Solutions for Problem sheet 5, PDE for finance

1) a) By definition,

$$u(\lambda x,t) = \max_{\alpha} \int_t^{\tau} e^{-\rho s} \alpha(\tilde{y}(s),s)^{\gamma} ds, \qquad (*)$$

where $\tilde{y}(s) \equiv \tilde{y}(\alpha,s)$ solves $\partial_s \tilde{y}(s) = r\tilde{y}(s) - \alpha(s,\tilde{y}(s))$ with initial condition $y(t) = \lambda x$, and this α being the control that in $(*)$ is maximized. In particular,

$$u(\lambda x,t) \geq \int_t^{\tau} e^{-\rho s} \alpha_0(y(s),s)^{\gamma} ds$$

for all possible controls α_0 and all $y(s)$ solving the ODE with that control α_0. With the suggested control $\tilde{\alpha}(t) = \lambda \alpha_x(t)$ (and $\alpha_x(t)$ the optimal control when starting in x),

we find $$u(\lambda x,t) \geq \int_t^{\tau} e^{-\rho s}(\lambda \alpha_x(s))^{\gamma} ds = \lambda^{\gamma} \int_t^{\tau} e^{-\rho s} \alpha_x(s) ds$$

$$\overset{\text{Since } \alpha_x \text{ is optimal!}}{=} \lambda^{\gamma} u(x,t).$$

But the same is true with $\frac{1}{\lambda}$ instead of λ, and thus

$$u(\tfrac{1}{\lambda}x,t) \geq \lambda^{-\gamma} u(x,t) \overset{y=\frac{1}{\lambda}x}{\Longrightarrow} u(y,t) \geq \lambda^{-\gamma} u(\lambda y,t)$$

$$\Rightarrow u(\lambda y,t) \leq \lambda^{\gamma} u(y,t)$$

Together, we find $u(\lambda x,t) = \lambda^{\gamma} u(x,t)$.

b) Clearly, $u(x,T)=0$ for all x, as there is no final payout.

Assume that we know $u(x,s)$ for some $s>t$ and all x.

For small δs,

$$u(x,s-\delta s) = \max_{\alpha} \int_{s-\delta s}^{\min(s,\tau)} e^{-\rho r}\alpha(y(r),r)^{\gamma}\,dr + \int_{\min(s,\tau)}^{\tau} e^{-\rho r}\alpha(y(r),r)^{\gamma}\,dr = (*)$$

Now $\int_{\min(s,\tau)}^{\tau} e^{-\rho r}\alpha(y(r),r)^{\gamma}\,dr = u(y(s),s)$. For $s<\tau$ this is by definition, and for $s>\tau$ both are zero (as then $y(s)=0$, and $u(0,s)=0$).

So, by approximating the integral with $\delta t \cdot$ „value of the integrand", we find

$$(*) \approx \max_{\alpha}\left[e^{-\rho s}\alpha(y(s),s)\,\delta s + u(y(s),s)\right] = (**).$$

Now do a Taylor expansion of the function $s \mapsto u(y(s),s)$ around $(s-\delta s)$:

$$u(y(s),s) \approx u(\underbrace{y(s-\delta s)}_{=x}, s-\delta s) + \partial_x u(y(s-\delta s), s-\delta s)\,\underbrace{\partial_s y(s-\delta s)}_{= r\underbrace{y(s-\delta s)}_{=x} - \alpha(\underbrace{y(s-\delta s)}_{=x}, s-\delta s)}\,\delta s$$

$$+ \partial_s u(y(s-\delta s), s-\delta s)\,\delta s.$$

Plug this into $(**)$ and cancel the terms $u(x,s-\delta s)$ to get

$$0 \approx \max_{\alpha}\Big[e^{-\rho s}\alpha(y(s),s) + \partial_x u(x,s-\delta s)\,(rx - \alpha(x,s-\delta s)) +$$

$$+ \partial_s u(x,s-\delta s)\Big]\,\delta s.$$

Divide by δs and let $\delta s \to 0$ to obtain the claim.

c) $u(x,t) = g(t)x^\gamma$, as $x \mapsto cx^\gamma$ is the only function having the scaling we found in a). Plug into the HJB-equation:

$$g'(t)\,x^\gamma + \max_{\alpha \geq 0}\left((rx-\alpha)\,g(t)\,\gamma x^{\gamma-1} + e^{-\rho t}\alpha^\gamma \right) = 0$$

Find the maximum by differentiating wrt α:

$$0 = -g(t)\gamma x^{\gamma-1} + \gamma e^{-\rho t}\alpha^{\gamma-1} \Rightarrow \alpha_* = \left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}} x$$

Plug this back in:

$$g'(t)\,x^\gamma + \left(r - \left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}}\right) g(t)\,\gamma x^\gamma + e^{-\rho t}\left(g(t)e^{\rho t}\right)^{\frac{\gamma}{\gamma-1}} x^\gamma = 0$$

Cancel x^γ, and simplify (using $\frac{1}{\gamma-1}+1 = \frac{\gamma}{\gamma-1}$):

$$0 = g'(t) + r\gamma\, g(t) - \gamma\left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}} g(t) + e^{-\rho t} g(t)e^{\rho t}\left(g(t)e^{\rho t}\right)^{\frac{1}{\gamma-1}}.$$

This gives (**). With $G = e^{\rho t}g$, we have

$$g'(t) = \left(e^{-\rho t}G(t)\right)' = -\rho e^{-\rho t}G(t) + e^{-\rho t}G'(t).$$

Then (**) becomes

$$0 = e^{-\rho t}\left(-\rho G + G'\right) + r\gamma e^{-\rho t}G + (1-\gamma)\,G^{\frac{1}{1-\gamma}}\,Ge^{-\rho t},$$

Cancelling $e^{-\rho t}$ gives (***). Finally for $H = G^{1/1-\gamma}$,

$G' = (H^{1-\gamma})' = (1-\gamma)H^{-\gamma}H'$. Then (***) becomes

$$0 = (1-\gamma)H^{-\gamma}H' + (r\gamma-\rho)H^{1-\gamma} + (1-\gamma)\left(H^{1-\gamma}\right)^{\frac{\gamma}{\gamma-1}}$$

$$= (1-\gamma)H^{-\gamma}H' + (r\gamma-\rho)H^{-\gamma}H + (1-\gamma)H^{-\gamma}.$$

Dividing by $(1-\gamma)H^{-\gamma}$ gives the desired equation for H.

d) The solution to $\partial_t H - \mu H + 1 = 0$, $H(T)=0$ is:

$$H(t) = \frac{1}{\mu}\left(1-e^{-\mu(T-t)}\right). \text{ So,}$$

$$G(t) = \left(\frac{1}{\mu}\right)^{\gamma-1}\left(1-e^{-\mu(T-t)}\right)^{1-\gamma}, \text{ and}$$

$$g(t) = e^{-\rho t}\left(\frac{1-\gamma}{\rho-r\gamma}\right)^{1-\gamma}\left(1-e^{-\frac{\rho-r\gamma}{1-\gamma}(T-t)}\right)^{1-\gamma}.$$

and $\alpha(x,t) = \frac{1-\gamma}{\rho-r\gamma}\left(1-\exp\left(-\frac{\rho-r\gamma}{1-\gamma}(T-t)\right)\right)x$.

2) a) Note that $v(x,t) = e^{\rho t}u(x,t)$, where

$$u(x,t) = \max_{\vec{\alpha}}\left(\int_t^T e^{-\rho s}h(y(s),\alpha(s)) + e^{-\rho T}g(y(T))\right).$$

So, u is just like the utility in a), except that $e^{-\rho T}g(x)$ is the final utility, the running utility h is general, and $\partial_s \vec{y}(s) = f(\vec{y}(s),\vec{\alpha}(s))$ is a more general ODE. This leads to the HJB equation

$$\partial_t u + \max_{\vec{\alpha}}\left(f(\vec{x},\vec{\alpha})\cdot\nabla u + e^{-\rho t}h(\vec{x},\vec{\alpha})\right) = 0 \qquad (**)$$

$u(x,T) = e^{-\rho T}g(x)$. So,

$$\partial_t v = \rho e^{\rho t}u(x,t) + e^{\rho t}\partial_t u(x,t),$$

$$\Rightarrow \partial_t u(x,t) = e^{-\rho t}\left(\partial_t v(x,t) - \rho v(x,t)\right), \text{ and}$$

$$\nabla u(x,t) = e^{-\rho t}\nabla v(x,t).$$

The desired equation follows from $(**)$ by dividing by $e^{-\rho t}$.

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2b) First we remark that

$r = s - \tau$

$$\tilde{V}(\vec{x}, t+\tau) = \max_{\vec{\alpha}} \int_{t+\tau}^{\infty} e^{-\rho(s-t-\tau)} h(\vec{y}(s), \vec{\alpha}(s))\, ds \overset{\downarrow}{=}$$

$$= \max_{\vec{\alpha}} \int_{t}^{\infty} e^{-\rho(r-t)} h(\vec{y}(r+\tau), \vec{\alpha}(r+\tau))\, dr = (*).$$

Here, $\vec{y}$ solves $\partial_s \vec{y}(s) = f(\vec{y}(s), \vec{\alpha}(s))$ with $\vec{y}(t+\tau) = \vec{x}$. $(**)$

Now let $\vec{\alpha}_0$ be any control from the set of which we take the maximum in $(*)$. Let $\vec{y}_0$ be the solution of $(**)$ with that $\vec{\alpha}_0$, and let $\vec{y}_{00}(s) = \vec{y}_0(s+\tau)$. Then $\vec{y}_{00}$ solves

$$\partial_s \vec{y}_{00}(s) = \partial_s \vec{y}_0(s+\tau) = f(\vec{y}_0(s+\tau), \vec{\alpha}_0(s+\tau)) = f(\vec{y}_{00}(s), \vec{\alpha}_0(s+\tau)), \quad (s \geq t)$$

with $\vec{y}_{00}(t) = \vec{x}$. By assumption, the control $s \mapsto \vec{\alpha}_0(s+\tau)$, $(s \geq t)$, is allowed. Thus for each allowed control $\vec{\alpha}_0$ in $(*)$ we have found an allowed control $\vec{\alpha}_{00}$ such that $h(\vec{y}_0(r+\tau), \vec{\alpha}_0(r+\tau)) = h(\vec{y}_{00}(r), \vec{\alpha}_{00}(r))$.

Thus $(**) \leq \max_{\vec{\alpha}_{00}} \int_{t}^{\infty} e^{-\rho(r-t)} h(\vec{y}_{00}(r), \vec{\alpha}_{00}(r))\, dr = \tilde{V}(\vec{x}, t)$.

So $\tilde{V}(\vec{x}, t+\tau) \leq V(\vec{x}, t)$. The same argument shows $\tilde{V}(\vec{x}, t-\tau) \leq V(\vec{x}, t)$, and equality follows.

Now since $\tilde{V}$ does not depend on t, the term $\partial_t \tilde{V}$ from a) vanishes, and we get $-\rho \tilde{V} + H(\vec{x}, \nabla_{\vec{x}} \tilde{V}) = 0$.