Chapter 2

The harmonic map flow

2.1 Definition of the flow

The harmonic map flow was introduced by Eells-Sampson in 1964; their work could be considered the start of the field of geometric flows. The flow takes a map u between closed Riemannian manifolds (M, g) and (N, G) of any dimension, and deforms it to make its energy decrease as quickly as possible; more precisely it performs L^2 gradient flow. The first variation formula from Lemma 1.5.1 gives us the equation

$$\frac{\partial u}{\partial t} = \tau_g(u) \tag{2.1.1}$$

for the flow. For smooth solutions, the energy then decreases according to

$$\frac{dE}{dt} = -\int_M |\tau_g(u)|^2 d\mu_g.$$

2.2 Nonpositively curved targets: The theorem of Eells-Sampson

The original work of Eells-Sampson [10] proved that if we impose an extra condition on the target, then the flow behaves very well, and finds a harmonic map.

Theorem 2.2.1 ([10, 16]). *Suppose that (N, G) is a closed Riemannian manifold of nonpositive sectional curvature, and (M, g) is any closed Riemannian manifold, of any dimension. Then given a smooth map $u_0 : M \to N$, there exists a unique smooth solution $u : M \times [0, \infty) \to N$ to the harmonic map flow (2.1.1) with $u(0) = u_0$, and there exists a harmonic map $u_\infty : M \to N$ such that*

$$u(t) \to u_\infty \qquad \textit{in } C^k \textit{ for any } k \textit{ as } t \to \infty.$$

Here $u(t)$ is shorthand for $u(\cdot, t)$.

Remark 2.2.2. In fact, the theorem above incorporates improved asymptotics due to Hartman [16]. The original theorem proved convergence $u(t_i) \to u_\infty$ for some sequence of times $t_i \to \infty$.

This theorem has been highly influential. The general method gives a way of transforming a very general object – here an arbitrary map – into a very special one – here a harmonic map. Similar flows now exist for many other geometric objects. For example, instead of deforming maps, Ricci flow deforms Riemannian metrics and curve shortening flow deforms curves within some ambient Riemannian manifold. We might see both of these flows in these lectures, tangentially.

Remark 2.2.3. The theorem above gives a way of deforming an arbitrary map to a homotopic *harmonic* map, when the target is nonpositively curved. If the harmonic map u_∞ were the *unique* harmonic map in its homotopy class, then we would be able to precisely identify a homotopy class of maps by its harmonic representative. This uniqueness does not quite hold, because, for example, one could consider harmonic maps from S^1 into a flat square torus T^2 and easily construct families of harmonic maps with different images. (Recall that harmonic maps from S^1 are geodesics.) The situation improves if one imposes *strict* negative sectional curvature on the target. Still we have nonuniqueness, because one can consider constant maps, which are clearly harmonic, and which can be deformed as different constant maps. Alternatively, one can consider harmonic maps whose image is an S^1 (for example, any geodesic from S^1) and then rotate the map around this S^1, keeping the image fixed. However, a result of Hartman [16] tells us that these cases are the only way that uniqueness can fail in the strictly negatively curved target case. We will exploit this in Chapter 4 when we talk about the space of hyperbolic metrics g on a surface M, and it will be useful to represent homotopy classes of maps $S^1 \mapsto (M, g)$ by geodesics.

A key idea in the proof of Theorem 2.2.1 is to consider the evolution of the energy density $e(u) := \frac{1}{2}|du|^2$. Once we have observed that the harmonic map flow is a *strictly parabolic* partial differential equation, and will admit solutions over some, possibly short, time interval $[0, \varepsilon)$, we mainly have to worry about the flow developing a singularity in finite time. The theory of parabolic regularity theory (e.g. using theory in [18]) will tell us that a singularity can only stop us if the energy density becomes unbounded. However, the energy density itself satisfies a nice parabolic PDE, according to the parabolic Bochner formula. When we inspect that PDE, we see that $(\frac{\partial}{\partial t} - \Delta)e(u)$ can be bounded linearly in terms of $e(u)$ plus a term that could be of order $e(u)^2$. Now that could lead to a blow-up of $e(u)$ as surely as the ODE $\frac{da}{dt} = a^2$ will blow up in general. On the other hand, in the case that the sectional curvature of the target is nonpositive, we find that this 'bad' term is negative, and will work to *prevent* energy blow-up rather than the other way round. See Q. 2.2.

In the early days of the harmonic map flow, there was (anecdotally) a naive hope that the flow would *never* develop singularities, even if the target was allowed to break the hypothesis of nonpositive curvature. However, there are good geometric reasons why this could not be the case, as we will see.

2.3 Two dimensional domains: The theory of Struwe

After this brief interlude where we allowed the domain to have arbitrary dimension, we now return to the case that (M, g) is a closed surface. (For this part of the theory, it is not important that the surface is oriented.) The theory of the harmonic map flow is particularly elegant and well-developed in this case.

In order to be able to handle nonsmooth maps most easily, it will be convenient to embed the target manifold (N, g) isometrically in some Euclidean space $\mathbb{R}^n$, as we know we can by the Nash embedding theorem. This way, the energy could be written

$$E(u) := \frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 d\mu_g,$$

i.e. in terms of the sum of the Dirichlet energies of the n components u^i. The harmonic map flow can then be seen to be governed by the equation

$$\frac{\partial u}{\partial t} = (\Delta u)^T, \tag{2.3.1}$$

where the superscript T means the projection onto $T_{u(x)}N \subset \mathbb{R}^n$, and one can compute that this leads to the system of n PDE

$$\frac{\partial u}{\partial t} = \Delta u + g^{ij} A(u)\left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\right) \tag{2.3.2}$$

where $A(u)$ is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^n$. Taking this viewpoint makes it easier to consider *weak* solutions $u \in W^{1,2}(M \times [0,\infty), N)$ (that is, $u \in W^{1,2}(M \times [0,\infty), \mathbb{R}^n)$ with $u(x) \in N \hookrightarrow \mathbb{R}^n$ for almost all $x \in M$).

Note that $W^{1,2}$ represents the Sobolev space of weakly differentiable maps with just enough regularity – first derivatives in L^2 – to be able to make sense of the energy E. On the other hand, the topology of $W^{1,2}$ is just weak enough so $W^{1,2}$ convergence does not imply C^0 convergence.

The following theorem relies on the fact that the domain is two dimensional. In it, we denote by $u(t)$ the restriction $u(\cdot, t)$.

Theorem 2.3.1. *(Struwe [32].) Given a smooth map $u_0 : M \to N$, there exists a weak solution $u \in W^{1,2}_{loc}(M \times [0,\infty), N)$ to (2.3.2) which is smooth in $M \times [0,\infty)$ except possibly on a finite set of singular points $S_1 \subset M \times (0,\infty)$, and has the properties:*

1. *$u(0) = u_0$;*
2. *$E(u(t))$ is a (weakly) decreasing function of t on $[0,\infty)$;*
3. *If the flow is smooth for $t \in [0,T)$, then it is the unique smooth solution over this time interval with the given initial data;*
4. *There exist a sequence of times $t_i \to \infty$ and a smooth harmonic map $u_\infty : M \to N$ such that*
   - *(a) $u(t_i) \rightharpoonup u_\infty$ in $W^{1,2}(M,N)$, and*
   - *(b) $u(t_i) \to u_\infty$ in $W^{2,2}_{loc}(M\backslash S_2, N)$, and in particular in $C^0_{loc}(M\backslash S_2, N)$, where S_2 is a finite set of point in M.*

   as $i \to \infty$.

This theorem takes a general map u_0 and deforms it to a harmonic map u_∞, somewhat like in Theorem 2.2.1 of Eells-Sampson. Amongst the differences are that we allow the possibility of singularities on the finite sets S_1 and S_2. We cannot hope to do away with both of these sets because then the flow would homotop an arbitrary map to a harmonic map, and this is impossible in general:

Theorem 2.3.2 (Eells-Wood [11]). *There does not exist any harmonic map of degree 1 from a torus T^2 to a sphere S^2, whichever metrics we take on the domain and target.*

We can deduce from this that the flow organises itself in such a way as to jump homotopy class at one of the singular points in S_1 and/or S_2. We see how it goes about this in the next section.

2.4 Bubbling

At each of the finite-time singular points in S_1 and infinite time singular points in S_2 we get *bubbling*. You can discover for yourself what this is in Q. 1.3. (We can assume, without loss of generality, that any points in S_1 or S_2 that can be removed without breaking Theorem 2.3.1 have been removed. That is, all points in S_1 and S_2 are genuine singular points.)

The first thing to remark is that at each of these singular points we have energy concentrating in the following senses.

Lemma 2.4.1. *If $(x,T) \in S_1 \subset M \times (0,\infty)$ is a singular point, then energy concentrates in the sense that*

$$\lim_{\nu\downarrow 0} \limsup_{t\uparrow T} E(u(t), B_\nu(x)) \neq 0.$$

Similarly, if $x \in S_2 \subset M$ is a singular point, then

$$\lim_{\nu\downarrow 0} \limsup_{t\to\infty} E(u(t), B_\nu(x)) \neq 0.$$

To clarify, we are using here the notation $E(u,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2$. Also, $B_\nu(x)$ is the geodesic ball centred at x, of radius ν.

To see what is going on at a singularity, where energy is concentrating, we have to *rescale*. When we do, we find a 'bubble'.

Theorem 2.4.2. *Suppose $u \in W^{1,2}_{loc}(M \times [0,\infty), N)$ is a flow from Theorem 2.3.1. Then without loss of generality, the sequence of times t_i can be chosen so that the following is also true. For each singular point $x_0 \in S_2 \subset M$, when we view u as a map in local isothermal coordinates on the domain with x_0 corresponding to the origin of the coordinates, there exist sequences $a_i \to 0 \in \mathbb{R}^2$ and $\lambda_i \downarrow 0$ as $i \to \infty$, and a nonconstant harmonic map $\omega : S^2 \to N$ (which we view as a map $\mathbb{R}^2 \cup \{\infty\} \to N$) such that*

$$u(a_i + \lambda_i y, t_i) \to \omega(y),$$

as functions of y, in $W^{2,2}_{loc}(\mathbb{R}^2, N)$ as $i \to \infty$.

It is the map ω arising as a blow-up limit near the singularity that we call a bubble. We can also find these bubbles at finite time singularities:

Theorem 2.4.3. *(Slight variant of Struwe [32].) Suppose $u \in W^{1,2}_{loc}(M \times [0,\infty), N)$ is a flow from Theorem 2.3.1, and $T \in (0,\infty)$ is a singular time. Then there exist times $t_i \uparrow T$ with the property that*

$$\|\tau(u(t_i))\|^2_{L^2(M)}(T - t_i) \to 0 \tag{2.4.1}$$

as $i \to \infty$. Moreover, for every singular point $(x_0, T) \in S_1 \subset M \times (0, \infty)$ at time T, when we view u as a map in local isothermal coordinates on the domain with x_0 corresponding to the origin, there exist sequences $a_i \to 0 \in \mathbb{R}^2$ and $\lambda_i \downarrow 0$ with $\lambda_i (T - t_i)^{-\frac{1}{2}} \to 0$ as $i \to \infty$, and a nonconstant harmonic map $\omega : S^2 \to N$ (which we view as a map $\mathbb{R}^2 \cup \{\infty\} \to N$) such that

$$u(a_i + \lambda_i y, t_i) \to \omega(y),$$

as functions of y, in $W^{2,2}_{loc}(\mathbb{R}^2, N)$ as $i \to \infty$.

In fact, in principle, there may be many bubbles developing near each singular point (cf. [34]) i.e. we might be able to take different sequences a_i and λ_i which lead to convergence to a different bubble. (Different meaning not just the old bubble distorted a bit.) It follows from Theorems 2.3.1 and 2.4.2 that

$$\lim_{t \to \infty} E(u(t)) - E(u_\infty) \geq E(\omega), \tag{2.4.2}$$

where ω is the bubble extracted at infinite time, and similarly from Theorems 2.3.1 and 2.4.3 that

$$\lim_{t \uparrow T} E(u(t)) - E(u(T)) \geq E(\omega), \tag{2.4.3}$$

for ω the bubble extracted at finite time. However, if we capture *all* the bubbles at each singular point and replace the right-hand side with the sum of all their energies, then it is a result of Ding-Tian [8] that we would have equality in (2.4.2) and (2.4.3). For more details, and the most refined result (based on the work of Struwe [32], Ding-Tian [8], Qing-Tian [21] and Lin-Wang [19]) see [35].

Note that the behaviour of the flow at the singularity can still be quite bad. For example, the flow map $u(T)$ at time T might be discontinuous, despite being in $W^{1,2}(M, N)$, and it is sometimes possible to change the sequence of times $t_i \uparrow T$ and end up with a different set of bubbles, even with different images. See [35] for details of these constructions.

Both types of singularity – finite and infinite time – can actually occur in practice.

An example of Chang-Ding [5] says that we can find a degree zero map from S^2 to S^2 that flows smoothly for all time, and develops two bubbles at infinite time (one degree 1 and one degree -1) with the map u_∞ from Theorem 2.3.1 being constant.

[draw a picture!!!!!]

Meanwhile, an example of Chang-Ding-Ye [6] says that we can find a degree one map from S^2 to S^2 that develops a singularity in finite time, when two bubbles are created (one degree 1 and one degree -1) leaving the flow to continue smoothly forever afterwards and converge to the identity map.

[draw a picture!!!!!]

We can now return to Theorem 2.3.2 to consider what the flow does with a degree 1 map from T^2 to S^2. What can happen (and will definitely happen for maps with energy close to their minimum possible value in their homotopy class) is that the flow generates one bubble as a singularity, leaving a map homotopic to a constant map. What is unclear is whether that happens in finite time, leaving a degree zero map that then flows on and converges to a constant map, or whether that happens at infinite time, leaving a constant map u_∞. This is unknown to this day.

2.5 No bubbling for nonpositively curved targets

A significant consequence of the singularity analysis of the previous section is that if we are dealing with a target manifold (N, G) for which there do not exist any nonconstant harmonic maps $S^2 \mapsto (N, G)$, then we can be sure that there there are never any singularities in the flow, even at infinite time, i.e. both S_1 and S_2 are empty. One such situation is:

Lemma 2.5.1. *Every harmonic map $S^2 \to (N, G)$ into a Riemannian manifold of nonpositive sectional curvature is a constant map.*

You are asked the following proof in Q. 2.4.

Proof. By lifting to the universal cover, we may assume that (N, G) is simply connected. The squared distance function $d^2(y_0, \cdot)$ to any fixed point $y_0 \in N$ is a strictly convex function because of the nonpositive curvature [2]. We pick any y_0 that lies within the image of the harmonic map, so the composition $d^2(y_0, u(x))$, where u is the harmonic map, takes the value zero somewhere. But this composition can be computed to be subharmonic by virtue of being the composition of a harmonic map and convex function. Therefore the composition is constant, and therefore identically zero. □

Remark 2.5.2. More generally, any harmonic map from any closed Riemannian manifold into any Riemannian manifold supporting a convex function is necessarily constant [12].

Thus we recover the original Eells-Sampson theorem (see Theorem 2.2.1 and Remark 2.2.2) in the 2D case as a toy instance of this argument.

2.6 Finding branched minimal immersions with the harmonic map flow

We saw at the end of the previous chapter/lecture that to find branched minimal spheres, we only have to find nonconstant harmonic maps from S^2. We will now give an example of how the harmonic map flow can achieve this.

Theorem 2.6.1. *Suppose the (closed) target N satisfies $\pi_2(N) \neq \{0\}$. Then for any target metric G, there exists a branched minimal immersion $S^2 \mapsto (N, G)$.*

Proof. Choose any homotopically nontrivial smooth map $u_0 : S^2 \to N$. By Struwe's Theorem 2.3.1, we can run the harmonic map flow starting at u_0. Either the flow exists for all time and converges in C^0 to our desired branched minimal immersion u_∞, or there exists a singularity at finite and/or infinite time. But if there is a singularity, then we must have a bubble ω according to Lemmata 2.4.2 and 2.4.3, which would itself give the desired branched minimal immersion. □

Remark 2.6.2. You can contrast this theorem with Q. 1.9 where we ask that the homotopy class of the harmonic map is prescribed.

On the other hand, if $\pi_2(N) = \{0\}$...

Theorem 2.6.3. *If $\pi_2(N) = \{0\}$, then we can* homotop *an arbitrary map u_0 (from an arbitrary closed surface) into a harmonic map.*

Proof. Again, we apply Struwe's Theorem 2.3.1. The claim is simply that the asymptotic harmonic map u_∞ is homotopic to u_0. For example, it is true that any flow with isolated point singularities does not change its homotopy class as we pass through the singular time. □

This latter theorem does not help us directly find branched minimal immersions: If the domain is S^2, then we will always be able to homotop u_0 to a constant map by hypothesis, and this is harmonic but of no help. On the other hand, if the domain is not S^2 then the harmonic map we found in Theorem 2.6.3 need not be weakly conformal, and therefore not necessarily a branched minimal immersion. The next chapter will attempt to rectify that.