MA908 Partial Differential Equations in Finance

Exercise Sheet 8: Stability and consistency of numerical schemes

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1. **The weighted average method**
Consider the heat equation $\partial_t u = \frac{\sigma^2}{2}\partial_x^2 u$ on $[0,1]$ with zero boundary conditions and initial condition $u_0(x)$. In the lecture we have seen the forward scheme (with $\mu = \frac{\sigma^2}{2}\frac{h}{h_x^2}$),

$$u_j^{n+1} - u_j^n = \mu(u_{j-1}^n + u_{j+1}^n - 2u_j^n),$$

and the backward scheme

$$u_j^{n+1} - u_j^n = \mu(u_{j-1}^{n+1} + u_{j+1}^{n+1} - 2u_j^{n+1}).$$

A natural next step is to mix the two schemes: for $0 \leqslant \theta \leqslant 1$ we put

$$u_j^{n+1} - u_j^n = \mu\Big(\theta(u_{j-1}^n + u_{j+1}^n - 2u_j^n) + (1-\theta)(u_{j-1}^{n+1} + u_{j+1}^{n+1} - 2u_j^{n+1})\Big)$$

is the *weighted average scheme.*

(a) Write the mixed scheme in the form

$$B_1 \boldsymbol{u}^{n+1} = B_2 \boldsymbol{u}^n,$$

with matrices B_1 and B_2. You should write B_1 and B_2 as a sum of the identity matrix and a multiple of the discrete Laplacian A from the lecture.

(b) Using the knowledge of the eigenvalues and eigenvectors of the discrete Laplacian, investigate for which θ the weighted average scheme is unconditionally stable, i.e. for which θ the matrix $B_1^{-1}B_2$ has no eigenvalues of absolute value greater than 1, for any value of μ. What does this imply for the approximations $\boldsymbol{u}^n$ to the true solution under the scheme?

(c) Investigate the order of consistency for the weighted average scheme. For this, compute the truncation error

$$T(x_j, t_{n+1/2}) = \frac{1}{h}\Big(u(x_j,t_{n+1}) - u(x_j,t_n)\Big)$$
$$-\frac{\sigma^2}{2h_x^2}\Big(\theta(u(x_{j-1},t_n) + u(x_{j+1},t_n) - 2u(x_j,t_n)) + (1-\theta)(u(x_{j-1},t_{n+1}) + u(x_{j+1},t_{n+1}) - 2u(x_j,t_{n+1}))\Big).$$

Here $u(x,t)$ is the true solution of the PDE. You should calculate $T(x_j,t_{n+1/2})$ by Taylor expanding $u(x,t)$ around $(x_j,t_{n+1/2})$, and using the approximate values of the expansion for the $u(x_j,t_n)$, $u(x_j,t_{n+1})$ etc. appearing in the truncation error. Use the PDE to cancel some terms.

(d) Show that for $\theta = \frac{1}{2}$ the scheme is second order consistent (you should have seem this as a result of the above question). This particular scheme is the famous *Crank-Nicholson scheme.*

2. **Irregularly spaced discretisation points:** Consider again the heat equation $\partial_t u = \partial_x^2 u$ on the interval $[0,1]$. Instead of disctretising x by regularly spaced points $\{(jh_x) : 0 \leqslant j \leqslant J\}$, we can also use arbitrary points

$$0 = x_0 < x_1 < x_2 < \ldots < x_J = 1.$$

The heat equation on the interval is then approximated by the forward scheme

$$\frac{1}{h}(u_j^{n+1} - u_j^n) = \frac{2}{\delta x_{j-1} + \delta x_j}\left(\frac{1}{\delta x_j}(u_{j+1}^n - u_j^n) - \frac{1}{\delta x_{j-1}}(u_j^n - u_{j-1}^n)\right),$$

where $\delta x_k = x_{k+1} - x_k$. Show that the truncation error is given by

$$T_j^n = \frac{h}{2}\partial_t^2 u(x_j,t_n) - \frac{1}{3}(\delta x_j - \delta x_{j-1})\partial_x^3 u(x_j,t_n) - \frac{1}{12}\Big((\delta x_j)^2 + (\delta x_{j-1})^2 - \delta x_j \delta x_{j-1}\Big)\partial_x^4(x_j,t_n) + \ldots,$$

where '...' means terms that get smaller faster when the δx_k and h go to zero. When does the term with the 1/3 prefactor vanish?