Notes on braid groups

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Abstract

This is an expository dissertation on some aspects of the study of braid groups. It is an expanded (and tidier) form of the notes for a graduate seminar I gave at the University of Warwick Mathematics Institute during the Spring Term of the 1997/98 academic year. This document is the 'director's cut', in the sense that it explores the topics in more detail than the original talk, during which there wasn't enough time to describe the representation theory of the braid groups.

It is based heavily on the books of Birman [2] and Hansen [5].

In the first chapter, we consider the geometric properties of braids, and some of their connections to knots and links. The second chapter is concerned with various descriptions of the braid group and some of its generalisations. The third chapter investigates the representation theory of the braid groups.

Chapter 1

Braids and Links

In this chapter we investigate some of the geometric properties of braids, looking, in particular, at the connections between braids and links.

1.1 Geometric braids

Consider Euclidean 3-space, denoted $\mathcal{E}^3$, and let $\mathcal{E}_0^2$ and $\mathcal{E}_1^2$ be the two parallel planes with z-coordinates 0 and 1 respectively. Let P_i and Q_i (where $1 \leqslant i \leqslant n$) be the points with coordinates $(i, 0, 1)$ and $(i, 0, 0)$ respectively, so that $P_1, \ldots, P_n$ lie on the line $y = 0$ in the upper plane, and $Q_1, \ldots, Q_n$ lie on the lint $y = 0$ in the lower plane.

A **braid** on n strings (often called an n**-braid**) consists of a system of n arcs $a_1, \ldots, a_n$ (the **strings** of the braid), such that a_i connects the point P_i in the upper plane to the point $Q_{\pi(i)}$ in the lower plane, for some permutation $\pi \in \mathrm{Sym}_n$. Furthermore:

(i) Each arc a_i intersects the plane $z = t$ once and once only, for any $t \in [0, 1]$.

(ii) The arcs $a_1, \ldots, a_n$ intersect the plane $z = t$ in n distinct points for all $t \in [0, 1]$.

In other words, an n-braid β consists of n strands which cross each other a finite number of times, do not intersect with themselves or any of the other strands, and travel strictly downwards, as depicted in figure 1.1.

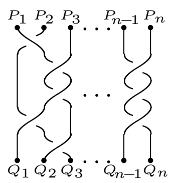


Figure 1.1: A geometric n-braid

The permutation π is called the **permutation** of the braid. If this permutation is trivial then β is said to be a **pure** (or **coloured**) braid.

Two braids β_0 and β_1 with the same permutation π are said to be **equivalent** or **homotopic** if there is a homotopy through braids β_t each with permutation π, where $t \in [0,1]$, from β_0 to β_1.

Alternatively, β_0 and β_1 are equivalent if there exists an ambient isotopy of $\mathcal{E}^3$ from β_0 to β_1 which fixes $\mathcal{E}_0^2$ and $\mathcal{E}_1^2$.

1.2 Closed braids, links and Markov's theorem

It transpires that there is a fundamental connection between braids and links. Define the **closure** $\widehat{\beta}$ of a braid β by identifying each of the points P_i in the plane $\mathcal{E}_1^2$ with the corresponding point Q_i in $\mathcal{E}_0^2$ — this is equivalent to joining the points P_i and Q_i by a series of concentric arcs as shown in figure 1.2. The closed braid $\widehat{\beta}$ is then said to be closed with respect to the axis ℓ.

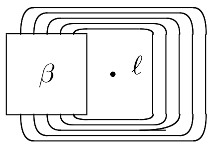


Figure 1.2: The closure $\widehat{\beta}$ of the braid β, with respect to the axis ℓ

Theorem 1.1 (Alexander 1923)
Every link is isotopic to a closed braid.

A stronger version of this result is due to Markov, and shows how two closed braid representatives of a given link are related to one another.

Let moves of type $\mathcal{R}$ and $\mathcal{W}$ be, as indicated in figures 1.2 and 1.2, the replacement of a segment of a closed braid $\widehat{\beta}$ by (respectively) two or three edges.

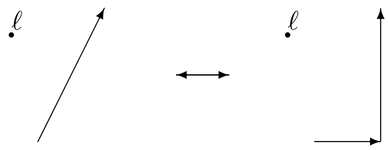


Figure 1.3: A Markov $\mathcal{R}$-move

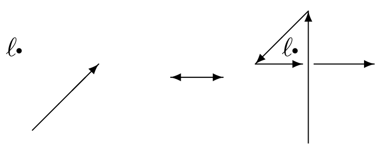


Figure 1.4: A Markov $\mathcal{W}$-move

Theorem 1.2 (Markov 1935)
If $\widehat{\alpha}$ and $\widehat{\gamma}$ are two isotopic closed braids, then there exists a finite sequence of closed braids

$$\widehat{\alpha} = \widehat{\beta_0}, \ldots, \widehat{\beta_s} = \widehat{\gamma}$$

such that each $\widehat{\beta_i}$ differs from $\widehat{\beta_{i-1}}$ by a single move of type $\mathcal{R}$ or type $\mathcal{W}$ or their inverses.

A proof of this is due to Birman [2].

A refinement of this result — a one-move version of Markov's theorem — was proved in 1997 by Rourke and Lambropoulou [7]. This theorem makes use of an operation called an $\mathcal{L}$-move, depicted in figure 1.2, where a string is cut and two more vertical strands are attached to the ends, passing either over or under all the other strands of the braid. There are thus two different types of $\mathcal{L}$-move, $\mathcal{L}_O$ and $\mathcal{L}_U$, but these are essentially the same operation.

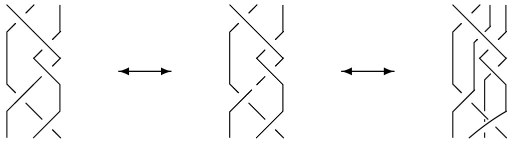


Figure 1.5: An $\mathcal{L}_U$-move on a braid

Chapter 2

Braid groups

2.1 Geometric braid groups

Given two n-braids, α and β, there is an obvious way of combining them to form a third n-braid $\alpha\beta$: attach β to the end of α as depicted in figure 2.1. This operation is called **composition**, and defines a group structure on the set of n-braids. The identity element is the braid formed from n parallel strands with no crossings, and the inverse β^{-1} of a braid β is formed by reflecting β in a horizontal plane.

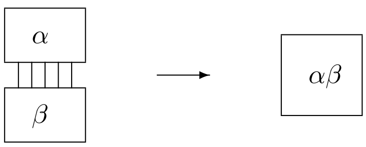


Figure 2.1: Composition of braids

This group, the n-string **braid group** is denoted by $\mathcal{B}_{\mathbf{n}}$. The subgroup $\mathcal{PB}_{\mathbf{n}}$ of $\mathcal{B}_n$ formed from braids with trivial permutation is the **pure** (or **coloured**) **braid group**.

2.2 Artin's presentation

Notice that any n-braid can be formed by a finite number of **elementary** braids $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i corresponds to the geometric n-braid formed by crossing the ith string over the $(i+1)$th string, as depicted in figure 2.2.

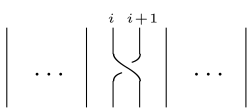
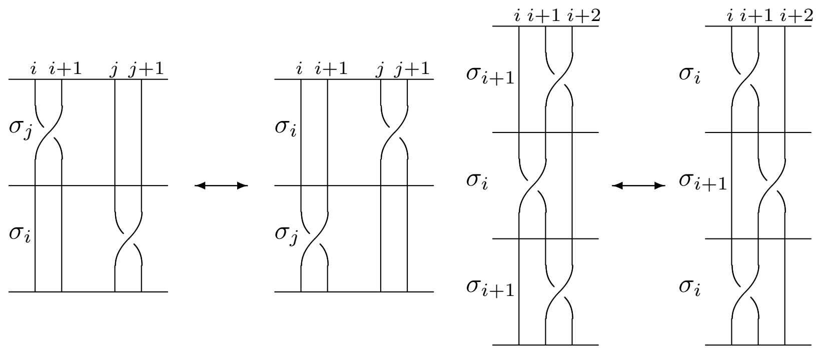


Figure 2.2: The elementary braid σ_i

We then notice that if i and j differ by more than one, then the elementary braids σ_i and σ_j

commute.

Furthermore, there is an analogue for braids of the third Reidemeister move for knots and links which, written in terms of the elementary braids, becomes $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.



$\sigma_i\sigma_j = \sigma_j\sigma_i$ $\qquad$ $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$

Figure 2.3: Relations in the elementary braids

The following theorem, due to Emil Artin, says that these two relations are sufficient to describe the n-string braid group:

Theorem 2.1 (Artin)
$\mathcal{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$

In other words, the n-string braid group is generated by generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to relations

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$.

(ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leqslant i \leqslant n-2$.

The pure braid group, $\mathcal{PB}_n$, can be considered as the subgroup of $\mathcal{B}_n$ consisting of braids which induce the identity permutation:

Theorem 2.2
The pure braid group $\mathcal{PB}_n$ has a presentation with generators:

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\ldots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\ldots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

where $1 \leqslant i < j \leqslant n$, and relations:

$$A_{rs}A_{ij}A_{rs}^{-1} = \begin{cases} A_{ij} & s < i \text{ or } j < r \\ A_{is}^{-1}A_{ij}A_{is} & i < j = r < s \\ A_{ij}^{-1}A_{ir}^{-1}A_{ij}A_{ir}A_{ij} & i < r < j = s \\ A_{is}^{-1}A_{ir}^{-1}A_{is}A_{ir}A_{ij}A_{ir}^{-1}A_{is}^{-1}A_{ir}A_{is} & i < r < j < s \end{cases}$$

The generators A_{ij} may be depicted geometrically as (pure) braids where the jth string passes behind the strings $(j-1), \ldots, (i+1)$, in front of the ith string and then behind the strings $i, \ldots, (j-1)$ back to the jth position.

2.3 Configuration spaces

We now look at another definition of these groups which leads to the study of a whole family of related groups.

Let M be a connected manifold of dimension two or greater, and let n be a positive integer. Now define $\mathcal{F}_n(M) := \{(x_1, \dots, x_n) \in M \times \dots \times M | x_i \neq x_j \text{ for } i \neq j\}$.

We can regard $\mathcal{F}_n(M)$ as a topological space by giving it the topology induced by the product topology on $M \times \dots \times M$, and since $\dim M \geqslant 2$ it is connected, and hence the homotopy groups $\pi_i(\mathcal{F}_n(M))$ are independent of the choice of basepoint.

This space $\mathcal{F}_n(M)$ is the **configuration space** of a set of n *ordered* points in M.

Now let $Q_m = \{q_1, \dots, q_m\}$ be a set of m fixed, pairwise different points in M (define Q_0 to be the empty set $\emptyset$), and define $\mathcal{F}_{m,n}(M) := \mathcal{F}_n(M \setminus Q_m)$.

With a moment's thought we see that $\mathcal{F}_{0,n}(M) = \mathcal{F}_n(M)$ and $\mathcal{F}_{m,1}(M) = M \setminus Q_m$.

The following theorem is due to Fadell and Neuwirth [3]:

Theorem 2.3 (Fadell/Neuwirth 1962)
Let n and r be integers such that $n \geqslant 2$ and $1 \leqslant r < n$. Then the canonical projection $p : \mathcal{F}_{m,n}(M) \to \mathcal{F}_{m,r}(M)$ mapping $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r)$ is a locally trivial fibration with fibre $\mathcal{F}_{m+r,n-r}(M)$.

There is an obvious right-action of the symmetric group Sym_n on the configuration space $\mathcal{F}_n(M)$, defined by permuting the coordinates of a point in $\mathcal{F}_n(M)$:

$\mu : \mathcal{F}_n(M) \times \mathrm{Sym}_n \to \mathcal{F}_n(M)$ with $((x_1, \dots, x_n), \sigma) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

We can think of the quotient space $\mathcal{C}_n(M) := \mathcal{F}_n(M)/\,\mathrm{Sym}_n$ defined by this action as being formed from $\mathcal{F}_n(M)$ by identifying all points whose coordinates differ by a permutation. This is the **configuration space** of a set of n *unordered* points in M.

2.4 Braid groups of manifolds

To see what all this has to do with braid groups, think about the fundamental groups of the configuration spaces $\mathcal{F}_n$ and $\mathcal{C}_n$.

The following result (due to Joan Birman[1]) suggests that the only really interesting cases of this question arise when M is a 2-manifold:

Theorem 2.4
Let M be a closed, smooth m-manifold. Then, for each $k \in \mathbb{Z}$, the inclusion map

$$i : \mathcal{F}_n(M) \hookrightarrow \prod_n M$$

induces a homomorphism

$$i_* : \pi_k(\mathcal{F}_n(M)) \to \prod_n \pi_k(M)$$

which is surjective if $\dim M > k$ and an isomorphism if $\dim M > k + 1$.

This means that, unless M is a 2-manifold, the fundamental group of $\mathcal{F}_n(M)$ is just a direct product of n copies of the fundamental group of the manifold M itself.

2.4.1 The braid group of the Euclidean plane $\mathcal{E}^2$

First of all, we consider the case where M is the Euclidean plane.

A corollary to theorem 2.3 states that:

Corollary 2.5 (Fadell/Neuwirth 1962)
$\pi_i(\mathcal{F}_{m,n}(\mathcal{E}^2)) = 0$ and, in particular, $\pi_i(\mathcal{F}_n(\mathcal{E}^2)) = 0$ for $i \geqslant 2$.

Which leads us to wonder what $\pi_1(\mathcal{F}_n(\mathcal{E}^2))$ is like, and whether it has any interesting or familiar structure.

The fundamental group $\pi_1(\mathcal{F}_n(\mathcal{E}^2))$ is generated by homotopy classes of loops in the (unordered) configuration space of the plane.

What does a closed loop in $\mathcal{F}_n(\mathcal{E}^2)$ look like? Well, a point in $\mathcal{F}_n(\mathcal{E}^2)$ is a set of n different, non-coincident points in the plane, so a closed loop in the ordered configuration space can be regarded as a kind of 'dance', in which the chosen n points move, smoothly, around each other, ending up back where they started.

If we look at a dance of this kind in the xy plane, with the z-axis depicting time, then we see that it is exactly a pure braid (since the points are ordered and distinct, and must therefore return to their starting positions). A homotopy class of this loop in the configuration space corresponds to an isotopy class of the related pure braid, and so the fundamental group $\pi_1(\mathcal{F}_n(\mathcal{E}^2))$ is isomorphic to the pure braid group $\mathcal{PB}_n$.

If, now, we replace $\mathcal{F}_n(\mathcal{E}^2)$ by $\mathcal{C}_n(\mathcal{E}^2)$, then without too much additional thought it should be clear that a closed loop in the unordered configuration space corresponds to a colourless (non-pure) braid, and so the fundamental group $\pi_1(\mathcal{C}_n(\mathcal{E}^2))$ is isomorphic to the usual braid group $\mathcal{B}_n$.

Now, let $a_2 = \sigma_1^2, a_3 = \sigma_1^{-1}\sigma_2^2\sigma_1, a_4 = \sigma_1^{-1}\sigma_2^{-1}\sigma_3^2\sigma_2\sigma_1, \ldots$

Geometrically, a_i is the pure braid formed by allowing the first strand to pass behind the second through to the $(i-1)^{\text{th}}$ strings, round the i^{th} strand, and back again.

So, $A_n(\mathcal{E}^2)$, the subgroup of $\mathcal{PB}_n(\mathcal{E}^2)$ generated by the braids $a_2, \ldots, a_n$, is isomorphic to the fundamental group of $\mathcal{F}_{n-1,1}(\mathcal{E}^2)$, and can be regarded as the group of pure braids in which only the first string does anything.

The group $D_n(\mathcal{E}^2)$, generated by $a_2, \ldots, a_n$ and $\sigma_2, \ldots, \sigma_{n-1}$, is the group of all braids which don't permute the first strand, that is, all braids with permutation π such that $\pi(1) = 1$.

We may now construct an exact sequence:

$$0 \longrightarrow A_n(\mathcal{E}^2) \xrightarrow{i} D_n(\mathcal{E}^2) \xrightarrow{j} \mathcal{B}_{n-1}(\mathcal{E}^2) \longrightarrow 0$$

In this sequence, i is the obvious inclusion homomorphism, and j, the **Chow homomorphism**, annihilates $A_n(\mathcal{E}^2)$. Geometrically, j can be regarded as the operation of removing and unthreading the first strand from the braid, thus leaving a braid on $n-1$ strings.

Now consider the homomorphism $\alpha : \mathcal{B}_n(\mathcal{E}^2) \to \mathrm{Sym}_n$, where each braid is mapped to its permutation, the elementary braid σ_i being mapped to the transposition $(i, i+1)$. Clearly $\ker \alpha = \mathcal{PB}_n(\mathcal{E}^2)$. Identify Sym_{n-1} with the subgroup of Sym_n leaving the first symbol fixed (that is, all permutations π such that $\pi(1) = 1$). Then $D_n(\mathcal{E}^2)$ is the preimage $\alpha^{-1}(\mathrm{Sym}_{n-1})$, and the diagram:

$$\begin{tikzcd} D_n(\mathcal{E}^2) \arrow[r, "j"] \arrow[dr, "\alpha"'] & \mathcal{B}_{n-1}(\mathcal{E}^2) \arrow[d, "\alpha"] \\ & \mathrm{Sym}_{n-1} \end{tikzcd}$$

commutes, and...

Proposition 2.6
The sequence

$$0 \longrightarrow A_n(\mathcal{E}^2) \xrightarrow{i} \mathcal{PB}_n(\mathcal{E}^2) \xrightarrow{j} \mathcal{PB}_{n-1}(\mathcal{E}^2) \longrightarrow 0$$

is exact. This is the **fundamental exact sequence** for $\mathcal{B}_n(\mathcal{E}^2)$.

In fact, more generally, if M is a closed, orientable 2-manifold, and *either* $n \geqslant 4$, *or* $n \geqslant 2$ and M is neither $\mathcal{P}^2$ nor $\mathcal{S}^2$, then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\mathcal{F}_{n-1,1}(M)) & \longrightarrow & \pi_1(\mathcal{F}_n(M)) & \longrightarrow & \pi_1(\mathcal{F}_{n-1}(M)) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & A_n(M) & & \mathcal{PB}_n(M) & & \mathcal{PB}_{n-1}(M) & & \end{array}$$

2.4.2 The braid group of the 2-sphere $\mathcal{S}^2$

The braid group of the 2-sphere is similar to the braid group of the Euclidean plane, except that the points move on $\mathcal{S}^2$ instead. An $\mathcal{S}^2$-braid may be depicted geometrically as a braid between two concentric spheres.

The group $\mathcal{B}_n(\mathcal{S}^2)$ is generated by the same generators σ_i and relations as $\mathcal{B}_n(\mathcal{E}^2)$, but with one additional relation:

(iii) $\sigma_1\sigma_2 \ldots \sigma_{n-1}\sigma_{n-1} \ldots \sigma_2\sigma_1 = 1$

This requirement says, geometrically, that the braid formed by taking the first string round behind all of the other strings and back in front of them, back to its starting position, is equivalent to the trivial braid.

By considering the geometric depiction of an $\mathcal{S}^2$-braid described above, we see that this is true, since the loop may be pushed off the inner sphere without tangling with any of the other strings.

As before, we can construct a fundamental exact sequence for $\mathcal{B}_n(\mathcal{S}^2)$:

$$0 \longrightarrow A_n(\mathcal{S}^2) \xrightarrow{i} \mathcal{PB}_n(\mathcal{S}^2) \xrightarrow{j} \mathcal{PB}_{n-1}(\mathcal{S}^2) \longrightarrow 0$$

The remark at the end of the previous subsection suggests that the braid groups of the 2-sphere and the projective plane might have some strange properties not shared by the braid groups of arbitrary 2-manifolds. This is further suggested by the following:

Theorem 2.7 (Newwirth)
If M is either $\mathcal{E}^2$ or any compact 2-manifold except $\mathcal{P}^2$ or $\mathcal{S}^2$ then neither $\mathcal{B}_n(M)$ nor $\mathcal{PB}_n(M)$ have any nontrivial elements of finite order.

So, is $\mathcal{B}_n(\mathcal{S}^n)$ torsion-free? Or can we find a nontrivial element of finite order?

Theorem 2.8 (Fadell/Newwirth 1962)
The word $\sigma_1\sigma_2\ldots\sigma_{n-1}$ has order $2n$ in $\mathcal{B}_n(\mathcal{S}^2)$.

This can be seen geometrically, with a little imagination. The word $\sigma_1\sigma_2\ldots\sigma_{n-1}$ corresponds to taking the first string over all the others to the nth position. If we do this n times, then each of the strings ends up back where it started, making a pure braid. If we then do the same thing a further n times (making $2n$ in total), each string winds round the remaining $n-1$ strings twice. We may then utilise a move known as the 'Dirac string trick' (qv [5] for a series of diagrams depicting this operation) to untangle all n strings, resulting in a trivial braid.

What are some of these groups $\mathcal{B}_n(\mathcal{S}^2)$ like? Notice that $\mathcal{B}_n(\mathcal{E}^2)$ is infinite for $n > 1$, but the previous theorem suggests that this might not necessarily be the case for the braid groups of the 2-sphere.

In fact:

$\mathcal{PB}_2(\mathcal{S}^2)$	$= 0$
$\mathcal{B}_2(\mathcal{S}^2)$	$= \mathbb{Z}_2$
$\mathcal{PB}_3(\mathcal{S}^2)$	$= \mathbb{Z}_2$
$\mathcal{B}_3(\mathcal{S}^2)$	is a ZS-metacyclic group of order 12

2.4.3 The braid group of the projective plane $\mathcal{P}^2$

We now consider the braid group of the projective plane. Recall that $\mathcal{P}^2$ is the 2-disc $\mathcal{D}^2$ with antipodal boundary points identified. It is not embeddable in $\mathbb{R}^3$ and is hence not particularly easy to visualise.

The group $\mathcal{B}_n(\mathcal{P}^2)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ (as for the braid groups of the plane and the 2-sphere), and $\rho_1, \ldots, \rho_n$, subject to the same relations as for $\mathcal{B}_n(\mathcal{E}^2)$, with the following additional relations:

(iii) $\sigma_i\rho_j = \rho_j\sigma_i$ if $|i-j| > 1$

(iv) $\rho_i = \sigma_i\rho_{i+1}\sigma_i$

(v) $\rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i = \sigma_i^2$

(vi) $\rho_1^2 = \sigma_1\sigma_2\ldots\sigma_{n-1}\sigma_{n-1}\ldots\sigma_2\sigma_1$

These generators each have a geometric interpretation — the σ_i may be regarded as the ith string passing in front of the $(i+1)$th string, and ρ_i may be depicted as the ith string moving forwards towards the boundary of $\mathcal{D}^2$ and then reappearing at the corresponding antipodal point before returning to its starting position.

As before, we get a fundamental exact sequence:

$$0 \longrightarrow A_n(\mathcal{P}^2) \xrightarrow{i} \mathcal{PB}_n(\mathcal{P}^2) \xrightarrow{j} \mathcal{PB}_{n-1}(\mathcal{P}^2) \longrightarrow 0$$

It transpires, also, that $\mathcal{B}_n(\mathcal{P}^2)$ has a torsion element:

Theorem 2.9 (van Buskirk 1966)
The word $\sigma_1 \dots \sigma_{n-1}$ has order $2n$ in $\mathcal{B}_n(\mathcal{P}^2)$

In addition, there are nontrivial braid groups of finite order, as with the $\mathcal{S}^2$ case:

$\mathcal{B}_1(\mathcal{P}^2)$	$= \mathbb{Z}_2$
$\mathcal{B}_2(\mathcal{P}^2)$	is dicyclic of order 16
$\mathcal{PB}_2(\mathcal{P}^2)$	is the quaternion group
$A_2(\mathcal{P}^2)$	$= \mathbb{Z}_4$
$\mathcal{B}_n(\mathcal{P}^2)$	is infinite for $n \geqslant 3$

2.5 Automorphisms of free groups

There is an alternative definition of the braid groups in terms of subgroups of $\operatorname{Aut}\mathcal{F}_n$, the group of (right) automorphisms of the free group of rank n.

Theorem 2.10 (Artin Representation Theorem)
Let $\mathcal{F}_n$ be the free group on n generators: $\langle x_1, \dots, x_n | \rangle$. Then $\mathcal{B}_n$ is isomorphic to the subgroup of $\operatorname{Aut}\mathcal{F}_n$ consisting of all right automorphisms $\overline{\beta}$ on $\mathcal{F}_n$ such that

$$x_i\overline{\beta} = A_i x_{\tau(i)} A_i^{-1}$$

$$(x_1 \dots x_n)\overline{\beta} = x_1 \dots x_n$$

where $1 \leqslant i \leqslant n, \tau \in \mathrm{Sym}_n$, and A_i is some word in $\mathcal{F}_n$.

Under this isomorphism, σ_i corresponds to an automorphism $\overline{\sigma_i}$ of $\mathcal{F}_n$, where

$$x_i\overline{\sigma_i} = x_i x_{i+1} x_i^{-1}$$

$$x_{i+1}\overline{\sigma_i} = x_i$$

$$x_j\overline{\sigma_i} = x_j$$

for all $j \neq i, i+1$.

The permutation τ for the automorphism $\overline{\beta}$ is the permutation of the braid β.

Justification

Identify $\mathcal{F}_n$ with the fundamental group of the n-punctured plane:

$$\mathcal{F}_n \cong \pi_1(\mathcal{E}_1^2 \setminus \{P_1, \dots, P_n\}, P_0) \cong \pi_1(\mathcal{E}_0^2 \setminus \{Q_1, \dots, Q_n\}, Q_0)$$

where $P_0 = (0, 0, 1)$ and $Q_0 = (0, 0, 0)$.

Thus, each generator $x_i \in \mathcal{F}_n$ corresponds to a loop, based at P_0, passing anticlockwise round P_i. Now consider a geometric braid $\beta \in \mathcal{B}_n$ in terms of the slab of $\mathcal{E}^3$ between $\mathcal{E}_0^2$ and $\mathcal{E}_1^2$ with the strings of the braid removed.

Then a braid β lifts to a map

$$\overline{\beta} : \mathcal{F}_n = \pi_1(\mathcal{E}_1^2 \setminus \{P_1, \dots, P_n\}, P_0) \to \mathcal{F}_n = \pi_1(\mathcal{E}_0^2 \setminus \{Q_1, \dots, Q_n\}, Q_0).$$

Geometrically, we visualise this by constructing the loop ℓ round the P_i corresponding to the word in $\mathcal{F}_n$, and then push ℓ down the braid.

Note that this is a single-valued mapping on homotopy classes and a homomorphism. Furthermore, it is a right automorphism — the inverse may be constructed by pushing the loop back up the braid again. The homotopy of braids says that if β_1 and β_2 are homotopic then $\overline{\beta_1} = \overline{\beta_2}$. The mapping $\mathcal{B}_n \to \operatorname{Aut} \mathcal{F}_n$ given by $\beta \mapsto \overline{\beta}$ is a homomorphism, since $\overline{\beta_1 \beta_2} = \overline{\beta_1}\overline{\beta_2}$.

Given that the word $x_1 \dots x_n$ corresponds to an anticlockwise loop round all the P_i, it will be unchanged by the automorphism given by any n-braid β: $(x_1 \dots x_n)\overline{\beta} = x_1 \dots x_n$.

Considering the action of the automorphisms $\overline{\sigma_i}$ (where σ_i is the ith elementary braid), we see that:

$$x_{i+1}\overline{\sigma_i} = x_i$$

$$x_i\overline{\sigma_i} = x_i x_{i+1} x_i^{-1}$$

$$x_j\overline{\sigma_i} = x_j$$

if $j \neq i, i+1$. □

Chapter 3

Representations of braid groups

In this chapter we provide a brief overview of Fox' free differential calculus, show how it may be used to construct matrix representations of automorphism groups of $\mathcal{F}_n$, and then look at two examples, namely Burau and Gassner's representations of, respectively, $\mathcal{B}_n$ and $\mathcal{PB}_n$.

3.1 Free differential calculus

Let $\mathcal{F}_n$ be a free group of rank n, with basis $\{x_1, \ldots, x_n\}$, and let ϕ be a homomorphism acting on $\mathcal{F}_n$, with $\mathcal{F}_n^\phi$ denoting the image of $\mathcal{F}_n$ under ϕ.

Now let $\mathbb{Z}\mathcal{F}_n^\phi$ denote the integral group ring of $\mathcal{F}_n^\phi$: an element of $\mathbb{Z}\mathcal{F}_n^\phi$ is a sum $\sum a_g g$, where $a_g \in \mathbb{Z}$ and $g \in \mathcal{F}_n^\phi$, with addition and multiplication defined by

$$\sum a_g g + \sum \mathcal{B}_g g = \sum (a_g + \mathcal{B}_g) g$$
$$\left(\sum a_g g\right)\left(\sum \mathcal{B}_g g\right) = \sum_g \left(\sum_h a_{gh^{-1}} \mathcal{B}_h\right) g$$

A homomorphism $\psi : \mathcal{F}_n^\phi \to \mathcal{F}_n^{\psi\phi}$ induces a ring homomorphism $\psi : \mathbb{Z}\mathcal{F}_n^\phi \to \mathbb{Z}\mathcal{F}_n^{\psi\phi}$. Later we will consider the cases where ψ is the abelianiser $\mathfrak{a}$ or the trivialiser $\mathfrak{t}$.

There is a well-defined mapping

$$\frac{\partial}{\partial x_j} : \mathbb{Z}\mathcal{F}_n \to \mathbb{Z}\mathcal{F}_n$$

given by

$$\frac{\partial}{\partial x_j}\left(x_{\mu_1}^{\varepsilon_1} \ldots x_{\mu_r}^{\varepsilon_r}\right) = \sum_{i=1}^{r} \varepsilon_i \delta_{\mu_i,j} x_{\mu_1}^{\varepsilon_1} \ldots x_{\mu_i}^{\frac{1}{2}(\varepsilon_i - 1)}$$

$$\frac{\partial}{\partial x_j}\left(\sum a_g g\right) = \sum a_g \frac{\partial g}{\partial x_j}$$

where $g \in \mathcal{F}_n, a_g \in \mathbb{Z}, \varepsilon_i = \pm 1$, and $\delta_{\mu_i,j}$ is the Kronecker δ.

The following properties follow from the definition:

Proposition 3.1

(i) $\frac{\partial x_i}{\partial x_j} = \delta_{i,j}$.

(ii) $\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{i,j} x_i^{-1}$.

(iii) $\frac{\partial (wv)}{\partial x_j} = \left(\frac{\partial w}{\partial x_j}\right) v^\tau + w\left(\frac{\partial v}{\partial x_j}\right)$.

3.2 Magnus representations

Let S_n be a free abelian semigroup with basis $\{s_1, \ldots, s_n\}$, let R be a ring, and let $A_0(R, S_n)$ be the semigroup ring of S_n with respect to R: elements in $A_0(R, S_n)$ are polynomials in non-negative powers of the s_i (which all commute), with coefficients in R.

Now define $\tau : \mathcal{F}_n \to M_2 A_0(\mathbb{Z}\mathcal{F}_n, S_n)$ as:

$$w \mapsto [w] = \begin{bmatrix} w & \sum_{j=1}^n \frac{\partial w}{\partial x_j} s_j \\ 0 & 1 \end{bmatrix}$$

In particular:

$$x_j \mapsto [x_i] = \begin{bmatrix} x_i & s_i \\ 0 & 1 \end{bmatrix}$$

(Since $\sum_{j=1}^n \frac{\partial x_i}{\partial x_j} s_j = \sum_{j=1}^n \delta_{ij} s_j = s_i$.)

If $w, v \in \mathcal{F}_n$ then $[wv] = [w][v]$. The mapping $w \mapsto [w]$ is a representation, the **Magnus representation**, of $\mathcal{F}_n$, and is not particularly interesting. If, though, we have a homomorphism ϕ acting on $\mathcal{F}_n$, and let

$$w \mapsto [w]^\phi = \begin{bmatrix} w^\phi & \sum_{j=1}^n \left(\frac{\partial w}{\partial x_j}\right)^\phi s_j \\ 0 & 1 \end{bmatrix}$$

then this is also a representation, the **Magnus ϕ-representation** of $\mathcal{F}_n$: $[\mathcal{F}_n]^\phi$ is the image of $\mathcal{F}_n$ under this homomorphism $\Phi : \mathcal{F}_n \to [\mathcal{F}_n]^\phi; w \mapsto [w]^\phi$.

We can generalise this representation Φ to representations of $\mathcal{F}_n$ by $k \times k$ upper-triangular matrices:

Define higher-order derivatives inductively (writing D_j for $\frac{\partial}{\partial x_j}$).

$$\begin{aligned} D_{i_1 i_2 \ldots i_q}(w) &= D_{i_q}(D_{i_1 i_2 \ldots i_{q-1}}(w)) \\ D(w) &= \sum_{i=1}^n D_i(w) s_i \\ D^{q+1}(w) &= D(D^q(w)) \end{aligned}$$

Then:

$$
\begin{aligned}
D^{q+1}(w) &= \sum_{1 \leqslant i_j \leqslant n} D_{i_1 \ldots i_q}(w) s_{i_1} \ldots s_{i_q} \\
D^q(uv) &= \sum_{p=1}^{q-1} (D^p(u)(D^{q-p}(v))^{\mathfrak{t}} + uD^q(v)
\end{aligned}
$$

where $\mathfrak{t}$ is the trivialiser.

Theorem 3.2 (Enright 1968)
Let ϕ be a homomorphism of $\mathcal{F}_n$ and let $(D^q(w))^\phi$ be the image of $D^q(w)$ under the ring homomorphism induced by ϕ.

Then, for $w \in \mathcal{F}_n$, let

$$
\{w\}^\phi = \begin{bmatrix} w^\phi & (D(w))^\phi & (D^2(w))^\phi & (D^3(w))^\phi & \cdots & (D^{k-1}(w))^\phi \\ 0 & 1 & (D(w))^{\mathfrak{t}} & (D^2(w))^{\mathfrak{t}} & \cdots & (D^{k-2}(w))^{\mathfrak{t}} \\ 0 & 0 & 1 & (D(w))^{\mathfrak{t}} & \cdots & (D^{k-3}(w))^{\mathfrak{t}} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
$$

Then $\Phi : w \mapsto \{w\}^\phi$ defines a representation of $\mathcal{F}_n$ in the ring $M_k(A_0(\mathbb{Z}\mathcal{F}_n, S_n))$ of $k \times k$ matrices over $A_0(\mathbb{Z}\mathcal{F}_n, S_n)$ for $k \geqslant 2$ and ϕ a homomorphism acting on $\mathcal{F}_n$.

Corollary 3.3
Let x_i be a basis element of $\mathcal{F}_n$. Then:

$$
x_i \mapsto \{x_i\}^{\mathfrak{t}} = \begin{bmatrix} 1 & s_i & 0 & \cdots & 0 \\ 0 & 1 & s_i & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & s_i \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}
$$

is a faithful matrix representation of $\mathcal{F}_n$ modulo the kth group of the lower central series of $\mathcal{F}_n$ over $A_0(\mathbb{Z}\mathcal{F}_n, S_n)$ for $k \geqslant 2$.

The point of all this is that we can now use this machinery of Magnus representations to study subgroups of $\operatorname{Aut}\mathcal{F}_n$, such as $\mathcal{B}_n$ or $\mathcal{PB}_n$.

Let ϕ be a homomorphism acting on $\mathcal{F}_n$ and let A_ϕ be any group of (right) automorphisms of $\mathcal{F}_n$ which satisfy

$$x\phi = x\alpha\phi$$

for all $x \in \mathcal{F}_n$ and $\alpha \in A_\phi$.

So, if ϕ is the abelianiser $\mathfrak{a}$, then A_ϕ might be the subgroup of $\operatorname{Aut}\mathcal{F}_n$ mapping each element into a conjugate of itself.

Any subgroup of $\operatorname{Aut}\mathcal{F}_n$ inducing the identity automorphism on $\mathcal{F}_n/\mathcal{F}_n'$ could do.

Now, if $\alpha \in A_\phi$, define $\tau : \alpha \mapsto \|\alpha\|^\phi = \left[\left(\frac{\partial(x_i\alpha)}{\partial x_j}\right)^\phi\right]$. Thus, τ defines a representation $A_\phi \to M_n(\mathbb{Z}\mathcal{F}_n^\phi)$.

For example:

Example 3.1
Let ϕ be $\mathfrak{t}$, the trivialiser $\mathfrak{t} : \mathcal{F}_n \to 1$, and $A_{\mathfrak{t}} be \operatorname{Aut}\mathcal{F}_n$. Then $\mathfrak{t}$ maps each element of $\operatorname{Aut}\mathcal{F}_n$ to an $n \times n$ matrix over $\mathbb{Z}$ where the (i, j)th element is the exponent sum of x_j in w_i. These matrices are invertible, and so have determinant ± 1, hence $\mathcal{F}_n^{\mathfrak{t}}$ is a subgroup of the unimodular group.

3.3 Burau's representation of $\mathcal{B}_n$

As noted before, $\mathcal{B}_n$ has a faithful representation as a group of automorphisms of $\mathcal{F}_n$, and hence we can regard $\mathcal{B}_n$ as a subgroup of $\operatorname{Aut}\mathcal{F}_n$.

Let $\mathbb{Z} = \langle t \rangle$ be the infinite cyclic group, and let $\psi : \mathcal{F}_n \to \mathbb{Z}; x_i \mapsto t$.

Then the corresponding representation, the **Burau representation** of $\mathcal{B}_n$ is given by:

$$\sigma_i \mapsto \|\sigma_i\|^\psi = \left[\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array}\right]$$

3.4 Gassner's representation of $\mathcal{PB}_n$

To represent the pure braid groups $\mathcal{PB}_n$, we can simply restrict the Burau representation of $\mathcal{B}_n$. But a more interesting representation exists, discovered by B.J. Gassner in 1961[4]:

Let ϕ be the abelianiser $\mathfrak{a}$. Then $\mathcal{PB}_n$ has a representation as a subgroup of $\operatorname{Aut}\mathcal{F}_n$ by the restriction of $\xi : \mathcal{B}_n \to \operatorname{Aut}\mathcal{F}_n$ to $\mathcal{PB}_n$.

Let $\mathcal{AF}_n$ be the free abelian group of rank n, with basis $\{t_1, \ldots, t_n\}$ and let $a : \mathcal{F}_n \to \mathcal{AF}_n$ be defined by $x_i a = t_i$.

The pure braid generators map a generator x_i of $\mathcal{F}_n$ into a conjugate of itself, so the requirement $x_i A_{rs} a = x_i a$ is satisfied for $1 \leqslant i \leqslant n$ and $1 \leqslant r < s \leqslant n$ if $\phi = \mathfrak{a}$.

So, we derive the **Gassner representation** of $\mathcal{PB}_n$:

$$(\!(A_{rs})\!)_{ij}^{\mathfrak{a}} = \begin{cases} \delta_{ij} & \text{if } s < i \text{ or } i < r \\ (1-t_i)\delta_{ir} + t_r\delta_{ij} & \text{if } s = i \\ (1-t_i)(\delta_{ij} + t_i\delta_{sj}) + t_i t_s \delta_{ij} & \text{if } r = i \\ (1-t_i)(1-t_s)\delta_{rj} - (1-t_r)\delta_{sj} + \delta_{ij} & \text{if } r < i < s \end{cases}$$

3.5 Fidelity

An, as yet, not completely answered question concerning the Burau and Gassner representations is whether they are faithful or not. In this section we state a number of partial answers to this question.

Theorem 3.4 (Magnus/Peluso 1969)
The Burau representation of $\mathcal{B}_3$ and the Gassner representation of $\mathcal{PB}_3$ are faithful.

Theorem 3.5
The Burau representation of $\mathcal{B}_4$ is faithful if and only if the matrix group generated by

$$\|\sigma_3\sigma_1^{-1}\|^{\psi} = \begin{bmatrix} -t & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t^{-1} \end{bmatrix}$$

and

$$\|\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\|^{\psi} = \begin{bmatrix} 1-t^{-1} & -t^{-1} & t^{-1} \\ 1-t^2 & -t^{-1} & 0 \\ 1 & -t^{-1} & 0 \end{bmatrix}$$

is free and has rank 2.

Theorem 3.6 (Moody 1991[6])
The Burau representation of $\mathcal{B}_n$ is not faithful for $n \geqslant 10$.

Subsequent work showed that:

Theorem 3.7
The Burau representation of $\mathcal{B}_n$ is not faithful for $n \geqslant 6$.

The question regarding the Burau representation of $\mathcal{B}_4$ and $\mathcal{B}_5$ has not as yet been settled.

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