PDE in Finance

Solutions to exercise sheet 2

$\boxed{1}$ For $d=2$, we have $\partial_x F(\vec{x}) = -\frac{1}{2\pi}\, \partial_x \left(\ln \left(\sqrt{x^2+y^2} \right)\right) =$

$$= -\frac{1}{2\pi} \frac{1}{\sqrt{x^2+y^2}}\, \partial_x \left(\sqrt{x^2+y^2}\right) = -\frac{1}{2\pi} \frac{x}{x^2+y^2}.$$

So $\partial_x^2 F(\vec{x}) = -\frac{1}{2\pi}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right)$

In the same way, we see

$$\partial_y^2 F(x,y) = -\frac{1}{2\pi}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right),$$

and thus

$$\Delta F(x,y) = -\frac{1}{2\pi}\left(\frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} \right) = 0.$$

For $d \geq 3$, we find

$$\partial_{x_i} F(\vec{x}) = \frac{1}{n(n-2)\,d(n)}\, \partial_{x_i} \frac{1}{\left(\sum_{j=1}^{n} x_j^2\right)^{(n-2)/2}} = \frac{1}{n(n-2)\,d(n)} \frac{-\left(\frac{n-2}{2}\right)\cdot 2x_j}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n-2}{2}+1}}$$

$$= \frac{-1}{n\,d(n)} \frac{x_j}{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2}}, \quad \text{and}$$

$$\partial_{x_i} F(\vec{x}) = \frac{-1}{n\,d(n)} \left(\frac{1}{\left(\sum_{j=1}^{n} x_j^2\right)^{n/2}} - \frac{\frac{n}{2}\cdot 2x_j^2}{\left(\sum_{j=1}^{n} x_j^2\right)^{\frac{n+2}{2}}} \right)$$

So $\Delta F(\vec{x}) = -\frac{1}{n\,d(n)}\left(\frac{n}{|\vec{x}|^n} - \frac{n\,|\vec{x}|^2}{|\vec{x}|^{n+2}} \right) = 0.$

2 The reasoning is again like last week's:

$$u(\vec{x}) = \mathbb{E}_{\vec{y}_0=\vec{x}}\left(u(\vec{y}_0)\right), \text{ and}$$

$$u(\vec{y}_{J(\vec{x})}) = \mathbb{E}_{y_0=\vec{y}_{J(\vec{x})}}(\ldots) = \Phi(\vec{y}_{J(\vec{x})}).$$

$$\text{So, } 0 = \mathbb{E}_{\vec{y}_0=\vec{x}}\left(\int_0^{J(\vec{x})} \psi(\vec{y}_s)\,ds + \underbrace{\Phi(\vec{y}_{J(\vec{x})})}_{=u(\vec{y}_{J(\vec{x})})} - u(\vec{y}_0)\right)$$

$$= \mathbb{E}_{\vec{y}_0=\vec{x}}\left(\int_0^{J(\vec{x})} \psi(\vec{y}_s)\,ds + \int_0^{J(\vec{x})} du(\vec{y}_s)\right).$$

We know that $du(\vec{y}_s) = \underbrace{\partial_t u}_{\equiv 0}(\vec{y}_s)\,ds + \mathcal{L}u(\vec{y}_s)\,ds +$

$$+ \sum_{i,j=1}^{n} g_{ij}(\vec{y}_s)(\partial_{x_i} u)(\vec{y}_s)\,dW_s^{(j)},$$

with $\mathcal{L}u = \sum_{i=1}^{n} f_i \partial_{x_i} u + \frac{1}{2}\sum_{i,j,k=1}^{n} g_{ik} g_{jk} \partial_{x_i}\partial_{x_j} u$,

by the Itō formula. The expectations of the stochastic integrals vanish, so u must solve

$$\mathcal{L}u + \psi = 0 \quad \text{on } D$$
$$u = \Phi \quad \text{on } \partial D.$$

[3] Assume that u solves

$$x \cdot \nabla u = \alpha u$$

Define $v: \mathbb{R} \to \mathbb{R}$ by $v(\lambda) = u(\lambda x)$.

Then $\frac{d}{d\lambda} v(\lambda) \overset{\text{chain rule}}{=} x \cdot \nabla u(\lambda x) = \frac{1}{\lambda}(\lambda x) \cdot \nabla u(\lambda x) =$

$$\overset{\text{PDE}}{=} \frac{1}{\lambda} \alpha u(\lambda x) = \frac{\alpha}{\lambda} v(\lambda).$$

So v solves the ODE $v' = \frac{\alpha}{\lambda} v$, with initial value $v(1) = u(1 \cdot x) = u(x)$. The unique solution to this ODE is $v(\lambda) = v(1) \cdot \lambda^\alpha$, and thus we must have

$$u(\lambda x) = v(\lambda) = v(1)\lambda^\alpha = \lambda^\alpha u(x).$$

[4] a) The PDE can be written as

$$b \cdot \nabla u = u \qquad \text{with } b = (1,1).$$

Finding a curve such that $\dot{\gamma}(s) = b$ is easy:

$$\gamma(s) = (s + r, s) \quad \text{for any } r \in \mathbb{R}.$$

Along γ, we find

$$\frac{d}{ds} v(s) := \frac{d}{ds} u(\gamma(s)) = b \cdot \nabla u(\gamma(s)) = u(\gamma(s)) \equiv v(s),$$

so $v(s) = e^s \cdot \text{const}$. By using

$v(0) = u(\gamma(0)) = u(r,0) \overset{\text{Bdary cond.}}{=} f(r)$,

we obtain $v(s) = e^s f(r)$.

In the original coordinates, [since $x = s+r$, $y = s \Rightarrow$
$\Rightarrow s = y$ and $r = x - s = x - y$], we have finally

$$u(x,y) = e^y f(x-y).$$

b) The initial data $u(x,x) = 1$ is on one of the curves $\gamma(s) = (s+r, s)$ from a), namely for $r=0$. We have

$\frac{d}{ds} v(s) = v(s)$ and $v(s) = u(\gamma(s)) = u(s,s) = 1$ for all s.

This is impossible (as $v'(s) = v(s)$ only admits $v(s) = c e^s \neq 1$), so there is no solution.

c) Same as in b), but now we get

$\frac{d}{ds} v(s) = v(s)$ and $v(s) = 0$. This has the solution $v(s) = 0$ for all s. But for all other lines $\gamma(s) = (s+r, s)$ $(r \neq 0)$ we have no prescribed value, and thus can choose any solution

$$v(s) = c(r) e^s, \text{ where } c(r) \text{ is arbitrary.}$$

So the solution is not unique.