TWIST-SPUN KNOTS

1. Knots and links

2. Racks and quandles

3. Classifying spaces and cohomology

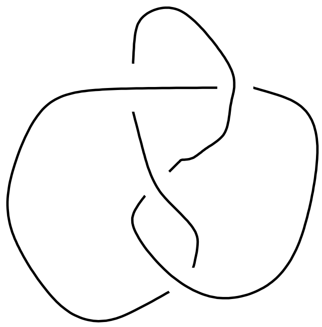
4. Twist-spun knots in $\mathbb{R}^4$

A (REALLY) QUICK TRIP THROUGH KNOT THEORY

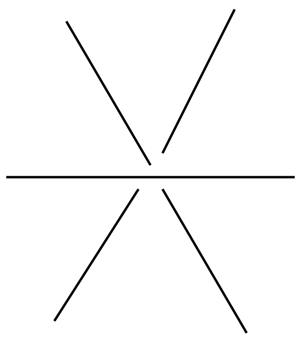
A **knot**: an embedding $S^1 \hookrightarrow S^3$.

A **link**: an embedding $\coprod S^1 \hookrightarrow S^3$.

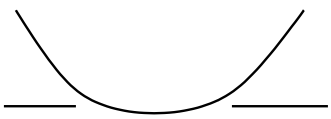
Depict knots and links by **diagrams**:



Don't allow triple points:



or non-transverse intersections:

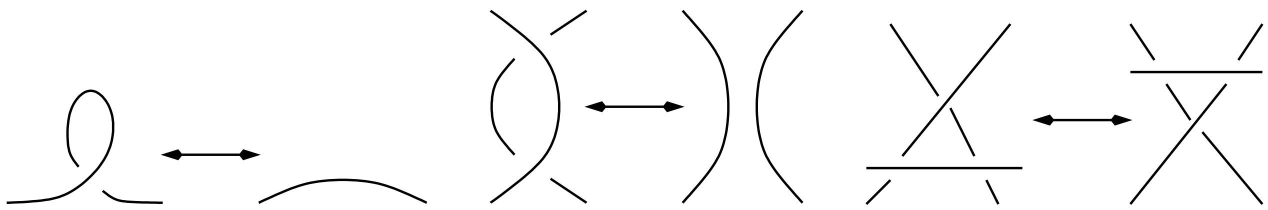


An **isotopy**: a homotopy of homeomorphisms.

regular isotopy: Don't allow arbitrary untwisting.

ambient isotopy: Do.

Reidemeister moves: Allowable alterations of diagrams. . .



Ω_1 Ω_2 Ω_3

THEOREM (Reidemeister 1932)

$$\left\{ \begin{array}{c} \text{ambient isotopy} \\ \text{classes of links} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{classes of diagrams} \\ \text{modulo } \Omega_1, \Omega_2, \Omega_3 \end{array} \right\}$$

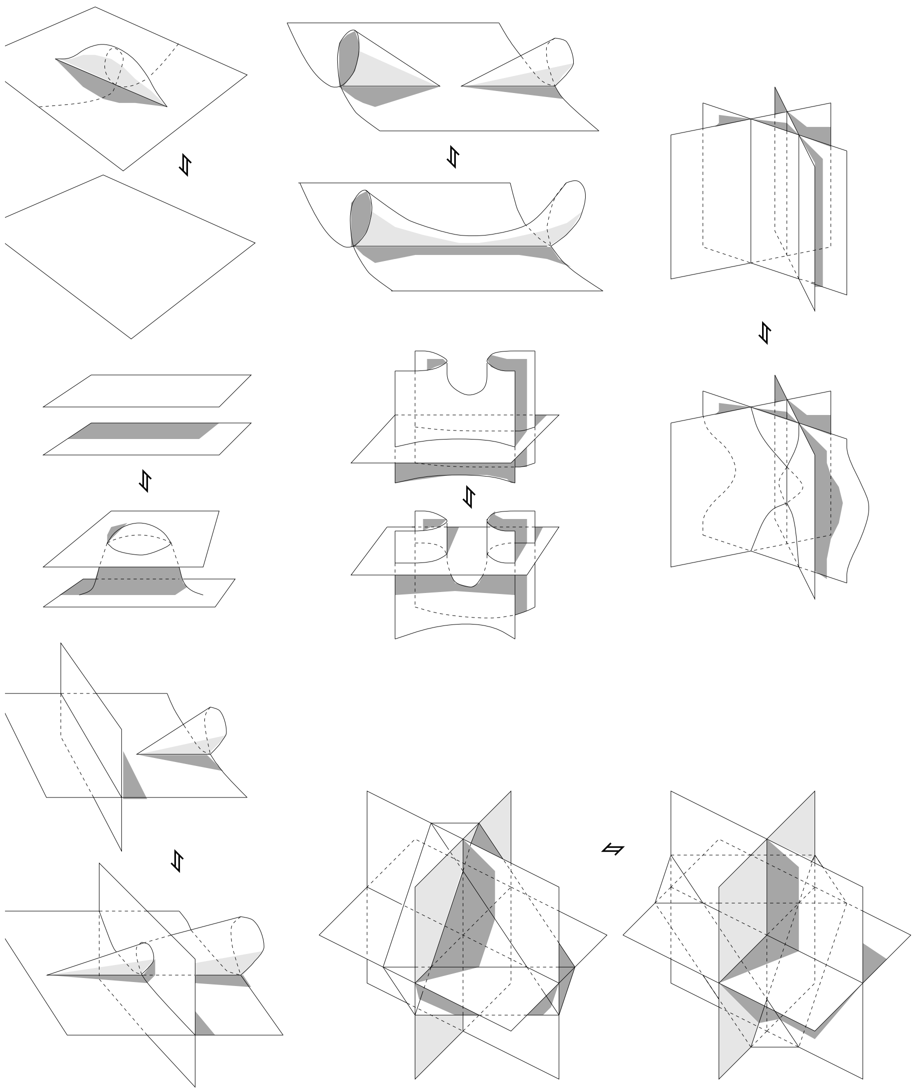
$$\left\{ \begin{array}{c} \text{regular isotopy} \\ \text{classes of links} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{classes of diagrams} \\ \text{modulo } \Omega_2, \Omega_3 \end{array} \right\}$$

KNOTTED SURFACES IN S^4

Can't knot a circle in 4-space, but can knot surfaces.

Study isotopy classes of embeddings $S^2 \hookrightarrow S^4$.

4-dimensional analogues of Reidemeister moves are the **Roseman moves**...



Racks and quandles

A **rack** is a set X, equipped with an asymmetric binary operation (usually written as exponentiation), such that:

R1 For any $a, b \in X$ there is a unique $c \in X$ such that $a = c^b$.

R2 $a^{bc} = a^{cb^c}$ for any $a, b, c \in X$.

A **quandle** is a rack which satisfies:

Q $a^a = a$ for all $a \in X$.

Originally studied by Conway and Wraith, later by Joyce, and recently by Fenn, Rourke and Sanderson.

EXAMPLES

The **trivial rack** T_n: rack structure defined on $\{0,\dots,n-1\}$ by setting $a^b := a$.

The **cyclic rack** C_n: rack structure defined on $\{0,\dots,n-1\}$ by setting $a^b := a+1 \pmod n$.

The **core rack** $\operatorname{Core} G$ of a group G:
Define $g^h := hg^{-1}h$.

The **conjugation rack** $\operatorname{Conj} G$ of a group G:
Define $g^h := hgh^{-1}$.

The **dihedral rack** $D_n = \{0,\dots,n-1\}$, with $p^q := 2q-p$. Generated by reflections in the dihedral group D_n.

Alexander quandles are modules over $\Lambda = \mathbb{Z}[t,t^{-1}]$ with rack structure given by $a^b := ta+(1-t)b$.

THE FUNDAMENTAL RACK Γ

Given a $(n-2)$-manifold L embedded in an n-manifold M, we can define the **fundamental rack** $\Gamma(L)$.

Say L is **framed** if there is a cross-section $\lambda : L \to \partial N(L)$ of the normal circle bundle.
Then $L^{+} = \text{im}(\lambda)$ is the **parallel manifold** to L.

Now let $\Gamma(L)$ consist of homotopy classes of paths in $M_0 = \overline{M \setminus N(L)}$, from a point in L^{+} to the basepoint $*$, where the initial point of the path is allowed to wander over L^{+}.

A point $p \in L^{+}$ lies in a unique meridional circle of $N(L)$ — define m_p to be the loop formed by following this circle in a positive direction.

Now, given two elements $a, b \in \Gamma(L)$, represented by paths α, β, respectively, we define

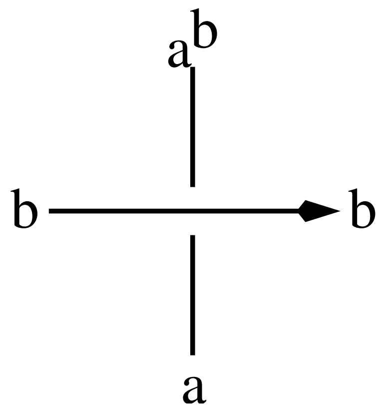
$$a^b := [\alpha \circ \overline{\beta} \circ m_{\beta(0)} \circ \beta]$$

THE FUNDAMENTAL QUANDLE Γ_q

The corresponding invariant of unframed codimension-2 embeddings is the **fundamental quandle** $\Gamma_q(L)$, in which the initial point of the path is allowed to wander all over $\partial N(L)$.

A PRESENTATION FOR $\Gamma(L)$

Attach a different label to each arc of a diagram for L. Each crossing point gives a relation as follows:



Then $\Gamma(L)$ is the free rack generated by all the labels, modulo all the crossing relations.

... AND $\Gamma_q(L)$

Do the same, but add in the relations $a^a = a$ for all $a \in X$.

COLOURING

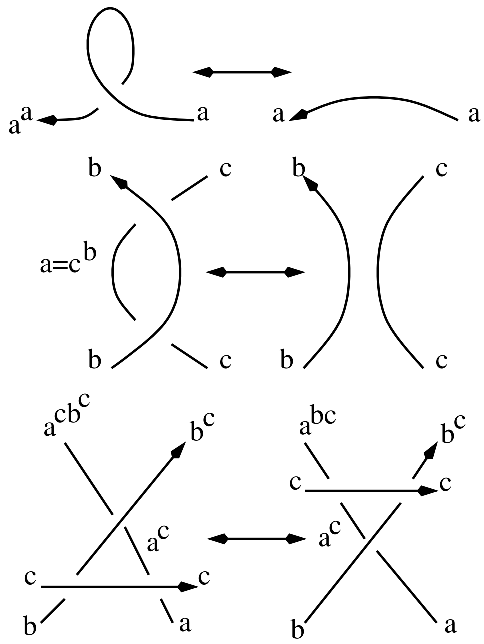
A **colouring** of a link L with a rack X is a consistent labelling of the arcs of L with elements of X, such that the required condition holds at each crossing.

In other words, a homomorphism $c : \Gamma(L) \to X$.

REIDEMEISTER RETURNS

The point of the definition of Γ and Γ_q is that the rack (and quandle) axioms correspond to the Reidemeister moves.

Γ can't see Ω_2 and Ω_3, and Γ_q can't see Ω_1 either.



R1: $a^a = a$

R2: $a = c^b$

R3: $a^{bc} = a^{cb^c}$

This works in higher dimensions, too.

THE RACK SPACE $\mathcal{B}X$

We can construct a classifying space $\mathcal{B}X$ for a rack X, analogous to the classifying space $\mathcal{B}G$ for a group G.

Briefly:

Take a 0-cell $*$.

Attach 1-cells $[x]$ for each element $x \in X$.

Then glue in 2-cells $[x, y]$ around the edges $[y][x^y][\overline{y}][\overline{x}]$.

Proceed by induction, gluing in n-cells $[x_1, \ldots, x_n]$ consistently.

Classifying spaces and (co)homology

Can define **cohomology** (and **homology**) of racks (and quandles), analogous to (co)homology of groups.

Given a rack X, and an abelian group A (written multiplicatively), form a **cochain complex** $C^*(X; A)$ as follows:

$$C^n(X; A) = \mathrm{Hom}(FA(X^n), A)$$

$$(\delta f)(x_0, \ldots, x_n) = \prod_{i=0}^{n} f(x_0^{x_i}, \ldots, x_{i-1}^{x_i}, x_{i+1}, \ldots, x_n)^{(-1)^i} \times \prod_{i=0}^{n} f(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^{-(-1)^i}$$

$$\cdots \xrightarrow{\delta^{n-1}} C^n(X; A) \xrightarrow{\delta^n} C^{n+1}(X; A) \xrightarrow{\delta^{n+1}} \cdots$$

A tedious calculation shows that $\delta^n \circ \delta^{n-1}$ is the trivial map, so this *is* a cochain complex.

Define $Z^n(X; A) = \ker(\delta^n)$ and $B^n(X; A) = \mathrm{im}(\delta^{n-1})$ (the groups of n-**cocycles** and n-**coboundaries**).

The cohomology $H^n(X;A) = Z^n(X;A)/B^n(X;A)$ is defined as per usual, and agrees with the (topological) cohomology of the rack space $\mathcal{B}X$:

$$H^n(C^*(X;A)) \cong H^n(\mathcal{B}X;A)$$

QUANDLE COHOMOLOGY

We are primarily interested in the cases where X is a quandle.

Consider a quotient $H_Q^*(X;A)$ of the rack cohomology:

$$P^n(X;A) = \{f \in C^n(X;A) \mid f(\vec{x}) = 1 \text{ for all } \vec{x} \text{ where } x_i = x_{i+1} \text{ for some } i\}$$

Define the groups of **quandle n-cocycles**, **quandle n-coboundaries**, and the **quandle cohomology** groups as follows:

$$\begin{aligned} Z_Q^n(X;A) &= Z^n(X;A) \cap P^n(X;A) \\ B_Q^n(X;A) &= B^n(X;A) \cap P^n(X;A) \\ H_Q^n(X;A) &= Z_Q^n(X;A)/B_Q^n(X;A) \end{aligned}$$

CHARACTERISTIC FUNCTIONS

The cohomology groups $H^n(X;A)$ and $H^n_Q(X;A)$ (with coefficients in $\mathbb{Z}, \mathbb{Z}_n$, or $\mathbb{Q}$) are generated by **characteristic functions**:

$$\chi_{\vec{x}}(\vec{y}) = \begin{cases} t & \text{if } \vec{x} = \vec{y} \\ 0 & \text{otherwise} \end{cases}$$

Here, t is the generator of the coefficient group, and $\vec{x}, \vec{y} \in X^n$.

This can be generalised to non-cyclic groups, but not today.

EXAMPLES

$H^2_Q(T_2;\mathbb{Z}) \cong \mathbb{Z}^2$, generated by $\chi_{(0,1)}$ and $\chi_{(1,0)}$.

$H^2_Q(D_3;\mathbb{Z})$ is trivial.

$H^3_Q(D_3;\mathbb{Z})$ is trivial, but $H^3_Q(D_3;\mathbb{Z}_3) \cong \mathbb{Z}_3$, generated by:

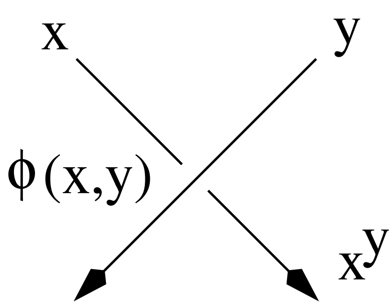
$$\begin{aligned} \eta \;=\; & \chi^{-1}_{(0,1,0)}\chi_{(0,2,0)}\chi^{-1}_{(0,2,1)} \\ & \chi_{(1,0,1)}\chi_{(1,0,2)}\chi_{(2,0,2)}\chi_{(2,1,2)} \end{aligned}$$

where $\eta^3 = 1$.

THE STATE SUM INVARIANT

Take a link diagram, and colour it with a given quandle X. That is, label each arc of the diagram consistently with elements of X.

Now take a cocycle $\phi \in H^2_Q(X; A)$. This may be applied to a crossing τ of the diagram by allowing it to act on the labels of the incoming arcs:



(Similarly, we can apply 3-cocycles to triple-points of knotted 2-spheres.)

Write the coefficient group A multiplicatively and then define the **state sum** $\Phi_\phi(L)$ corresponding to this cocycle as:

$$\sum_{\mathcal{C}} \prod_{\tau} \phi(x, y)^{\varepsilon(\tau)}$$

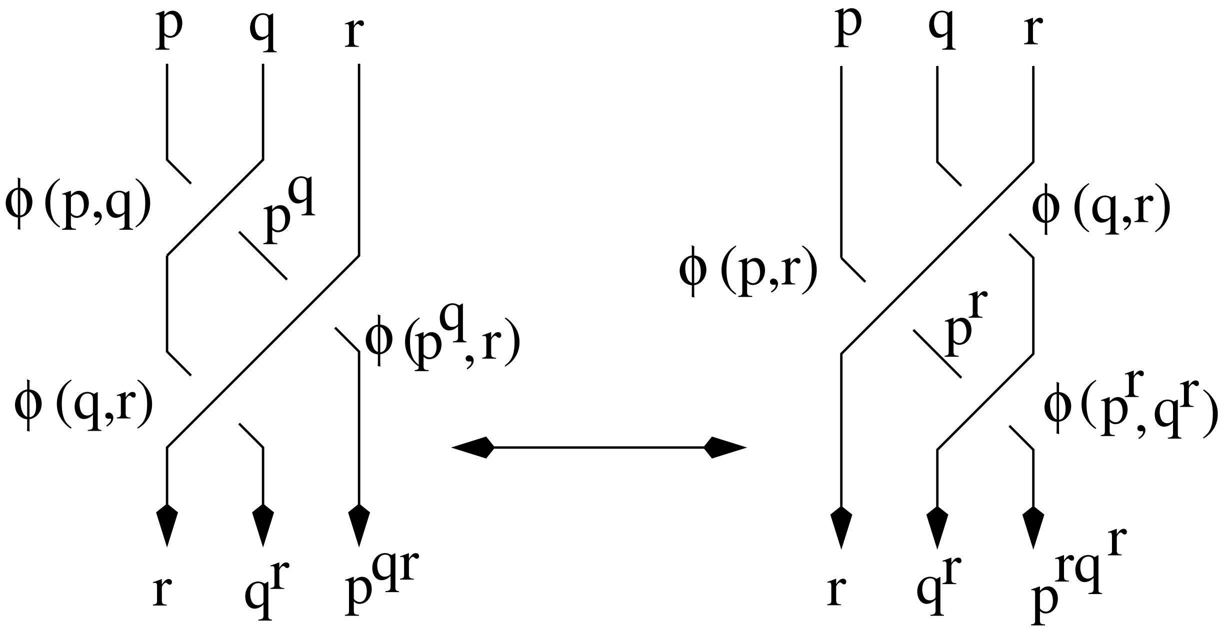
where $\varepsilon(\tau)$ is the sign of the crossing τ.

Note that this sum is taken over all colourings $\mathcal{C} : \Gamma(L) \to X$, and all crossings τ.

The definition of a 2-cocycle says that

$$\phi(p,r)\phi(p^r,q^r) = \phi(p,q)\phi(p^q,r)$$

This is equivalent to invariance of Φ_ϕ under Ω_3:



For a 3-cocycle,

$$\phi(p,q,r)\phi(p^r,q^r,s)\phi(p,r,s) = \\ \phi(p^q,r,s)\phi(p,q,s)\phi(p^s,q^s,r^s)$$

which is equivalent to the tetrahedral Roseman move.

The weird definition of *quandle* cocycles assures invariance under Ω_1, making Φ_ϕ an ambient isotopy invariant.

AN EXAMPLE

We can colour any knot or link with the two-element trivial quandle T_2 – the colour of an under-arc is unchanged at crossings.

Let $\phi = \chi_{(0,1)} \in H^2_Q(T_2;\mathbb{Z})$ to define a cocycle invariant Φ for classical links.

If K is a knot then $\Phi(K) = 2$.
If $L = K_1 \cup K_2$ is a two-component link, then

$$\Phi(L) = 2(1 + t^{\mathrm{lk}(L)})$$

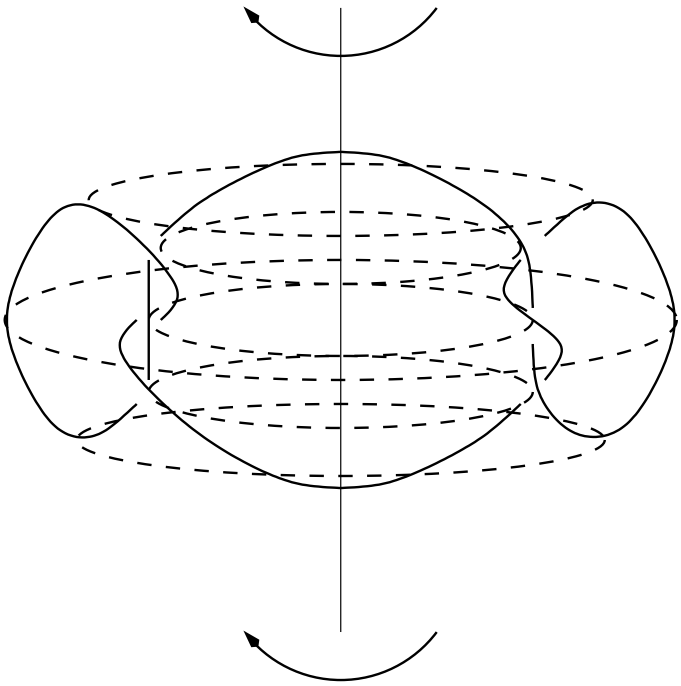
If $L = K_1 \cup K_2 \cup K_3$ is a three-component link then

$$\Phi(L) = 2(1 + \sum_{1 \leqslant i < j \leqslant 3} t^{\mathrm{lk}(L) - \mathrm{lk}(K_i, K_j)})$$

So cocycle invariants derived from trivial quandles would seem to give linking information.

SPUN KNOTS

Can create a class of (possibly knotted) embedded 2-spheres in S^4 by **spinning**:



TWIST-SPUN KNOTS

Rotate the knot (an integral number of times) during the spinning process.

THE TWICE TWIST-SPUN TREFOIL

Spin the trefoil, twisting it 720° as you do so.
This gives an orientable embedded 2-sphere S in 4-space.

Orient S and calculate the state sum invariant corresponding to the cocycle

$$\eta = \chi^{-1}_{(0,1,0)}\chi_{(0,2,0)}\chi^{-1}_{(0,2,1)} \chi_{(1,0,1)}\chi_{(1,0,2)}\chi_{(2,0,2)}\chi_{(2,1,2)}$$

in $H^3_Q(D_3;\mathbb{Z}_3)$.

Let S^+ be S with the positive orientation, and S^- be S with the negative orientation.

Then

$$\Phi_\eta(S^+) = 3 + 6t$$

but

$$\Phi_\eta(S^-) = 3 + 6t^2$$

So

$$S^+ \not\approx S^-$$

That is, the twice twist-spun trefoil is not ambient isotopic to its inverse.