2.3. **The heat equation.** A function $u(x,t)$ solves the *heat equation* if (with $\sigma > 0$)

$$\partial_t u - \frac{1}{2}\sigma^2 \Delta u = 0. \tag{2.10}$$

In the simplest case, (2.10) is supposed to hold for all $x \in \mathbb{R}^n$, and all $t > 0$, and there is an initial condition $u(x,0) = u_0(x)$. We have seen this equation already: if in (1.3) we put $F = 0$ and $G = \sigma^2$, then we obtain (2.10). In other words, the heat equation is the Kolmogorov backward equation for Brownian motion. It is one of the most important equations in physics, as it models heat flow (hence the name), diffusion of liquids, and many more things.

From Black-Scholes to heat: We will now show that the BSPDE can be transformed into a heat equation by a change of variables. Recall the BSPDE given in (1.9):

$$\partial_t P + \frac{1}{2}\sigma^2 x^2 \partial_x^2 P + b(x\partial_x P - P) = 0, \tag{2.11}$$

with final condition $P(x,T) = \Phi(x)$. To understand how anybody could guess the variable transform that we are going to use, note that in (2.11), x need to be positive, as it is a stock price; and, that we have a final condition at T. In contrast, in (2.10), we have $x \in \mathbb{R}$ and an initial condition. So the least we would have to do to connect the two is to invert time, and to map the nonnegative x into something on all of $\mathbb{R}$. The latter is just what the logarithm does, and an additional hint for uisng it would be that geometric BM behaves like the exponential of BM itself.

After these explanations, the following transformation may seem a bit less arbitrary: we put

$$y = \ln x \text{ (so } x = \mathrm{e}^y \text{)}, \quad \text{and } \tau = \frac{1}{2}\sigma^2(T-t).$$

Then we put

$$v(y,\tau) = P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau) \qquad (= P(x,t)).$$

Let us try whether v solves the heat equation:

$$\partial_\tau v(y,\tau) = -\frac{2}{\sigma^2}\partial_2 P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau) = -\frac{2}{\sigma^2}\partial_t P(x,t),$$

$$\partial_y v(y,\tau) = \mathrm{e}^y\, \partial_1 P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau) = x\partial_x P(x,t),$$

$$\partial_y^2 v(y,\tau) = (\mathrm{e}^y)^2 \partial_1^2 P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau) = \mathrm{e}^y\, \partial_1 P(\mathrm{e}^y, T - \frac{2}{\sigma^2}\tau) = x^2\partial_x^2 P(x,t) - x\partial_x P(x,t).$$

Above, $\partial_1 P$ means the function that one gets from P by differentiating with respect to the first argument. Note that this is different from $\partial_y P(\mathrm{e}^y, \ldots)$ since this would invoke a chain rule, and also better than $\partial_x P(\mathrm{e}^y, \ldots)$, where the reader is asked to guess that the first argument is somehow connected to the letter x. This example shows that the inherited notation for derivatives is not satisfactory (it uses a dummy variable explicitly), but unfortunately it is very deeply entrenched in mathematics and there is no hope to overcome it.

Back to the calculation. We find

$$\begin{aligned}\partial_\tau v - \partial_y^2 v &= -\frac{2}{\sigma^2}\partial_t P - x^2\partial_x^2 P - x\partial_x P\\ &= -\frac{2}{\sigma^2}\left(\partial_t P + \frac{\sigma^2}{2}x^2\partial_x^2 P + \frac{\sigma^2}{2}x\partial_x P\right)\\ &= -\frac{2}{\sigma^2}\left(-b(x\partial_x P - P) + \frac{\sigma^2}{2}x\partial_x P\right)\\ &= -\frac{2}{\sigma^2}\left((-b+\sigma^2/2)\partial_y v + bv\right).\end{aligned}$$

So v solves

$$\partial_\tau v - \partial_y^2 v + (1-\frac{2b}{\sigma^2})\partial_y v + \frac{2b}{\sigma^2}v = 0. \tag{2.12}$$

This is not quite yet the heat equation. To proceed, let us put $k = 2b/\sigma^2$, and

$$u(y,\tau) = \mathrm{e}^{-\alpha y - \beta\tau}\, v(y,\tau),$$

thus $v(y,\tau) = \mathrm{e}^{\alpha y + \beta\tau}\, u(y,\tau)$. Then (2.12) becomes

$$(\beta u + \partial_\tau u) - (\alpha^2 u + 2\alpha\partial_y u + \partial_y^2 u) + (1-k)(\alpha u + \partial_y u) + ku = 0.$$

Then $\partial_y u$ terms vanish if $-2\alpha + (1-k) = 0$, and the u terms vanish if $\beta - \alpha^2 + (1-k)\alpha + k = 0$. This gives

$$\alpha = \frac{1-k}{2}, \qquad \beta = -\frac{(k+1)^2}{4}.$$

With this choice of α, β, the function u indeed solves the heat equation

$$\partial_\tau u - \partial_y^2 u = 0, \qquad u(y,0) = \mathrm{e}^{\frac{1}{2}(k-1)y}\,\Phi(\mathrm{e}^y). \tag{2.13}$$

So to solve the BS-PDE, we have to solve (2.13) to get u, then get v from u, and then undo the change of variables to find $P(x,t) = v(\ln x, \frac{1}{2}\sigma^2(T-t))$. This will work for any payoff-function Φ, provided we can solve the heat equation with the corresponding initial condition. This we can indeed do:

Solution for the whole-space heat equation:

The function

$$f(\boldsymbol{x},t) = \frac{1}{(2\pi\sigma^2 t)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2\sigma^2 t}}\, f_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \tag{2.14}$$

solves the heat equation (2.10) with initial condition $f(\boldsymbol{x},0) = f_0(\boldsymbol{x})$. The function

$$F(\boldsymbol{x},t) = \frac{1}{(2\pi\sigma^2 t)^{n/2}}\,\mathrm{e}^{-\frac{|\boldsymbol{x}|^2}{2\sigma^2 t}}$$

is called *fundamental solution* of the heat equation. It actually also solves the heat equation for $t > 0$, except when $\boldsymbol{x} = 0$.

Remarks: F is the transition density of Brownian motion, i.e. $\mathbb{P}_0(\mathcal{W}_t \in A) = \int_A F(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}$. This is no accident, but is related to the time reversibility of Brownian motion and the Kolmogorov equations. We will not discuss this further in the present lecture.

Also, we need to place some restrictions on the initial condition f_0 for the solution to make sense. In fact, f_0 need not be continuous, but it must not grow too fast at infinity. If $f_0(\boldsymbol{x}) \leqslant M\,\mathrm{e}^{c|\boldsymbol{x}|^2}$, then the solution (2.14) at least exists for finite time. If $f_0(\boldsymbol{x}) \leqslant M\,\mathrm{e}^{c|\boldsymbol{x}|^{2-\delta}}$, for some small $\delta > 0$, then the solution (2.14) exists for all times. Translating the latter condition back to the Black-Scholes coodinates gives

$$u(y,0) = \mathrm{e}^{-\frac{1}{2}(k-1)y}\,\Phi(\mathrm{e}^{y}) \leqslant M\,\mathrm{e}^{c|y|^{2-\delta}} \Leftrightarrow x^{-\frac{1}{2}(k-1)}\Phi(x) \leqslant M\,\mathrm{e}^{c(\ln x)^{2-\delta}}.$$

This works fine if $\Phi(x) < |x|^r$ for any $r > 0$, but will fail if Φ grows exponentially at infinity.

2.4. **Solution of the heat equation on a half space.** This solution will be useful for the pricing of barrier options. We want to solve

$$\begin{aligned} &\partial_t u = \frac{1}{2}\partial_x^2 u = 0 \qquad \text{for } t > 0, x > 0. \\ &u(x,0) = g(0), \quad u(0,t) = \phi(t). \end{aligned} \tag{2.15}$$

It seems at first that the boundary data needs to fit together for this equation to make sense; more precisely, when $\lim_{t\to 0}\phi(t) \neq \lim_{x\to 0} g(x)$, then it seems that we want on the one hand a function that is twice differentiable in x and differentiable in t (for the PDE to make sense), but on the other hand is discontinuous at the boundary. We will however see that this is not a problem. To solve (2.15), let us split it into two easier problems.

Proposition: Assume that v solves

$$\partial_t v = \frac{1}{2}\partial_x^2 v, \qquad v(x,0) = g(x), v(0,t) = 0, \tag{2.16}$$

and that w solves

$$\partial_t w = \frac{1}{2}\partial_x^2 w, \qquad w(x,0) = 0, w(0,t) = \phi(t). \tag{2.17}$$

Then $u = v + w$ solves (2.15).

Proof. It is clear that u fulfils the boundary conditions, and that it solves the PDE follows from the fact that the derivatives can be dsitributed onto v and w, who individually solve the heat equation. □

It will turn out to be an advantage if in (2.17), we have $\lim_{t\to 0}\phi(t) = 0$; then the formula for the solution will be easier to make sense of. This can be easily achieved: we replace $\phi(t)$ with $\phi(t) - \phi(0)$ in (2.17), and $g(x)$ with $g(x) - \phi(0)$ in (2.16). To the solution $\tilde{u}$ that we obtain from this we only have to add $\phi(0)$, which then solves the original equation.

Solution of (2.16):

We use a reflection trick: Let us put

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0, \\ -g(-x) & \text{if } x < 0. \end{cases}$$

We now solve the whole space problem with initial condition $\tilde{g}$, using (2.14). The result is

$$v(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{|x-y|^2}{2t}} \tilde{g}(y)\,\mathrm{d}y = (*).$$

We now change integration variables from y to $-y$, with the result

$$(*) = \frac{1}{\sqrt{2\pi t}} \int_{\infty}^{-\infty} \mathrm{e}^{-\frac{|x+y|^2}{2t}} \tilde{g}(-y)\,\mathrm{d}y == \frac{-1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{|(-x)-y|^2}{2t}} \tilde{g}(y)\,\mathrm{d}y = -v(-x,t)$$

So, the solution has the same symmetry as the boundary condition for all times! In particular, $v(0,t) = -v(0,t)$, which only leaves the possibility $v(0,t) = 0$. Thus v restricted to $x > 0$ indeed solves (2.16).
The above solution can be written in terms of the fundamental solution $F(x,t) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$. Namely, simple manipulations show that

$$v(x,t) = \int_0^{\infty} G(x,y,t) g(y)\,\mathrm{d}y, \qquad \text{with } G(x,y,t) = F(x-y,t) - F(x+y,t).$$

Note the striking similarity to the Greens function we found in the solution to (2.6). G is indeed the Greens function for the heat equation. This will become clear when we consider the
Solution to (2.17)**:**
As (2.17) is a bit like the boundary value problem (2.6), we can expect a similar solution formula, and there is indeed one: the function

$$w(x,t) = \int_0^t \partial_y G(x,y,t-s)|_{y=0} \phi(s)\,\mathrm{d}s \tag{2.18}$$

solves (2.17). The partial derivative of G can of course be computed, leading to

$$w(x,t) = \int_0^t \frac{x}{(t-s)\sqrt{2\pi(t-s)}} \mathrm{e}^{-\frac{x^2}{2(t-s)}} \phi(s)\,\mathrm{d}s.$$

The proof of this formula goes roughly as follows: Since F solves the heat equation, so does $(x,t) \mapsto G(x,y,t)$ for all y, and also $(x,t) \mapsto \partial_y G(x,y,t-s)$ for all s (just exchange the order of derivatives). But when two or more functions solve the heat equation, (or, any linear equation), then all weighted sums of these function solve the same equation (just distribute the derivatives), and this even applies to convergent sums of infinitely many terms, and even integrals. So w as given by (2.18) does solve the heat equation. For the boundary conditions: naively, $w(x,0) = 0$ as the range of the integration is zero. However, we have to approach this limit coming from positive t, which makes it less trivial. Likewise, the limit of w as $x \to 0$ needs to be studied carefully, and it needs to be shown that it converges to $\phi(t)$. This is beyond the scope of the present lecture and will not be done here.
Pricing a barrier option
A barrier option changes its value suddenly when the asset process $\boldsymbol{y}_t$ hits a predefined barrier. For example, a down-and-out call with barrier X will be worthless if the stock falls below X before maturity. Otherwise, it behaves like a normal call.

Of course, such options are very little different from gambling in a casino, and encourage massive market manipulation to temporarily suppress a stock price, and should not be legal. But this is not out concern here, we are trying to price them, assuming that no manipulation takes place. In that case, interestingly, the Black-Scholes PDE gives a fair price, so they are not fundamentally different from vanilla options. The procedure goes like this: We start with the BSPDE with boundary condition zero at asset value X. We do the variable transform to turn this into a heat equation with zero boundary condition at a suitably modified place. We then solve this heat equation using the theory above. Finally we transform back to the Black-Scholes coordinates. The result for a down-and-out call with barrier X is

$$V(x,t) = V_0(\ln x, \frac{1}{2}\sigma^2(T-t)) - \Big(\frac{x}{X}\Big)^{1-k} V_0(\ln \frac{X^2}{x}, \frac{1}{2}\sigma^2(T-t)),$$

with $k = 2r/\sigma^2$, and where V_0 is the value of a vanilla option with the same strike price. The details will be worked out on an exercise sheet. Just notice that the formula makes sense: when the stock price x is much larger than the barrier price X, the price is almost that of a vanilla option, while it is almost zero if the stock price x is close to X.