PDE Finance, Sheet 3, Solutions

① a) By the solution formula,

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{2t}} \mathbb{1}_{\{y>0\}}\, dy =$$

$$= \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} e^{-\frac{|x-y|^2}{2t}}\, dy \overset{\boxed{z=x-y}}{=} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-\frac{|z|^2}{2t}}\, dz =$$

$$\boxed{\begin{array}{l} y = \frac{z}{\sqrt{t}} \Rightarrow dz = \sqrt{t}\, dy \\ z = -\infty \Rightarrow y = -\infty \\ z = x \Rightarrow y = \frac{x}{\sqrt{t}} \end{array}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{t}} e^{-\frac{|y|^2}{2}}\, dy = N\left(\frac{x}{\sqrt{t}}\right).$$

b) $$\partial_x u(x,t) = \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2t}} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}.$$

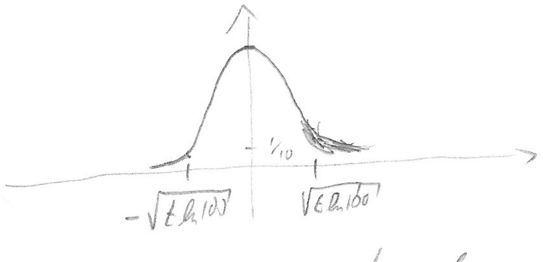
maximal at $x=0$; value $\frac{1}{\sqrt{2\pi t}}$, $\xrightarrow[t\to 0]{} \infty$

$$\partial_x u(x,t) \le \frac{1}{10} \max \partial_x u(x,t) \iff e^{-\frac{|x|^2}{2t}} \le \frac{1}{10} \iff$$

$$\iff \frac{|x|^2}{2t} \le \ln 10 \iff |x| \le \sqrt{2t \ln 10} = \sqrt{t \ln 100} \approx$$

$$\approx 4.6\sqrt{t}.$$

Graph:



c) $v(x,t) = \int_{-\infty}^{x} u(y,t)\, dy$. Differentiation under the integral (allowed for $\varepsilon \le t < \infty$, any ε) gives $\partial_t v(x,t) = \int_{-\infty}^{x} \partial_t u(y,t)\, dy$

$$\overset{PDE}{=} \int_{-\infty}^{x} \partial_y^2 u(y,t)\, dy = \partial_y^2 \int_{-\infty}^{x} u(y,t)\, dy = \partial_y^2 v(x,t).$$

So v solves the heat eqn.

Since $\lim_{t\to 0} u(x,t) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$,

we need $\lim_{t\to 0} v(x,t) = \int_{-\infty}^{x} 1_{\{y>0\}}\,dy = \max(0,x)$.

This is the initial condition.

The approach to 0 as $x\to-\infty$ is very fast, faster than exponential:

For any $\gamma>0$, there is $C\in\mathbb{R}$ such that if $|x|>C$, then $e^{-\frac{|x|^2}{2t}} < e^{-\gamma|x|}$. So, $u(x,t) < \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-x/\sqrt{t}} e^{-\gamma|y|}\,dy = \frac{1}{2\pi\gamma} e^{-\frac{\gamma}{\sqrt{t}}x}$

$\Rightarrow v(-x,t) < \int_{-\infty}^{-x} u(y,t)\,dy \leq \int_{-\infty}^{-x} \frac{1}{2\pi\gamma} e^{-\frac{\gamma}{\sqrt{t}}|y|}\,dy = \frac{\sqrt{t}}{2\pi\gamma^2} e^{-\frac{\gamma}{\sqrt{t}}|x|}$.

So indeed, v decays to 0 faster than $e^{-\gamma|x|}$ for any $\gamma>0$.

For $x\to\infty$, $u(x,t)\longrightarrow$ const, so $v(x,t)\sim$ const $\cdot\, x$. It grows linearly.

$v(0,t) = \int_{-\infty}^{0} u(y,t)\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dy \int_{-\infty}^{y/\sqrt{t}} e^{-\frac{z^2}{2t}}\,dz$. This looks horrible, but

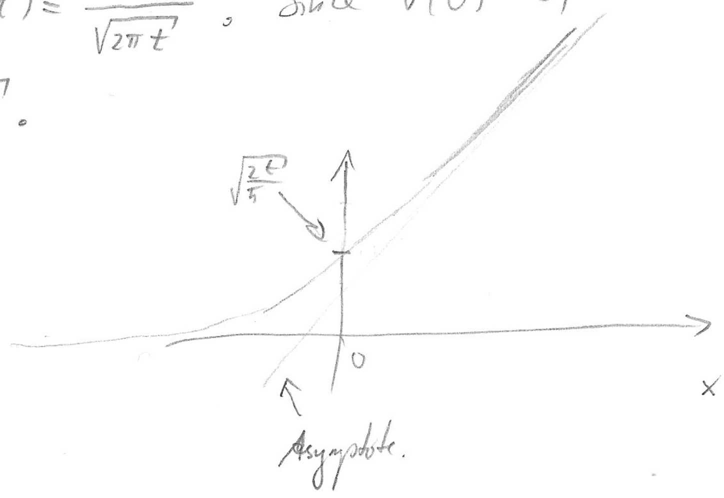
we can use a better idea: remember that v solves $\partial_t v(x,t) = \partial_x^2 v(x,t)$.

But $\partial_x^2 v(x,t) = \partial_x u(x,t) = \partial_x \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{x} e^{-\frac{|z|^2}{2t}}\,dz = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}}$.

For $x=0$, we find $\partial_t v(0,t) = \frac{1}{\sqrt{2\pi t}}$. Since $v(0)=0$, this means

$v(0,t) = \int_0^t \frac{1}{\sqrt{2\pi s}}\,ds = \sqrt{\frac{2}{\pi}}\sqrt{t}$.

Graph of v:

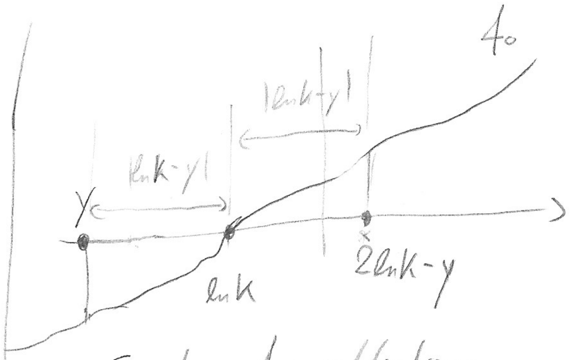


2 a) Initial condition is

$$u(y,0) = e^{-\frac{1}{2}(k-1)y}(e^y - K)^+ = u_0(y)$$

b) The initial condition is zero if $e^y < K$, or equivalently if if $y < \ln K$. We convert the heat equation on the half space to one on full space by reflecting the b.c. across $y = \ln K$, i.e. the new b.c. is

$$f_0(y) = u_0(y) - u_0(2\ln K - y).$$



Equation for reflection:
$y < \ln K < y'$, y and y' same distance from $\ln K$
$\Rightarrow \ln K - y = y' - \ln K \Rightarrow$
$\Rightarrow y' = 2\ln K - y.$

By symmetry, the solution to

$$\partial_\tau u - \partial_y^2 u = 0, \quad (y \in \mathbb{R}) \qquad (*)$$
$$u(y,0) = f_0(y)$$

is equal to zero at $y = \ln K$.

So its part for $y > \ln K$ solves the boundary value problem.

c) Both heat equation and BSPDE are linear, thus if f_1, f_2 solve one of them, then $f_1 + f_2$ also does.

So to get P, the solution of the BSPDE, we can (by b)):

1) Solve the full space heat eqn. with initial conditions

$$u_0(y) \text{ and } u_0(2\ln K - y).$$

Call the solutions $u_1(y,\tau)$ and $u_2(y,\tau)$, respectively.

2) Transform them back: since $u(y,\tau) = P(e^y, T - \frac{2}{\sigma}\tau)e^{-\alpha y - \beta\tau}$,

we find

$$P_1(x,t) = e^{+\alpha y(x)} e^{+\beta \tau(t)} u_1(\underbrace{y(x)}_{=\ln x}, \tau(t)) \overset{(*)\ x=e^y}{=} C(x,t), \text{ where}$$

$C(x,t)$ is the standard solution formula for the BSPDE for a European vanilla call.

$$P_2(x,t) = e^{\alpha y(x) + \beta \tau(t)} u_2(y(x), \tau(t)) =$$

$$= e^{\alpha y(x) + \beta \tau(t)} u_1(2\ln k - y(x), \tau(t))$$

$$= e^{\alpha y(x) + \beta \tau(t)} e^{-\alpha(2\ln k - y(x)) - \beta\tau(t)} C(e^{2\ln k - y(x)}, t)$$

$$= K^{-2\alpha} e^{2\alpha \ln x} C\left(\frac{K^2}{x}, t\right) = \left(\frac{x}{k}\right)^{1-k} C\left(\frac{K^2}{x}, t\right)$$

$\boxed{3}$ a) By the reflection trick, we can solve

$$\partial_t w = \frac{1}{2}\partial_x^2 w, \quad w(x,0) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

So, $$w(x,t) = \frac{1}{\sqrt{2\pi t}} \int e^{-\frac{|x-y|^2}{2t}} \left(1_{\{y>0\}} - 1_{\{y<0\}}\right) dy$$

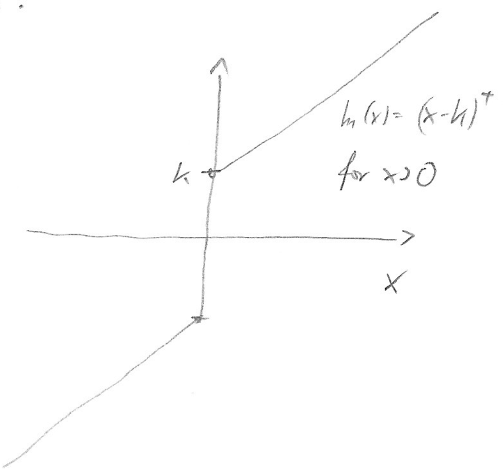
$= u(x,t) - u(-x,t)$, where u solves the equation of question 1a).

b) Same trick:

$$w(x,t) = \frac{1}{\sqrt{2\pi t}} \int e^{-\frac{|x-y|^2}{2t}} \left((y-k)^+ - (-y-k)^+\right) dy$$

$= v(y,t) - v(-y,t)$, v as in 1b).

c) Here we need to „cut" the initial condition at $x=0$, but otherwise it is the same trick:



Put $h_k(x)=(x-K)^+\,1_{\{x>0\}}$. Then

$$w(x,t)=\frac{1}{\sqrt{2\pi t}}\int e^{-\frac{|x-y|^2}{2t}}\big(h(y)-h(-y)\big)\,dy.$$

Near $(0,0)$ this becomes very steep!

d) Here we consider $\tilde{w}(x,t)=w(x,t)-1$.

Then $\tilde{w}$ still solves $\partial_t\tilde{w}=\frac{1}{2}\partial_{xx}\tilde{w}$, and

$\tilde{w}(0,t)=0$, $\tilde{w}(x,0)=-1$. So, by the reflection trick,

$$\tilde{w}(x,t)=-\int e^{-\frac{|x-y|^2}{2t}}\big(1_{\{y>0\}}-1_{\{y<0\}}\big)\,dy=u(-x)-u(x).$$

So $w(x,t)=u(-x)-u(x)+1$.

Interpretations: a) see sheet. b) European knockout call for Brownian motion with knockout value 0 and strike $k>0$.

c) Same as b) with $K<0$

d) Option that pays 1 if asset goes below 0, and nothing otherwise.