Exercises for Tuesday April 15th

Riemann-Roch and adjunction formula

Let X be a smooth projective surface embedded by its anticanonical divisor, i.e. such that $K_X = -H$ where H is a hyperplane section. For example, X could be a cubic surface in $\mathbb{P}^3$ or the intersection of two quadrics in $\mathbb{P}^4$.

Exercise 1. *Let C be a smooth curve in X. Use the adjunction formula to show that $C^2 \geq -1$.*

Exercise 2. *Conversely, let D be a divisor satisfying $D^2 \geq -1$ and $D.H > 0$. Use the Riemann–Roch theorem to show that the linear equivalence class of D contains an effective divisor.*

Exercise 3. *Let D be a divisor on X such that nD is principal for some positive integer n. Show that $D^2 = 0$. Using the Riemann–Roch theorem, deduce that $D \sim 0$ and therefore that $\operatorname{Pic} X$ is torsion-free.*

Picard group of del Pezzo surfaces

Exercise 4. *We will blow up $\mathbb{A}^2(x, y)$ in the point $P = (0, 0)$. Consider the surface X in $\mathbb{A}^2(x, y) \times \mathbb{P}^1(s, t)$ given by $tx = sy$. Let π denote the projection $\pi\colon X \to \mathbb{A}^2$. Set $E = \pi^{-1}(P)$. Show that E is isomorphic to $\mathbb{P}^1$ and that π induces an isomorphism from $X - E$ to $\mathbb{A}^2 - \{P\}$. We call X the blow-up of $\mathbb{A}^2$ in P. Note that each direction at P is determined by a line given by $t_0 x = s_0 y$, so the directions in $\mathbb{A}^2$ are parametrized by $\mathbb{P}^1(s, t)$ and we can think of X as "the surface obtained by replacing P by all directions at P". Consider the curve $C \subset \mathbb{A}^2$ given by $y^2 = x^3 - x^2$. Show that C has a node as singularity at P. Show that $\pi^{-1}(C)$ consists of E and another component, say C'. We call C' the strict transform of C. Is C' still singular?*

Exercise 5. *Show that if the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in r points, then no six of them lie on a conic.*

Exercise 6. *Show that if the del Pezzo surface X is the blow-up of $\mathbb{P}^2$ in 8 points, then they do not lie on a singular cubic that has its singularity at one of the 8 points.*

Exercise 7. *Suppose X is the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points $P_1, \ldots, P_r$ in general position. Then the strict transforms of the following curves in $\mathbb{P}^2$ are all exceptional curves:*

(1) *a line through 2 of the P_i,*
(2) *a conic through 5 of the P_i,*
(3) *a cubic passing through 7 of the P_i such that one of them is a double point (on that cubic),*
(4) *a quartic passing through 8 of the P_i such that three of them are double points,*
(5) *a quintic passing through 8 of the P_i such that six of them are double points,*
(6) *a sextic passing through 8 of the P_i such that seven of them are double points and one is a triple point.*

Exercise 8. *Determine the number of exceptional curves on a del Pezzo surface of degree d for each d (getting two possibilities for degree 8).*

Exercise 9. *Let C, D be a curves and suppose that the genus of C is at most one. Where can the surface $C \times D$ appear in the classification of surfaces?*

Exercise 10. *Compute the Hodge numbers of $\mathbb{P}^2$.*

Exercise 11. *Compute the Hodge numbers of $C_1 \times C_2$, where C_1 and C_2 are smooth curves of genus g_1 and g_2 respectively.*

Exercise 12. *Compute the Hodge numbers of smooth curves of degree d in $\mathbb{P}^2$.*

Exercise 13 ($\star$). *Compute the Hodge numbers of smooth surfaces in $\mathbb{P}^4$ that are intersections of two hypersurfaces of degree d and e.*

Exercise 14. *Check that the graphs in Table 1 are correct and label the vertices by exceptional curves on $\bar{X}$ so that the number of edges between two distinct vertices equals the intersection number of the corresponding exceptional curves.*

Exercise 15. *Construct an example of a del Pezzo surface of degree eight defined over $\mathbb{Q}$, containing no rational points. (Hint: think about quadric surfaces in $\mathbb{P}^3_{\mathbb{Q}}$.)*

Exercise 16. *Let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be the quadric defined by*

$$a_0X_0^2 + a_1X_1^2 + a_2X_2^2 + a_3X_3^2 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}^*$. Assume X contains a rational point. Show that there is an isomorphism $X \simeq \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ defined over $\mathbb{Q}$ if and only if $a_0a_1a_2a_3$ is a square.

Find a form X of $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ with a rational point, for which the splitting is not defined over $\mathbb{Q}$. Determine a birational map of X to $\mathbb{P}^2$ defined over $\mathbb{Q}$.

Exercise 17. *Prove that a del Pezzo surface of degree seven containing a k-rational point is k-rational, considering the contraction of the "middle" exceptional curve. (Hint: the surface obtained by the contraction is a del Pezzo surface; what is its degree?)*

Exercise 18. *Analyze the cases in which X is a del Pezzo surface of degree six and it has a k-rational point p lying on some exceptional curve.*

Exercise 19. *Let X be a del Pezzo surface of degree five and assume that $X(k) \neq \emptyset$. Show that X is birational to $\mathbb{P}^2_k$ over k.*

Exercise 20. *Construct an example of a del Pezzo surface of degree four defined over $\mathbb{Q}$, containing no rational points.*

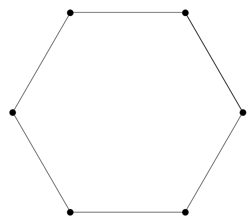
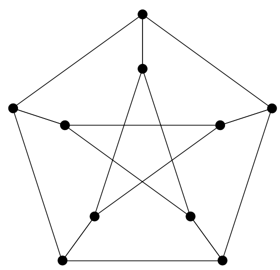
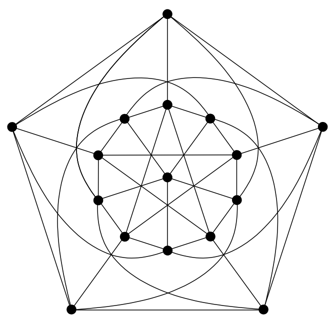
Degree of X	Graph G_X
8	
7	
6	
5	
4	

Table 1. Small graphs of exceptional curves