Classification and the Minimal Model Program; the Hodge diamond

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Let k be a field and let X be a smooth projective variety defined over k. The aim of the Minimal Model Program is to find a variety X', a *minimal model*, which reflects as much as possible the properties of the variety X, and which is as simple as possible. The criterion to decide whether a variety is simple enough to be called a minimal model is based on intersection properties of the canonical divisor with curves on X. In order to simplify the statements, we shall assume for this talk that the base field k is algebraically closed and that $\mathrm{char}(k) = 0$.

Minimal models (of surfaces). The definition of a minimal model involves intersection numbers. Recall the definition of a nef divisor.

Definition 1. *A divisor N on X is nef if for all curves C in X we have $N \cdot C \geq 0$.*

An ample divisor H is certainly nef: in this case all intersection numbers of curves with H are strictly positive. Also, an irreducible curve C on X such that $C \cdot C = C^2 \geq 0$ is nef: for all curves D not containing C as a component, the intersection number $C \cdot D$ is non-negative more or less by construction.

Let $K = K_X$ denote a canonical divisor on X, and recall that K is only well-defined up to linear equivalence: this is not a problem, since all that we are going to need are intersection numbers. The divisor K is a finite linear combination of codimension one subvarieties of X. One of the starting points of the Minimal Model Program was the following remarkable implication: if the canonical divisor of X has negative intersection number with a curve $C \subset X$, then each point of C is contained in a rational curve. Varieties containing many rational curves have special properties; such varieties are not going to be minimal models.

Definition 2. *A smooth projective variety X is a* minimal model *if K_X is a nef divisor.*

Note that this definition works for varieties X of all dimensions. In particular, if we take X to be a smooth curve, then X is a minimal model if and only if X has genus at least one. Thus every smooth projective curve different from $\mathbb{P}^1$ is a minimal model, which agrees with our intuitive notion that $\mathbb{P}^1$ is the "easiest" curve.

For surfaces X the situation is more intricate. The first main difference with curves is that it is no longer true that a birational map between two smooth projective surfaces can be extended to an isomorphism. The universal example of why this happens is the *blow-up*. Given a surface X and smooth point $p \in X$, the blow-up of X at p is a surface $Bl_p(X)$ together with a morphism $\pi \colon Bl_p(X) \to X$. The morphism π is an isomorphism above all points of X different from p and the fiber E of π above p is isomorphic to $\mathbb{P}^1$. Intuitively the blow-up of X at p is obtained by replacing the point p by all the tangent directions at p: the tangent directions at p of X are parameterized by $E \simeq \mathbb{P}^1$, which is the fiber of π at p. The fiber E of π at p is often called the *exceptional divisor*. It is a fact (see Proposition **??**) that $K_{Bl_p(X)} \cdot E = -1$; therefore $Bl_p(X)$ is not a minimal model. It follows that if X is a minimal model, X is not the blow-up of another smooth surface, and this partially justifies the terminology. Thus in our search for minimal models we can restrict our attention to surfaces that are not blow-ups of smooth surfaces; such surfaces are classically called *minimal*.

Theorem 3. *Suppose that X is a minimal surface that is not a minimal model; then either $X \simeq \mathbb{P}^2$, or X is ruled, that is there is a morphism $b\colon X \to C$, where C is a smooth curve and the fibers of b are isomorphic to $\mathbb{P}^1$.*

The next theorem lists the minimal models of surfaces in increasing order of complexity, mentioning for each type the classical name of the corresponding surface. We give some further properties and some examples after the statement.

Theorem 4. *Let X be a smooth projective surface and a minimal model (and hence also minimal).*

(1) *Suppose that $(K_X)^2 = 0$.*
   - *If K_X is linearly equivalent to zero, then X is either an abelian surface or a K3 surface.*
   - *If K_X is numerically equivalent to zero, but not linearly equivalent to zero, then either X is an Enriques surface or X is a bielliptic surface.*
   - *If K_X is not numerically equivalent to zero, then X is an elliptic surface.*

(2) *Suppose that $(K_X)^2 > 0$, then X is a surface of general type.*

An abelian surface is a smooth projective variety together with the structure of an algebraic group, e.g. the Jacobian of a genus two curve. A K3 surface is a simply-connected surface with trivial canonical divisor, e.g. a smooth quartic surface in $\mathbb{P}^3$. An Enriques surface is a quotient of a K3 surface by a fixed-point free involution. A bielliptic surface is a quotient of a product of two elliptic curves by some special finite group acting on both curves. An elliptic surface is a surface admitting a morphism to a curve, with genus one curves as general fibers.

The Hodge diamond. Given a variety X, let $\Omega^p_X := \bigwedge^p \Omega_X$ the p-the exterior power of the sheaf of differential one-forms. The dimensions $h^{p,q}$ of the vector spaces $\mathrm{H}^q\big(X, \Omega^p_X\big)$ are called the *Hodge numbers*. If X is a smooth projective variety defined over the complex numbers, then the singular cohomology $\mathbb{C}$-vector spaces $\mathrm{H}^k\big(X, \mathbb{C}\big)$ admit a direct sum decomposition

$$\mathrm{H}^k\big(X, \mathbb{C}\big) = \bigoplus_{p+q=k} \mathrm{H}^q\big(X, \Omega^p_X\big)$$

called the *Hodge decomposition*. If X has dimension n, then the Hodge numbers satisfy the identities

$$\begin{aligned} h^{p,q} &= h^{q,p} \\ h^{n-p,n-q} &= h^{p,q} \end{aligned}$$

coming from the fact that $\mathrm{H}^q\big(X, \Omega^p_X\big) = \overline{\mathrm{H}^p\big(X, \Omega^q_X\big)}$ and from Poincaré duality, respectively. Moreover we also have the identity

$$e_{top}(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}$$

where e_{top} is the topological Euler characteristic of the topological space associated to X, with the induced Euclidean topology.

Typically, the Hodge numbers are written in the following way:

$$
\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
 & & \dots & \dots & \dots & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
$$

which takes the name *Hodge diamond*, for obvious reasons! We have $h^{0,0} = h^{n,n} = 1$. For a surface the Hodge diamond is

$$
\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{2,1} & & h^{1,2} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{1,0} & & h^{0,1} & \\
 & & h^{0,0} & &
\end{array}
$$

where $\chi\big(X, \mathcal{O}_X\big) = h^{0,0} - h^{0,1} + h^{0,2} = 1 - h^{0,1} + h^{0,2}$ is the Euler characteristic of the structure sheaf and $p_a := h^{0,2} - h^{0,1}$ is the arithmetic genus.

Smooth surfaces in $\mathbb{P}^3$. Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d; we wish to determine the Hodge numbers of X. First, by one exercise of one of the previous lectures we know that $h^{1,0} = 0$ and hence $h^{0,1} = h^{2,1} = h^{1,2} = 0$; thus we have

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
h^{0,2} & & h^{1,1} & & h^{0,2} \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

To compute the remaining two Hodge numbers we are going to compute $\chi\big(X, \mathcal{O}_X\big)$ and $e_{top}(X)$.

The Euler characteristic $\chi\big(X, \mathcal{O}_X\big)$ is the evaluation at $n = 0$ of the Hilbert polynomial $\chi\big(X, \mathcal{O}_X(n)\big)$, which, for n large enough, coincides with the dimension of the degree n part of the graded ring $k[X_0, X_1, X_2, X_3]/(F)$, where X_0, X_1, X_2, X_3 are homogeneous coordinates on $\mathbb{P}^3$ and F is a non-zero homogeneous polynomial of degree d, vanishing along X. The space of homogeneous polynomials of degree n in $\mathbb{P}^3$ has dimension $\binom{n+3}{3}$. The dimension of the space of polynomials of degree n vanishing along X has dimension $\binom{n+3-d}{3}$, since such polynomials are exactly the multiples of F. Thus the homogeneous part of degree n of $k[X_0, X_1, X_2, X_3]/(F)$ has dimension

$$\binom{n+3}{3} - \binom{n+3-d}{3} = \frac{1}{2}dn^2 + \frac{1}{2}(4d - d^2)n + \frac{1}{6}(d^3 - 6d^2 + 11d)$$

for $n \geq d$. Since the expression on the right of the last equation is a polynomial, it is the Hilbert polynomial of $X \subset \mathbb{P}^3$; evaluating at $n = 0$ and subtracting one we find $h^{0,2} = \binom{d-1}{3}$.

To compute the topological Euler characteristic of X, we shall follow two strategies: the first uses an identity called Noether's formula; the second uses Chern classes.

Noether's formula is the following identity:

$$\chi\big(X, \mathcal{O}_X\big) = \frac{1}{12}(K_X^2 + e_{top}(X)).$$

In our case K_X is linearly equivalent to $(d-4)H_X$, where H_X is the restriction of a plane to X. Since $H_X^2 = d$, we find $e_{top}(X) = 12\chi\big(X, \mathcal{O}_X\big) - d(d-4)^2$. This allows us to conclude that $h^{1,1} = \frac{d(2d^2-6d+7)}{3}$.

Alternatively, we use the sequences

$$0 \to \Omega_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(-1)^4 \to \mathcal{O}_{\mathbb{P}^3} \to 0$$

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbb{P}^3}|_X \to \Omega_X \to 0.$$

Both sequences were introduced in the previous lecture. The first is the Euler sequence. The second comes from the fact that any cotangent vector to $\mathbb{P}^3$ induces a cotangent vector to X; the kernel of this morphism is generated by the differential of the equation of X. Denote by H the hyperplane class in $\mathbb{P}^3$ and by H_X its restriction to X; taking total Chern classes, we find that

$$1 + c_1(\mathbb{P}^3) + c_2(\mathbb{P}^3) + c_3(\mathbb{P}^3) = (1-H)^4 = 1 - 4H + 6H^2 - 4H^3$$

$$1 + c_1(X) + c_2(X) = \frac{(1+c_1(\mathbb{P}^3)+c_2(\mathbb{P}^3)+c_3(\mathbb{P}^3))|_X}{(1-dH_X)} = (1 - 4H_X + 6H_X^2)(1 + dH_X + d^2H_X^2).$$

We have $c_2(X) = e_{top}(X)$, and, since X has degree d, also $d[point] = H_X^2$; combining everything we conclude that

$$e_{top}(X) = d^3 - 4d^2 + 6d$$

and finally the Hodge diamond of X is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ \binom{d-1}{3} & & \frac{d(2d^2-6d+7)}{3} & & \binom{d-1}{3} \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Exercise 5. *Compute the Hodge numbers of* $\mathbb{P}^2$.

Exercise 6. *Compute the Hodge numbers of* $C_1 \times C_2$*, where* C_1 *and* C_2 *are smooth curves of genus* g_1 *and* g_2.

Exercise 7. *Compute the Hodge numbers of smooth curves of degree* d *in* $\mathbb{P}^2$.

Exercise 8 ($\star$)**.** *Compute the Hodge numbers of smooth surfaces in* $\mathbb{P}^4$ *that are intersections of two hypersurfaces of degree* d *and* e.