Riemannian manifold must be weakly conformal.
Hint: The lowest-tech way of seeing this is to consider the Hopf differential in stereographic charts centred at the north and south poles (i.e. centred at antipodal points).

1.9. **Open question:** Is it true that for every compact Riemannian manifold N and every continuous map $u_0 : S^2 \to N$, there exists a harmonic map u that is homotopic to u_0?
Comment: You can contrast this with Theorems 2.2.1 and 2.3.2.