

Exercises for Thursday April 17th

BRAUER GROUPS

Exercise 1. Check that $\mathbb{H}_{\mathbb{R}}$ is a division algebra.

Exercise 2. Prove that if $c \in k^*$, then the four quaternion algebras $(a, b)_k$, $(b, a)_k$, $(ac^2, b)_k$, and $(a, bc^2)_k$ are isomorphic.

Prove that the quaternion algebra $(a^2, b)_k$ is not a division algebra.

Prove that the quaternion algebra $(a, b)_k$ is a matrix algebra if and only if the Hilbert symbol $(a, b)_k$ equals one.

Exercise 3. Prove that if k is algebraically closed, then $\text{Br}(k) = (0)$. (Hint: It suffices to show that there are no non-trivial division algebras over k . If D is a division algebra over k , let $d \in D$ and consider $k(d)$.)

Exercise 4. Find an explicit isomorphism between $\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $M_2(\mathbb{C})$.

Exercise 5. Prove that the algebra $\mathbb{H}_{\mathbb{R}}$ has order two in $\text{Br}(\mathbb{R})$. Prove also that if $(a, b)_k$ is a quaternion algebra over a field k , then $2[(a, b)_k] = 0$ in the Brauer group of k .

Exercise 6. Let v be a valuation of \mathbb{Q} . Show that the value of the invariant at v of the quaternion algebra $(a, b)_{\mathbb{Q}}$ is equal to the Hilbert symbol $(a, b)_v$ (with the obvious group isomorphism between $\{1, -1\}$ and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$).

Exercise 7. Prove that quaternion algebras are cyclic algebras.

NORMS

Exercise 8. Let X be a smooth projective variety over \mathbb{Q} . Let $f \in k(X)^\times$ be a rational function on X , and suppose that there exists a quadratic extension $k = \mathbb{Q}(\sqrt{d})$ of \mathbb{Q} such that the divisor (f) is a norm from k – that is, there is some divisor $D \in \text{Div } X_k$ such that $(f) = N_{k/\mathbb{Q}}D$.

If K is any field containing \mathbb{Q} , consider the map

$$(1) \quad X(K) \rightarrow \frac{K^\times}{NK(\sqrt{d})^\times}$$

defined by f . A priori, this is only defined away from the zeros and poles of f .

- a) Show that multiplying f by a norm does not affect this map. In other words, let $g \in k(X_k)^\times$ be a rational function on X_k and show that $fN_{k/\mathbb{Q}}(g)$ defines the same map as f .
- b) Let $P \in X(K)$ be any point of X . Show that there exists a function $g \in k(X_k)^\times$ such that $fN_{k/\mathbb{Q}}(g)$ has neither a zero nor a pole at P . Deduce that the map (1) is well-defined on the whole of $X(K)$.
- c) Taking $K = \mathbb{Q}_v$ in (1), we get a map which we can equally well write using the Hilbert symbol:

$$X(\mathbb{Q}_v) \rightarrow \{\pm 1\}, \quad x \mapsto (d, f(x))_v.$$

Multiplying these maps, we get a map from the product of all the $X(\mathbb{Q}_v)$ to $\{\pm 1\}$ as follows:

$$\prod_v X(\mathbb{Q}_v) \rightarrow \{\pm 1\}, \quad (x_v) \mapsto \prod_v (d, f(x_v))_v.$$

Show that the diagonal image of $X(\mathbb{Q})$ in $\prod_v X(\mathbb{Q}_v)$ must lie in the kernel of this map (i.e. the inverse image of 1).

- d) Consider the example of Birch and Swinnerton-Dyer given in the lecture. The surface is defined by

$$\begin{cases} uv = x^2 - 5y^2 \\ (u+v)(u+2v) = x^2 - 5z^2 \end{cases}$$

Show the the function $f = u/(u+v)$ has the property described above, with $k = \mathbb{Q}(\sqrt{5})$. Write down a few other functions giving the same norm group map (1) as f .

GALOIS COHOMOLOGY

Exercise 9. Show that if M is a G -module, and the action of G on M is trivial, then we have $H^1(G, M) = \text{Hom}(G, M)$.

Exercise 10. Show that a 1-cocycle $\varphi: G \rightarrow M$ is determined by the choice of $m = \varphi(\sigma)$. Show that there is an isomorphism $H^1(G, M) \rightarrow {}_N M / \Delta(M)$.

Exercise 11. Let C_n be a cyclic group of order n , acting on \mathbb{Z}^n by permuting the coordinates cyclically. Find a sublattice L that G also acts on (a sub- G -module), such that $(\mathbb{Z}^n)^G \rightarrow (\mathbb{Z}^n/L)^G$ is not surjective. What is the cokernel?

Exercise 12. Suppose k is a number field and μ_n is the group of all n -th roots of unity in \bar{k} . Show that there is a natural isomorphism $k^*/(k^*)^n \rightarrow H^1(k, \mu_n)$.

Exercise 13. Let E be an elliptic curve of a field k without complex multiplication, i.e., the endomorphism ring of E is isomorphic to \mathbb{Z} . What is the automorphism group of E ? Show that the twists of E are in one-to-one correspondence with the elements of k^*/k^{*2} .