MA908L Partial Differential Equations in Finance

Exercise Sheet 0: Test your skills!

Volker Betz January, 2012.

---

1. **Differentiation**

   (a) Differentiate the following functions:

   $$f(x) = \mathrm{e}^{-x^2}, \quad f(x) = a^x, \quad f(x) = x^{(x^2)}, \quad f(x) = \frac{1}{1+x^2}$$

   (b) **Multidimensional chain rule:** Let $F : \mathbb{R}^2 \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Write down the multidimensional chain rule for $\frac{d}{dt}F(f(t), g(t))$. Now apply this to compute the derivative to $h(t) = \int_0^t \mathrm{e}^{-st} \cos(s)\,\mathrm{d}s$, and more generally to

   $$h(t) = \int_0^t \phi(s,t)\,\mathrm{d}s, \qquad h(t) = \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \phi(s-s').$$

   (c) **The gradient:** For $f : \mathbb{R}^n \to \mathbb{R}$ the gradient is defined as

   $$\nabla f(\boldsymbol{x}) = \Big(\partial_{x_1} f(\boldsymbol{x}), \dots, \partial_{x_n} f(\boldsymbol{x})\Big) \in \mathbb{R}^n.$$

   (Here, $\boldsymbol{x} = (x_1, \dots, x_n)$.) Let $\ell_a = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) = a\}$ be the level line of level a to f. Prove that $\nabla f(\boldsymbol{x})$ is perpendicular to a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ if and only if γ is a level line.

   (d) **The Laplace operator:** For a function $f : \mathbb{R}^n \to \mathbb{R}$, the Laplace operator is defined through

   $$\Delta f(\boldsymbol{x}) = \sum_{i=1}^n \partial_{x_i}^2 f(\boldsymbol{x}).$$

   Calculate the result of applying the Laplace operator to

   $$(x,y) \mapsto f(x,y) = \mathrm{e}^{-\frac{x^2+y^2}{2t}},$$

   where $t > 0$ is a parameter. Try to find a function $h(t)$ such that

   $$\partial_t h(t)\,\mathrm{e}^{-(x^2+y^2)/2t} = \frac{1}{2}\Delta_{(x,y)} h(t)\,\mathrm{e}^{-(x^2+y^2)/2t}.$$

   (e) **Harmonic functions:** A function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ with $\Delta f(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in D$ is called harmonic in D. Can you find easy nonzero harmonic functions in all dimensions? Now show that in three dimensions, the function $f(x) = \frac{1}{|x|}$ is harmonic for all $x \neq 0$. Here

   $$|x| = \sqrt{x_1^2 + \ldots + x_n^2}$$

   is the Euclidean norm in $\mathbb{R}^n$. Confirm that $1/|x|$ is not harmonic in dimensions different from three.

2. **Itô calculus:**
   Recall Itô's formula: Assume that a stochastic process $\boldsymbol{y}_t = (y_t^{(1)}, \dots, y_t^{(n)})$ satisfies the stochastic differential equation

   $$\mathrm{d}y_t^{(i)} = F_i(\boldsymbol{y}_t, t)\mathrm{d}t + G_i(\boldsymbol{y}_t, t)\,\mathrm{d}\mathcal{W}_t^{(i)},$$

   where $F_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $G_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are smooth enough functions, and $\mathcal{W}_t^{(i)}$ are independent Brownian motions. Then for a smooth function $h : \mathbb{R}^n \times \mathbb{R}$, the stochastic process $h_t = h(\boldsymbol{y}_t, t)$ solves the stochastic differential equation

   $$\mathrm{d}h_t = \partial_t h(\boldsymbol{y}_t, t)\,\mathrm{d}t + \sum_{j=1}^n \partial_{x_j} h(\boldsymbol{y}_t, t)\,\mathrm{d}y_t^{(j)} + \frac{1}{2}\sum_{j,k=1}^n (\partial_{x_j}\partial_{x_k} h(\boldsymbol{y}_t, t))\,\mathrm{d}\boldsymbol{y}_t^{(j)}\mathrm{d}\boldsymbol{y}_t^{(k)}.$$

   In the last term above, recall that $(\mathrm{d}\mathcal{W}^{(j)})^2 = \mathrm{d}t$ while all other mixed terms are zero.

(a) Apply the Itô-formula to

$$dy_t = F(y_t)\,\mathrm{d}t + G(y_t)\mathrm{d}\mathcal{W}_t$$

and $h(y_t) = \mathrm{e}^{y_t}$. How do we have to choose F and G to obtain geometric Brownian motion

$$\mathrm{d}h_t = \mu h_t \mathrm{d}t + \sigma h_t \mathrm{d}\mathcal{W}_t$$

with constant drift μ and constant variance σ?

(b) Now apply Itôs formula to

$$y_t^{(i)} = \sigma \mathcal{W}_t^{(i)},$$

and general twice differentiable $h : \mathbb{R}^n \to \mathbb{R}$. Can you give a condition so that the usual chain rule $\mathrm{d}h(\boldsymbol{y}_t) = \sum_{j=1}^n \partial_{x_j} h(\boldsymbol{y}_t)\,\mathrm{d}y_t^{(j)}$ holds in this case?

(c) Consider again geometric Brownian motion

$$\mathrm{d}y_t = \mu y_t \mathrm{d}t + \sigma y_t \mathrm{d}\mathcal{W}_t.$$

Can you identify the stochastic process $h_t = 1/y_t$?