PDE for Finance, Solutions for Sheet 7

1 a) For $n=0$, we have $z_1 \le C z_0$.

Now assume that the claim holds for some $n \in \mathbb{N}_0$.

I.e. assume $z_n \le D \dfrac{C^n - 1}{C-1} + z_0 C^{n+1}$ $(*)$.

Then

$$z_{n+1} \overset{\text{Assumption}}{\le} C z_n + D \overset{(*)}{\le} C\left(D \frac{C^n - 1}{C-1} + z_0 C^{n+1}\right) + D$$

$$= D\left(\frac{C^{n+1} - C}{C-1} + 1\right) + z_0 C^{n+2}$$

$$= D\left(\frac{C^{n+1} - C + C - 1}{C-1}\right) + z_0 C^{n+2} = D \frac{C^{n+1} - 1}{C-1} + z_0 C^{n+2}.$$

So if the claim holds for some $n \in \mathbb{N}$, it also holds for the next bigger integer. Since it holds for $n=0$, it holds for all $n \in \mathbb{N}$.

b) $$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3)$$

$$= y(t_n) + h F(y(t_n), t_n) + \frac{h^2}{2} \frac{d}{dt}\left(F(y(t), t)\right)\Big|_{t=t_n} + O(h^3)$$

$$= y(t_n) + h F(y(t_n), t_n) + \frac{h^2}{2}\Big[(\partial_y F)(y(t_n), t_n) \underbrace{y'(t_n)}_{= F(y(t_n), t_n)} + (\partial_t F)(y(t_n), t_n)\Big] + O(h^3)$$

$$= y(t_n) + h F(y(t_n), t_n) + \frac{h^2}{2}\left(\partial_y F(y(t_n), t_n) F(y(t_n), t_n) + \partial_t F(y(t_n), t_n)\right) + O(h^3)$$

c) $\phi(h) := F\left(y(t_n) + hF(y(t_n), t_n),\ t_n + h\right).$

$$\phi(h) = \phi(0) + h\,\phi'(0) + O(h^2)$$

$$= F\left(y(t_n) + hF(y(t_n), t_n),\ t_n\right) +$$

$$h\cdot\left(\left[(\partial_y F)\left(y(t_n), t_n\right)\right]\cdot F\left(y(t_n), t_n\right)\right) + \partial_t F\left(y(t_n), t_n\right)\Big) + O(h^2)$$

$$\Rightarrow h\cdot\left((\partial_y F)(y(t_n), t_n)\ F(y(t_n), t_n) + \partial_t F(y(t_n), t_n)\right) =$$

$$= \underbrace{F\left(y(t_n) + hF(y(t_n), t_n),\ t_n + h\right)}_{=\phi(h)} - \underbrace{F\left(y(t_n), t_n\right)}_{\phi(0)} + O(h^2)$$

Plug this into the expression in b) to get

$$y(t_{n+1}) = y(t_n) + \frac{h}{2}F\left(y(t_n), t_n\right) + \frac{h}{2}F\left(y(t_n) + hF(y(t_n), t_n),\ t_n + h\right) + O(h^3).$$

So the second order scheme is

$$y_{n+1} = y_n + \frac{h}{2}\left(F(y_n, t_n) + F\left(y_n + hF(y_n, t_n),\ t_{n+1}\right)\right)$$

with $t_n = hn$.

2) We discretize time with step size h and space with step size h_x. We write $t_n = t_0 + nh$, for $0 \le n \le N$, with $N = \frac{T - t_0}{h}$. Thus, $t_N = T$. We write $x_j = h_x \cdot j$.

Now, $\partial_t u(x_j, t_n) = \frac{1}{h}\left(u(x_j, t_n) - u(x_j, t_{n-1})\right) + O(h^2)$ (*).

For the x-derivatives, we use central differences since there is no prefered direction of space. We write

$$\partial_x u(x_j, t_n) = \frac{1}{2}\left(\frac{1}{h_x}\left[u(x_{j+1}, t_n) - u(x_j, t_n)\right] - \frac{1}{h_x}\left[u(x_j, t_n) - u(x_{j-1}, t_n)\right]\right)$$

$$= \frac{1}{2}\frac{1}{h_x}\left(u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\right).$$

$$\partial_x^2 u(x_j, t_n) = \frac{1}{h_x^2}\left(u(x_{j+1}, t_n) + u(x_{j-1}, t_n) - 2u(x_j, t_n)\right) + O(h_x^2)$$

(standard approximation to the second derivative!)

In the HJB-equation, this gives

$$u(x_j, t_{n-1}) = u(x_j, t_n) - h\,\partial_t u(x_j, t_n) =$$

$$= u(x_j, t_n) - h\left(x_j r\,\partial_x u(x_j, t_n) + \frac{1}{2}\lambda^2 \frac{(\partial_x u(x_j, t_n))^2}{\partial_x^2 u(x_j, t_n)}\right)$$

$$= u(x_j, t_n) - \frac{h}{h_x} x_j r\,\frac{1}{2}\left(u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\right) -$$

$$- \frac{h}{2}\lambda^2 \frac{\frac{1}{4h_x^2}\left(u(x_{j+1}, t_n) - u(x_{j-1}, t_n)\right)^2}{\frac{1}{h_x^2}\left(u(x_{j+1}, t_n) + u(x_{j-1}, t_n) - 2u(x_j, t_n)\right)} + \text{errors}$$

So the scheme is

$$u_j^{n-1} = u_j^n - \frac{h}{2h_x} x_j r\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{h}{8}\lambda^2 \frac{\left(u_{j+1}^n - u_{j-1}^n\right)^2}{u_{j+1}^n + u_{j-1}^n - 2u_j^n}$$

The last term of this scheme looks harmless at first sight – it even seems that the middle term is the more difficult one, as it constrains $h \lesssim h_x$.

The problem with the final term is that for small h_x, the quantity $u_{j+1}^n + u_{j-1}^n - 2u_j^n$ is of order h_x^2, so in the last term we are essentially dividing by zero – this is numerically terrible. Of course, the term $(u_{j+1}^n - u_{j-1}^n)^2$ is also of order h_x^2, but still this is a numerical instability.

Note that due to the final condition, we chose the scheme so that we compute u_j^{n-1} from u_j^n.