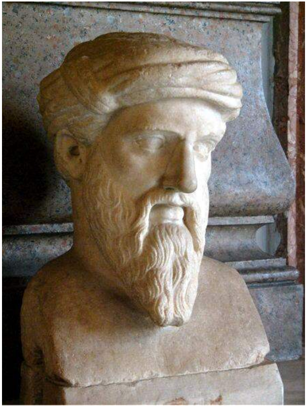
Unsolvable Problems

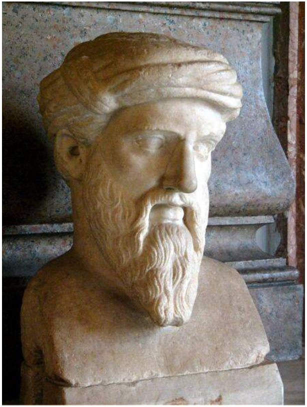
Olympus 2012

Nicholas Jackson

Easter 2012



- Samos c.570BC – Metapontum c.495BC
- Mathematician, philosopher, scientist, mystic



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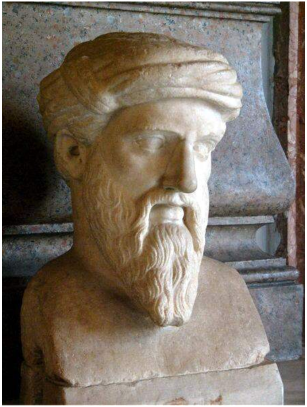

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- Wrote on the moon using blood and a mirror
- Believed that mathematics underpinned all reality

Rational numbers

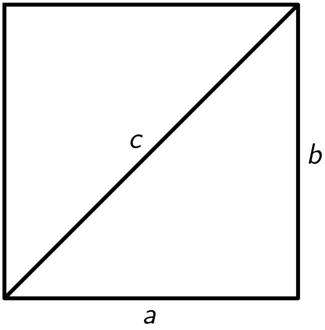
In particular, the Pythagoreans believed that rational numbers were the foundations of the universe.

Definition

A rational number is one that can be expressed as a quotient $\frac{p}{q}$ where

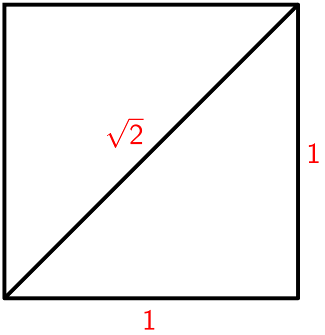
- p and q are both integers,
- $q \neq 0$, and
- p and q have no common factors.

Denote the set of rational numbers by $\mathbb{Q}$.



Pythagoras' Theorem says:

$$a^2 + b^2 = c^2$$



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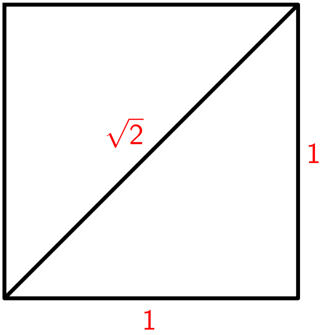
Or

$$c = \sqrt{1^2 + 1^2}.$$

The existence of $\sqrt{2}$



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$$a^2 + b^2 = c^2$$

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So $\sqrt{2}$ exists and, amongst other things, can be constructed with straightedge and compasses.

LEMMA

If n^2 is even, then so is n.

Lemma

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Proof.

If $n = 2m+1$ is odd, then so is

$$n^2 = (2m+1)^2 = 4m^2+4m+1 = 2(2m^2+2m)+1.$$

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If $n = 2m$ is even, then so is

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These are the only two possibilities, so n^2 is even only when n is. $\square$

Theorem (Hippasus of Metapontum?)

$\sqrt{2}$ is not a rational number.

Irrationality of $\sqrt{2}$

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Proof.

Suppose it is rational after all: $\sqrt{2} = \frac{p}{q}$, where p and q are integers, $q \neq 0$, and p and q have no common factors.

THEOREM (HIPPASUS OF METAPONTUM?)

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PROOF.

Suppose it is rational after all: $\sqrt{2} = \frac{p}{q}$, where p and q are integers, $q \neq 0$, and p and q have no common factors.

Then $\frac{p^2}{q^2} = 2$.

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Therefore, by the Lemma, p is even, so we can rewrite it as $p = 2r$.

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Therefore, by the Lemma, q is even.
So both p and q are even.
But this contradicts our original statement that p and q have no common factors.
So $\sqrt{2}$ can't be expressed as a rational number. $\square$

This is called proof by contradiction or *reductio ad absurdam*.

> *. . . and* reductio ad absurdum, *which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers* the game.
>
> G H Hardy, *A Mathematician's Apology*

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Algebraic irrational numbers

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$$x^2 - 2 = 0 \qquad x^2 - 3 = 0$$

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Call irrational numbers of this type algebraic.

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Doubling a Cube Construct, in finite time, using only compasses and straightedge, a cube with twice the volume of a given one.

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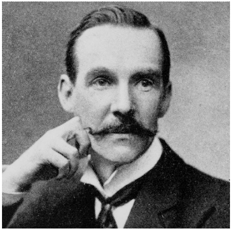
> **Meton** *With the straight ruler I set to work to inscribe a square within this circle; in its centre will be the market-place, into which all the straight streets will lead, converging to this centre like a star, which, although only orbicular, sends forth its rays in a straight line from all sides.*
>
> Aristophanes, *The Birds* (414BC)

Problem

Find a polynomial equation with rational coefficients which has π as a solution.

Historical accounts in:

- **Jean-Étienne Montucla**, *Histoire des Récherches sur la Quadrature du Cercle* (1754)
- **Ernest Hobson**, *Squaring the Circle: A History of the Problem* (1913)



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Phase 1 (– c.1650): Attempts at geometric construction

- Rhind Papyrus $\pi \approx \frac{256}{81} \approx 3.1605$
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Phase 2 (c.1650 – c.1750): Use of calculus to approximate π

- John Wallis (1616–1705): $\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots$
- Gottfried Leibniz (1646–1716): $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

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Phase 3 (c.1750 – c.1890): Detailed study of nature of π

- Joseph Liouville (1809–1882): Existence of non-algebraic (transcendental) numbers (1840)
- Charles Hermite (1822–1901): e is transcendental (1873)
- Ferdinand von Lindemann (1852–1939): π is transcendental (1882)

Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.

II Chronicles 4:2 ($\approx$ I Kings 7:23)

Theorem (Lindemann–Weierstrass)

If $x_1, \ldots, x_n$ are distinct real or complex algebraic numbers, and $p_1, \ldots, p_n$ are algebraic numbers, at least one of which is nonzero, then

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So π can't be algebraic, and must be transcendental. □

COROLLARY

No square can be constructed with the same area as a given circle.

Many enthusiastic amateurs attempted to square the circle, even after Lindemann's work.

- **William Myers**, *The Quadrature of the Circle, the Square Root of Two, and the Right-Angled Triangle* (1873)
- **Rufus Fuller**, *A Double Discovery: The Square of the Circle* (1893)
- . . . and many more.

I consider myself as having made my report, and being discharged from further attendance on the subject. I will not, from henceforward, talk to any squarer of the circle, trisector of the angle, duplicator of the cube, constructor of perpetual motion, subverter of gravitation, stagnator of the earth, builder of the universe, &c.

– Augustus de Morgan, *A Budget of Paradoxes*

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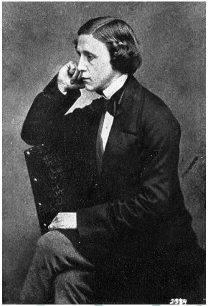
One of these – the chief one, I think – is the old old problem of 'Squaring the Circle', which has certainly wasted many a human life. Whether it has actually driven any one mad, I know not – most of its victims were, I fancy, partly crazed before they entered on the quest – but it clearly has the power of demolishing such slender reasoning powers as they may ever have chanced to possess. – Charles Dodgson, *Curiosa Mathematica I*



L'Académie a pris, cette année, la résolution de ne plus examiner aucune folution des problèmes de la duplication du cube, de la trifection de l'angle, ou de la quadrature du cercle, ni aucune machine annoncée comme un mouvement perpétuel.

– Histoire de l'Académie royale des sciences (1775) 61

Indiana House Bill 246 (1897)
A Bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the Legislature of 1897.

Section 1

Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is entirely wrong, as it represents the circle's area one and one-fifth times the area of a square whose perimeter is equal to the circumference of the circle. This is because one fifth of the diameter fails to be represented four times in the circle's circumference. For example: if we multiply the perimeter of a square by one-fourth of any line one-fifth greater than one side, we can in like manner make the square's area to appear one-fifth greater than the fact, as is done by taking the diameter for the linear unit instead of the quadrant of the circle's circumference.

Section 2

It is impossible to compute the area of a circle on the diameter as the linear unit without trespassing upon the area outside of the circle to the extent of including one-fifth more area than is contained within the circle's circumference, because the square on the diameter produces the side of a square which equals nine when the arc of ninety degrees equals eight. By taking the quadrant of the circle's circumference for the linear unit, we fulfill the requirements of both quadrature and rectification of the circle's circumference. Furthermore, it has revealed the ratio of the chord and arc of ninety degrees, which is as seven to eight, and also the ratio of the diagonal and one side of a square which is as ten to seven, disclosing the fourth important fact, that the ratio of the diameter and circumference is as five-fourths to four; and because of these facts and the further fact that the rule in present use fails to work both ways mathematically, it should be discarded as wholly wanting and misleading in its practical applications.

Section 3
In further proof of the value of the author's proposed contribution to education and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country. And be it remembered that these noted problems had been long since given up by scientific bodies as insolvable mysteries and above man's ability to comprehend.

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- Drafted by Dr Edward Johnston Goodwin (c.1828–1902), of Solicitude, Indiana.
- Persuaded his local representative Taylor I Record to sponsor the bill.
- Introduced in the Indiana House of Representatives on 18 Jan 1897.
- Referred to the Committee on Canals. . .
- . . . who referred it to the Committee on Education.
- Favourable report 2 Feb 1897.
- Second and Third Readings 5 Feb 1897, passed to Senate.
- Referred to Committee on Temperance 11 Feb 1897.
- Favourable report on 12 Feb 1897.

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A member then showed the writer a copy of the bill just passed and asked him if he would like an introduction to the learned doctor, its author. He declined the courtesy with thanks, remarking that he was acquainted with as many crazy people as he cared to know.

C A Waldo, *What Might Have Been*, Proc. Indiana Acad. Sci. 26 (1916) 445–446

E J Goodwin, *Quadrature of the circle*, Amer. Math. Monthly 1 (1894) 246–247

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All of these are wrong.

Goodwin also 'solved' the problems of trisecting the angle and doubling the cube:

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(B) Duplication of the Cube:
Doubling the dimensions of a cube octuples its contents, and doubling its contents increases its dimensions twenty-five plus one percent.

This is a close approximation: $\sqrt[3]{2} \approx 1.2599 \approx 1.26$.

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They can be solved using more sophisticated methods:

- Both can be solved using origami.
- Both can be solved using a marked ruler.
- Both can be solved using the neusis construction.
- An angle can be trisected using an infinite sequence of bisections:
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \cdots$$
- An angle can be trisected using a piece of string, a ‘tomahawk’, a linkage, etc.
- Menaechmus (380–320 BC) doubled the cube using conic sections.

Both problems proved unsolvable (with compasses and straightedge) in 1837 by Pierre Wantzel (1814–1848).

> *He worked usually during the evening, not going to bed until late at night, then reading, and only sleeping poorly for a few hours, alternately abusing coffee and opium; until he married, he took his meals at odd and irregular hours. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death.*
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> *A mathematician is a machine for turning coffee into theorems.*
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> – Paul Erdös

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$$\cos 60^\circ = \tfrac{1}{2} = 4\cos^3 20^\circ - 3\cos 20^\circ.$$

Write $x = 2\cos 20^\circ$, then this becomes

$$x^3 - 3x - 1 = 0.$$

THEOREM

Any complex number z which is constructible by compasses and straightedge from 0 and 1 must be algebraic of degree 2 over $\mathbb{Q}$.

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In other words, we need to find a quadratic polynomial

$$ax^2 + bx + c$$

where a, b and c are rational, and which is a factor of $x^3 - 3x - 1$.

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Equivalently, we need to find a linear factor

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That is, we need a rational root of $x^3 - 3x - 1$: a rational number $x = \frac{p}{q}$ such that

$$\left(\tfrac{p}{q}\right)^3 - 3\left(\tfrac{p}{q}\right) - 1 = 0.$$

Theorem (Rational Root Theorem)

Suppose

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0.$$

where $a_n, \ldots, a_0$ *are integers. Then any rational solution* $x = \frac{p}{q}$ *must satisfy:*

- p *is an integer factor of* a_0, *and*
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So $x^3 - 3x - 1$ doesn't have a rational root. And the angle $\frac{\pi}{3} = 60^\circ$ can't be trisected with compasses and straightedge.

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$$\begin{aligned} (1)^3 - 2 &= -1 & \qquad (-1)^3 - 2 &= -3 \\ (2)^3 - 2 &= 6 & \qquad (-2)^3 - 2 &= -10 \end{aligned}$$

So $x^3 - 2$ doesn't have a rational root, and hence the cube can't be doubled in volume using just compassess and straightedge.

This is an application of a branch of mathematics called Galois Theory, named after the French mathematician Évariste Galois (1811–1832).

- Studied advanced mathematics as a teenager.
- Failed entrance exam to the École Polytechnique, instead went to the École Normale (1828).
- Published four papers in 1829–1830, on continued fractions, number theory and solutions of polynomial equations.
- Expelled from the École Normale in 1831, arrested and imprisoned for republican activism.
- Died following a pistol duel in May 1832.