INVARIANT TENSORS AND DIAGRAMS

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Abstract. In this paper we first give three known examples of strict pivotal categories defined by a finite presentation. Then in the final section we give some of the relations for a conjectural strict pivotal category. If this category can be constructed it gives a ribbon category which specialises to the category of representations of the quantised enveloping algebra of a simple Lie algebra for any simple Lie algebra. This is the quantum group version of the universal Lie algebra introduced in [52] (available on his home web page).

1. Introduction

This paper is a contribution to the conjecture of Deligne on the exceptional series of Lie groups. This conjecture and some supporting numerology is given in [7], [12], [13].

The starting part for this conjecture is the observation that the dual Coxeter number parametrises the five exceptional Lie groups by five points on a line and there are some remarkable rational functions which specialise to the dimensions of irreducible representations at the five points. The conjecture of Deligne is that there is a Tannakian category which explains these dimension formulae. Since the conjecture was made the numerical evidence supporting it has been significantly strengthened. The most general dimension formulae are given in [14] and [36]. These dimension formulae are derived from the Weyl dimension formula and so also give quantum dimension formulae. The paper [53] studies the R-matrices which satisfy the Yang-Baxter equation (with spectral parameter) and gives further support for this conjecture. The nilpotent orbits in the exceptional groups are studied in [37] where it is shown that various numerical data associated to these orbits is well-behaved as a function of the parameter. Finally, the exceptional series was discovered independently in [15] by regarding the Potts model and dilute Potts model as one parameter families of (non-rational) conformal field theories and then studying a massive but integrable perturbation.

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In this paper we have in mind a modification of Deligne's conjecture. We conjecture that there is a ribbon category which explains the quantum dimension formulae and which specialises to give a positive solution to Deligne's conjecture. Our contribution is to define a ribbon category, essentially by a finite presentation, which has some of the properties required by the conjecture. In fact the main property which is required and that we are not able to establish is the property that every Hom-set is finitely generated as a module over the coefficients.

The original (and unpublished) work of Vogel suggests a stronger result. Namely, he introduces a projective plane with the symmetry group of permutations of the three coordinates and associates an orbit of this symmetry group to each simple Lie algebra. Then he shows that there are some universal representations derived from the adjoint representation which have dimensions given by rational functions. In this paper we give quantum analogues of these dimension formulae.

The relations that we discuss are the diagrammatic versions of $6j$-symbols. Thus one way of viewing the results in this paper is that we give some universal $6j$-symbols. Since the eigenvalues of the braid matrices are known this is equivalent to giving a six dimensional irreducible representation of the three string braid group. In this paper we have concentrated on the Vogel plane rather than exceptional series and the other rows of the Freudenthal magic square. One reason for this is that the exceptional series and the previous row are both lines in the Vogel plane and so the results for these cases can be obtained by specialisation. A second reason is that the representations of the three string braid groups in these cases have dimension at most five. These representations are given explicitly in [50] and this also determines the $6j$-symbols. A third reason is that the rows of the Freudenthal magic square are studied in [53] and the result that these R-matrices satisfy the Yang-Baxter can be interpreted as giving several relations of the form considered in this paper.

The idea of considering a category depending on parameters instead of a family is not new. The basic idea is to consider the following construction. Let V be a finite dimensional representation of a Hopf algebra H. Define a category $\mathrm{Inv}(V)$ to have objects $r = 0, 1, 2, \ldots$ and morphisms

$$\mathrm{Hom}(r, s) = \mathrm{Hom}_H(\otimes^r V, \otimes^s V)$$

This is a K-linear tensor category and if V is self-dual then it is pivotal. The centraliser algebras are the endomorphism algebras in this category, namely

$$\mathrm{End}(r) = \mathrm{End}_H(\otimes^r V)$$

For the groups $SL(n)$ let V be the fundamental representation of dimension n. Then the categories $\mathrm{Inv}(V)$ have been studied independently of n for a long time. For example, for all $n > 0$, the algebras $\mathrm{End}(\otimes^r V)$ are quotients of the group algebra of the symmetric group on r letters. This is known as Schur duality. These are symmetric tensor categories but are not rigid, since the representation V is not self-dual. There is a q-analogue of the category which is constructed using the HOMFLY skein relations. The centraliser algebras in this category are the Hecke algebras.

Similarly for the remaining classical groups let V be a finite dimensional vector space with a non-degenerate inner product which is either symmetric or anti-symmetric. Let G be the group of linear automorphisms of V which preserve the inner product. Then the categories $\mathrm{Inv}(V)$ can be studies in a uniform way using the Brauer category which has a parameter and the centraliser algebras are quotients of the Brauer algebras. This category has a q-analogue which is constructed using the Kauffman skein relations. The centraliser algebras in this case are quotients of the Birman-Wenzl algebras.

There is a new feature that appears in this example but not in the above examples. Let $\mathcal{F}$ be the category of framed tangles. Then if V is self-dual there is a pivotal functor $\mathcal{F} \to \mathrm{Inv}(V)$. Let K be the coefficient ring of $\mathrm{Inv}(V)$ and extend this to a K-linear functor $K\mathcal{F} \to \mathrm{Inv}(V)$. In the above examples this functor is surjective but in this paper the image is a proper pivotal subcategory. Denote this image by $\mathrm{Inv}'(V)$. This description of this subcategory is natural in the quantum group setting where a braid group acts on a tensor power whereas in the classical setting this only gives an action of the symmetric group. However there is an alternative description of the subcategory obtained in the classical limit. In this description tensor powers can be decomposed using idempotents in the group algebra of the symmetric groups and also by taking eigenspaces of the quadratic Casimir. The distinction between $\mathrm{Inv}(V)$ and $\mathrm{Inv}'(V)$ is particularly significant for the non-reductive groups and Lie algebras which occur in this paper.

The categories $\mathrm{Inv}(V)$ are defined as quotients of the free K-linear category on the category of framed tangles. The results of this paper can be viewed as skein relations in the sense that all the relations given in this paper can be rewritten as linear relations on framed tangles. However this is not the approach we adopt. Our approach is much closer in spirit to the work in [29] on rank two Lie algebras. The category we start with is a category of trivalent graphs with two types of edges and two types of trivalent vertex. Our relations are then linear relations on these diagrams. This gives a finite presentation of a ribbon category. This category has all the properties one would expect except that we have not so far been able to show that for each $r, s > 0$ the K-module $\mathrm{Hom}(r, s)$ is finitely generated.

The categories $\mathrm{Inv}(V)$ and $\mathrm{Inv}'(V)$ are not additive and do not have split indempotents However there are two formal extensions which construct categories with these properties and which are universal for extensions with these properties. In particular since these constructions are natural if the original category has additional structure such as being tensor, pivotal, braided, or ribbon then these extensions will have the same properties. These two constructions are discussed in [20, Chapter 2. Exercises].

The first process extends the category to an additive category. This new category has objects lists of objects and morphisms given by rectangular arrays of morphisms. The original category can be regarded as a subcategory of this extension. This inclusion has the universal property that any functor from the original category to an additive category factors through this extension.

Next let C be an additive category and e an idempotent in the ring $\mathrm{End}_C(c)$ for some object c. Then e is split if there are morphisms $c \to d$ and $d \to c$ such that $c \to d \to c = e$ and $d \to c \to d = 1$. There is a category with objects pairs (c, e) where c is an object of C and e is an idempotent in $\mathrm{End}_C(c)$. The morphisms are given by

$$\mathrm{Hom}(e, e') = e\,\mathrm{Hom}_C(c, c')e'$$

This category is an extension of C since there is a natural inclusion of C which sends c to $(c, 1)$. All idempotents split in this extension and this extension is universal with this property.

2. Diagram calculus

The idea of using a diagram notation for invariant tensors is now well-established. This is fundamental for the applications of quantum groups to low dimensional topology. This was the original motivation for this work and is described in [47], [48] and [51]. However this idea was previously used in particle physics, for example in [28], [9], [10], [10]. This idea was then used by Penrose in his work on spin networks, see [6] and [27]. More recently, it has been shown in [17], how to define the Young projection operators using diagrams and then establish their basic properties. This idea was also known in the particle physics community where it seems to have been regarded as an aid to organise calculations rather than as a fundamental point of view. In this paper we present an alternative and show that it is possible to perform calculations without reverting to the components.

In this paper we discuss abstract tensor categories. These are now well-established in the mathematical community. Since our approach may also be of interest in the particle physics community we now give an indication of how this paper might be rewritten in an index notation. One potential application to particle physics is that the main aim of [45] is to give a generalisation of pure lattice gauge theory where the role of the gauge group is played by a Tannakian category.

In the classical approach to tensors a tensor is a multi-dimensional array of scalars. A tensor of valence (p, q) is denoted by

$$T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}$$

where each index takes a value from 1 to n. An equivalent way of saying this is that we choose an ordered basis B of the vector space V. Then T is a function from B^{p+q} to the ring of scalars. These scalars are then required to transform in a specified way under a change in the choice of ordered basis. These transformation rules distinguish the upper and lower indices. This is the simplest case where there is a single vector space V. More generally one has a number of vector spaces $\{V(\lambda) : \lambda \in \Lambda\}$ and then we choose an ordered basis $B(\lambda)$ for each vector space $V(\lambda)$. The valence of the tensor now consists of two ordered lists of elements of Λ, $[\lambda_1\lambda_2 \dots \lambda_p]$ and $[\lambda_1\lambda_2 \dots \lambda_q]$ which are called the upper and lower indices. Given two such ordered lists a tensor T of this valence is now taken to be the multidimensional array corresponding to a function from

$$(B(\lambda_1) \times B(\lambda_2) \times \dots \times B(\lambda_p)) \times (B(\lambda_1) \times B(\lambda_2) \times \dots \times B(\lambda_q))$$

to the ring of scalars. Again these scalars are required to transform in a specified way under a change in the choice of any of the ordered bases. These transformation rules distinguish the upper and lower indices.

Then there are number of operations one can perform with tensors. First the set of tensors with fixed valence is a module over the ring of scalars. This means that any linear combination of tensors all of the same valence and with scalar coefficients is also a tensor of the same valence. The second operation is the outer product or the tensor product. This operation takes two tensors of arbitrary valence and constructs a third tensor whose components are all products of the components of the two original tensors. The valence of this tensor is given by concatenating lists. This operation is bilinear over the ring of scalars. The other basic operation is contraction. Let T be a tensor with upper indices $\lambda(T)$ and lower indices $\mu(T)$ and let S be a tensor with upper indices $\lambda(S)$ and lower indices $\mu(S)$. Then if we are given lists μ, ν and λ such that we have factorisations

$$\mu(T) = \mu.\nu \quad \text{and} \quad \lambda(S) = \nu.\lambda$$

then we can form the contraction along ν to get a tensor with upper indices $\lambda.\lambda(T)$ and lower indices $\mu.\mu(S)$. Similarly if we have factorisations

$$\mu(T) = \nu.\mu \quad \text{and} \quad \lambda(S) = \lambda.\nu$$

then again we can contract along ν and this gives a tensor with upper indices $\lambda.\lambda(T)$ and lower indices $\mu(S).\mu$. The outer product can be seen as the special case of either of these by taking ν to be the empty list.

The final operation that we include is the duality opertaion. This is a generalisation of taking the transpose of a matrix. This operation is an involution since taking the dual of the dual gives back the original tensor. First we have an involution on the set of indices, called the dual. Then the upper indices of the dual are given by taking the duals of the lower indices in reverse order and similarly the lower indices are given by taking the duals of the upper indices in reverse order.

In the situation we are interested in all the vectors spaces $\{V(\lambda) : \lambda \in \Lambda\}$ are finite dimensional representations of an algebraic group. Then the algebraic group acts on each module of tensors of a given valence and a tensor is called invariant if it is invariant under this action. For each valence we only consider the submodule of invariant tensors. These submodules are closed under the operations of outer product, contraction and taking the dual.

The definition of an abstract strict tensor category is an abstraction of this situation in which we now longer require that there is a vector space $V(\lambda)$ for $\lambda \in \Lambda$. Instead we simply take Λ as an indexing set. Then we take the same notion of valence and we are required to specify a module of the ring of scalars for each valence. Then we are required to specify operations of outer product and contraction. These operations are further required to satisfy conditions which formally express the properties of these operations in the situation in which we do have components. This is equivalent to the definition of a linear strict tensor category. The composition is the special case of contraction in which $\mu(T) = \nu = \lambda(S)$ and the tensor product is the outer product. Conversely given the category we can define the operations of contraction by taking the composition of $\mathrm{Id}(\lambda) \otimes T$ with $S \otimes \mathrm{Id}(\mu)$.

In this paper we consider categories of invariant tensors not just as tensor categories but as pivotal categories. The main additional data this requires is a duality operation. A pivotal category requires additional data which in the index notation correspond to raising and lowering operators. The definition of a pivotal category and definitions of the closely related concepts of autonomous and sovereign categories is given in [21].

Now we come to the issue of calculations. The calculations we are interested in take the following form. We are given a finite list of invariant tensors. Then the operations of outer product and contraction can be used to construct further tensors. More specifically, these operations can be used to construct totally contracted tensors. This means that all indices are contracted or equivalently that the valence is given by two empty lists. Any such tensor is in fact a scalar and is independent of any choice of basis. Our objective is to give a method for computing these scalars. Of course if the tensors are in fact invariant tensors of an algebraic group then these scalars can be computed use the components and many calculations have been carried out by this method. In this paper we give some examples of cases where these calculations can be carried out using tensor identities. The idea is that in a particular example we may know some identities between invariant tensors. This means that we know of some linear combinations of invariant tensors all of the same valence which are zero. Then if we are asked to evaluate some totally contracted tensor we might try to simplify this expression using the known tensor identities before we resorted to using the components. In this paper we study some examples in which we can show that we can use a finite set of tensor identities to evaluate any totally contracted tensor.

A description of the diagrammatic notation for tensor calculus and a discussion of the advantages and disadvantages is given in [46, Chapter 2 and Appendix]. If we compare his situation with ours we note the following differences. First Penrose is considering what in this paper would be called tensor fields. From our point of view this means that the entries of a tensor are functions. This is not an obstacle but does mean that Penrose also has various operations such as the covariant derivative and the Lie derivative which are are not applicable to our algebraic context. In our context we may have more than one vector space. The main point though is that in this application the diagrams are simply an alternative notation for the usual tensor notation. These are two notations and any expression in one notation can easily be translated to an equivalent expression in the other language. Consequently the choice of notation is simply a matter of convenience.

The contents of this paper consist of the following. The first section gives a summary of the construction of a diagram category as a free strict pivotal category. The next two sections give the best known examples of finitely presented strict pivotal categories. These are associated with the classical groups. These examples have been included to illustrate the general methods. The novelty is that these categories are usualy constructed using skein relations which gives braided categories. Here we give presentations in terms of planar diagrams which does not give the braiding. The third section discusses the construction of the category of invariant tensors of the exceptional simple Lie group G_2. This example is important as the presentation can be shown to be confluent. The main novelty here is that we give a bijection between the irreducible diagrams and lattice walks. The proof we sketch that this is a bijection requires no knowledge of representation theory (except that we expect the reader to recognise that this is the expected answer). The final section gives some numerology for all simple Lie algebras. This is work in progress. The idea of this project is to regard these as relations for a finitely presented strict pivotal category. The conjecture is that these relations together with further relations that remain to be found define a strict pivotal category such that every Hom set is a finitely generated free module. This would imply the conjecture in [12] by specialising.

3. Diagram categories

In this section we introduce diagram categories. This construction gives examples of abstract strict pivotal categories. This discussion is based on [22].

Definition 3.1. *A graph is a finite one complex, Γ, with a distinguished finite subset, $V(\Gamma)$, called vertices such that the complement $\Gamma/V(\Gamma)$ is a disjoint union of circles and open intervals. The closure of a component of the complement $\Gamma/V(\Gamma)$ is called an edge.*

If a vertex v has an open neighbourhood whose closure is the join of v with k distinct points then we say that v is a vertex of degree k. In this paper we will only consider graphs with vertices of degrees one and three. Denote the set of vertices of degree one by $\partial\Gamma$.

Definition 3.2. *A standard rectangle is a rectangle drawn on the page with two horizontal and two vertical edges. The horizontal edges are called the top and the bottom edge. Denote the rectangle by R and the union of the top and bottom edges by ∂R. Then a diagram is a piecewise-linear embedding $(\Gamma, \partial\Gamma) \to (R, \partial R)$. We require that all points of the complement $\Gamma/\partial\Gamma$ are mapped to points in the interior of the rectangle.*

FIGURE 1. Contraction

Specifically we are only allowing planar diagrams. Equivalence of diagrams is given by isotopy which preserves the edges of the rectangle setwise. This notion of equivalence means that a equivalence class of diagrams is combinatorial in the sense that it is determined by a finite amount of data.

Next we consider labelling of graphs. Let Λ be a set with an involution $*$. In the examples the set Λ is finite. Then a labelling of a graph consists of an orientation of each edge together with a label from Λ. If we reverse the orientation on an edge and replace the label by the dual label then we obtain an equivalent labelling. In practice if an edge has a self-dual label then it is not given an orientation.

Then labelled diagrams are the morphisms of an strict pivotal category. In the category theory language; the composition of two morphism is given by putting one rectangle on top of the other; the tensor product is given by putting two rectangles side by side; the dual of a morphism is given by rotating the diagram through half a revolution. In terms of the contraction operations these are given by putting one rectangle on top of the other but without aligning the sides as shown in Figure 1.

Instead of allowing all labellings we specify a set of allowed labellings at each trivalent vertex. The labelled diagrams such that the labelling at each trivalent vertex is one of the allowed labellings is a pivotal subcategory. Our point of view is that these should be considered as free strict pivotal categories on the set of trivalent vertices with allowed labelling. The justification for this is that they have the following universal property. Suppose that we have specified a finite set Λ with an involution and a finite set of labelled trivalent verticesi, T. Then the above discussion constructs from this finite data set a strict pivotal category $\mathcal{G}$. Let $\mathcal{P}$ be any strict pivotal category. Assume we are given a map of sets with involution from Λ to the objects of $\mathcal{P}$ and a map of sets from T to the morphisms of $\mathcal{P}$ such that each trivalent vertex is sent to a tensor with the correct valence. Then these set maps extend uniquely to a strict pivotal functor $\mathcal{G} \to \mathcal{P}$. This result has not been stated in this form in the literature but the proof is given by using the methods of [54] and [22].

Now that we have a notion of a free strict pivotal category we also have the notion of a finitely presented strict pivotal category. In this paper our relations and categories are linear so we first take the free linear category on the free strict pivotal category. This simply means that we allow formal linear combinations of allowed labelled diagrams. Then the relations consist of a finite set of morphism in this category. Then we take the intersection of all ideals which contain these relations and which are closed under the operations of taking the tensor product or contraction with an arbitrary tensor and taking duals. The category defined by the presentation is the quotient by this ideal and is a strict pivotal category by construction. Each of the next three sections gives a finite presentation.

Although this defines a category there is a further property which is required of a finite presentation in order for it to be of interest. This is the property that the module of tensors, or morphisms, of fixed valence should be a finitely generated module. In the examples in which this is known to hold these modules are also free. This could be imposed as a requirement or it could be relaxed to the condition that each of these modules is projective (or even just flat). If we let I be the trivial representation or identity object then $\mathrm{End}(I)$ is a commutative ring. In the present context a general element of this ring is a formal linear combination of totally contracted tensors. As a ring it is the polynomial ring on the set of connected irreducible totally contracted tensors. Thus the requirement that $\mathrm{End}(I)$ is a finitely generated module implies that all totally contracted tensors are reducible.

This condition is similar to the condition that a finitely presented algebra be finite dimensional. This condition is undecidable in general but may be shown to hold in special cases. One of the special cases in which this can be decided is if the finite presentation is confluent. This is a very powerful property of a finite presentation and if it holds it also gives a basis of the algebra. The definition of confluence can be extended to finite presentations of strict pivotal categories. The example of G_2 is an example of a confluent presentation.

4. Hecke algebras

This section is based on the skein relations for the HOMFLY polynomial given in [19] and [24]. These skein relations are relations on the category of oriented tangles. These relations determine a quotient which is a ribbon category by construction. The coefficient ring depends on two parameters. For each positive integer n there is a specialisation of this ring to the ring of Laurent polynomials $\mathbb{Z}[q, 1/q]$ and a functor of ribbon categories from the specialisation of the category to the category of representations of $U_q(\mathfrak{sl}(n))$, the quantised enveloping algebra of the simple Lie algebra $\mathfrak{sl}(n)$. These skein relations involve diagrams with crossings. Here we give alternative relations for planar diagrams. There is a different set of generators and relations given in [44].

The starting point for this alternative approach is the observation that there is an isomorphism

$$V \otimes V^* \cong \mathfrak{g} \oplus 1$$

where V^* is the dual representation, $\mathfrak{g}$ is the adjoint representation and 1 is the trivial representation.

Now consider trivalent graphs with two types of edges. One type of edge is oriented and corresponds to the fundamental representation. The other type of edge is unoriented and corresponds to the adjoint representation. There is just one type of trivalent vertex. The usual coefficient ring is given by taking, $\mathbb{Z}[q, z, 1/zq]$, the ring of Laurent polynomials in two indeterminates q and z. Here we take theven blow-up by adjoining an indeterminate δ and the relation

$$\delta(q - q^{-1}) = (z - z^{-1}) \tag{1}$$

This ring is a blow-up since we have two specialisations. If we invert $(q - q^{-1})$ then we can define

$$\delta = \frac{(z - z^{-1})}{(q - q^{-1})}$$

On the other hand if we specialise to $q = \pm 1$ and $z = \pm 1$ then we have the ring $\mathbb{Z}[\delta]$. There are elements $[an + b]$ which under these specialisations give

$$\frac{(z^a q^b - z^{-a} q^{-b})}{(q - q^{-1})} \qquad \text{and} \qquad a\delta + b$$

This avoids treating these as two special cases.

This ring has an involution given by $q \mapsto q^{-1}$, $z \mapsto z^{-1}$, and $\delta \mapsto \delta$. The category of invariant tensors is defined over the subring fixed by this involution. This is a pivotal category. In order to define a braided tensor category the larger ring is needed. In fact the pivotal category of invariant tensors is defined over the polynomial subring $\mathbb{Z}[(z + z^{-1}), \delta]$.

The basic relations are given in Figures 2 and 3.

The four diagrams in Figure 4 are four vectors in a three dimensional vector space. The symmetry requirements give the form of this relation and then the confluence requirement determines the coefficient.

(2)

(3)

(4)

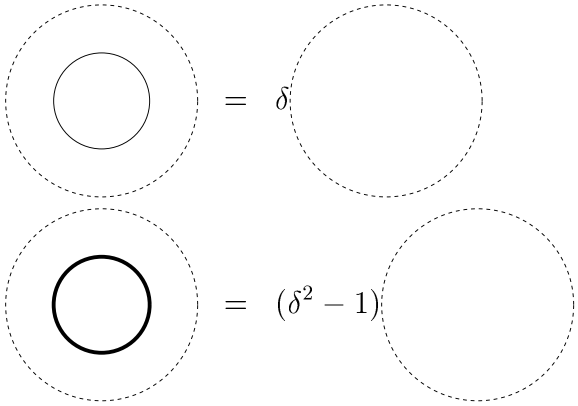


Figure 2. Closed loops

(5)

(6)

(7)

(8)

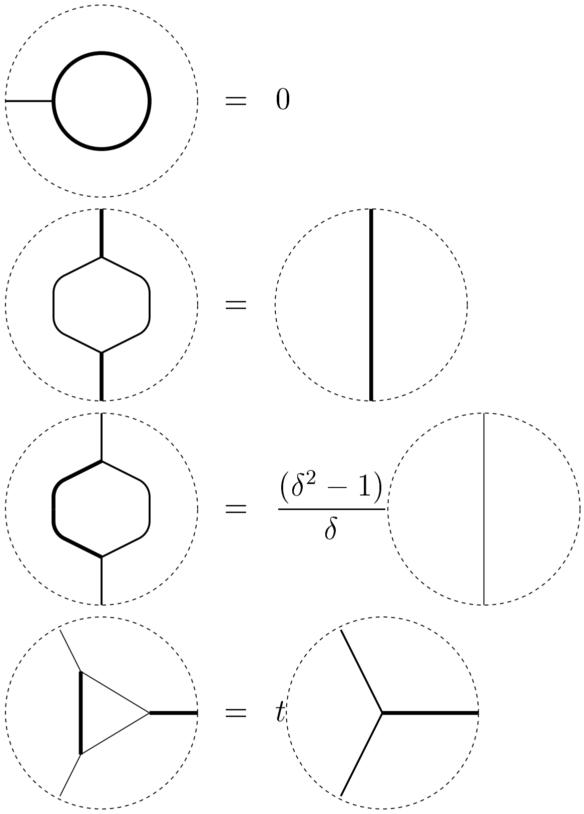


Figure 3. Basic relations

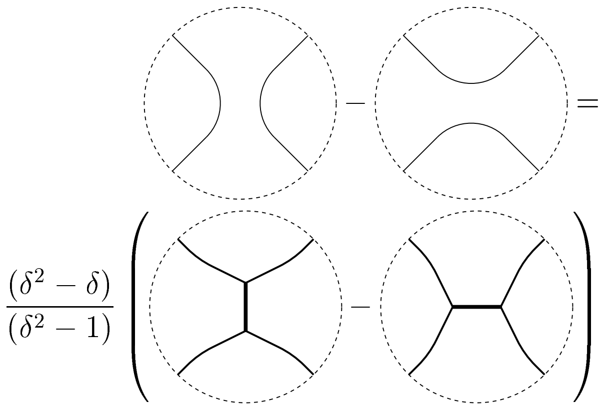


FIGURE 4. Linear relation

From this point of view the Hecke algebra is given by the following presentation. The generators of $H(n)$ are $H_1, H_2, \dots, H_{n-1}$ and the defining relations are

$$\begin{aligned} H_i^2 &= (z+z^{-1})H_i + (\delta^2-1) \\ H_iH_{i+1}H_i - H_i &= H_{i+1}H_iH_{i+1} - H_{i+1} \\ H_iH_j &= H_jH_i \end{aligned}$$

5. Birman-Wenzl-Murakami algebras

This is the second oldest and most studied example of categories of invariant tensors. Here we take a vector space V with a non-degenerate inner product which may be symmetric or anti-symmetric. Then the automorphism group is an orthogonal group if the inner product if the inner product is symmetric and is a symplectic inner product if the inner product is anti-symmetric. These are known as the classical groups. Then again the category of invariant tensors can be constructed in two stages. First construct the Brauer category. This is a linear category over the polynomial ring $\mathbb{Z}[\delta]$. Then the second optional stage is given by specialising δ to $\dim(V)$ if the inner product is symmetric and to $-\dim(V)$ if the inner product is anti-symmetric and also adding some further relations. The centraliser algebras are constructed in [4]. and independently by Murakami.

The more recent point of view on this is to construct the category of invariant tensors as the quotient of the category of unoriented tangles by the Kauffman (or Dubrovnik) skein relations. The centraliser algebras in this case are the Birman-Wenzl or Murakami algebras. In the

(9)

(10)

(11)

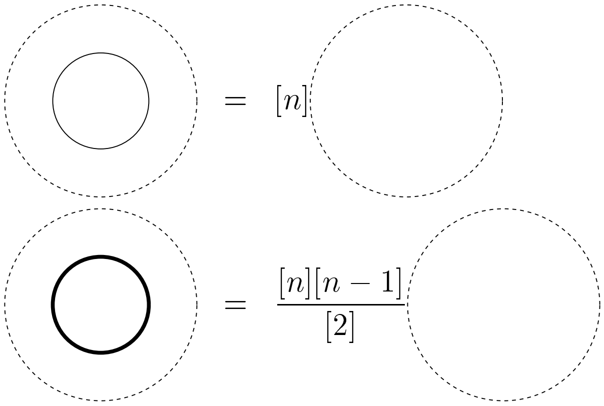


FIGURE 5. Closed loops

usual approach these algebras are defined over a two variable ring of Laurent polynomials. These algebras do not specialise to the Brauer algebras. Instead it is better to define the Birman-Wenzl algebras over the blow-up of the Laurent polynomial ring defined in (1).

In this section we revisit this approach. Our aim is to develop an equivalent description of the invariant theory but using trivalent graphs and not crossings. The starting point for this alternative approach is the observation that $V \otimes V$ decomposes into three distinct irreducible representations. One of these is the trivial representation as V admits an invariant inner product (by construction). Another of the irreducible representations is the adjoint representation so we consider trivalent graphs with two types of edge. One type of edge corresponds to the fundamental representation and the other to the adjoint representation. Since both representations are self-dual there are no orientations on the edges.

The relations are given in the following Figures 5 and 6.

The dimensions of the representations are given in [32], [16], [39], [33].

It is clear that there must be a linear relation between the four diagrams in the relation in Figure 6 since these are four vectors in the three dimensional vector space $\mathrm{End}(V \otimes V)$. Furthermore the relation must take this form since if we rotate all diagrams through a quarter revolution we must get the same relation. This just leaves one coefficient undetermined. This coefficient can be determined by confluence.

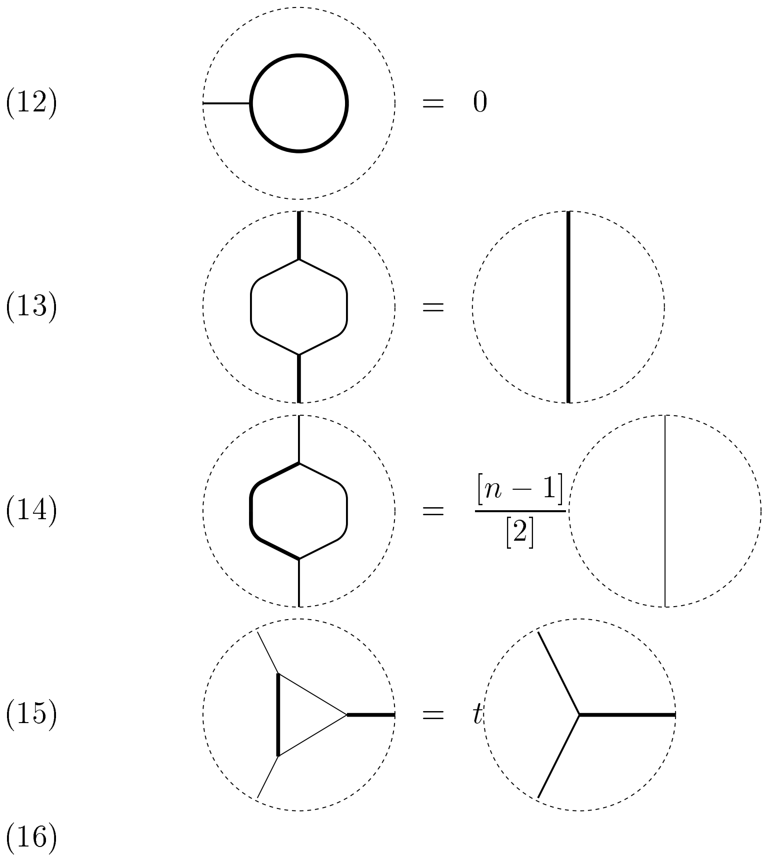


FIGURE 6. Basic relations

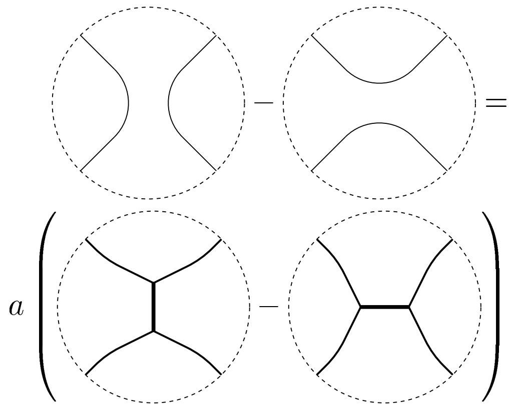
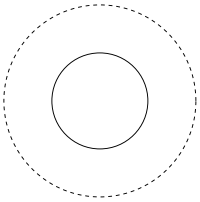


FIGURE 7. Linear relation



$$= (\delta^2 - 3)(\delta^3 + \delta^2 - 2\delta - 1)$$

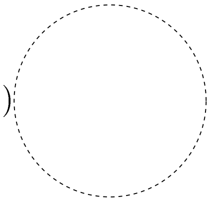


FIGURE 8. Closed loop

6. Trivalent graphs and G_2

In this section we take H to be the quantised enveloping algebra $U_q(G_2)$ and V to be the fundamental representation of dimension seven. This example was constructed by Greg Kuperberg in [29], [35], [25] (and [26]).

The original construction of G_2 came before the Cartan-Killing classification of simple Lie algebras. Let O be the Cayley numbers. These are an eight dimensional vector space with a (non-associative) multiplication. Any Cayley number can be written uniquely as the sum of a real Cayley number and an imaginary Cayley number. The real Cayley numbers are the scalar multiples of the unit and are a one dimensional vector space and the imaginary Cayley numbers are a seven dimensional vector space. Multiplying two imaginary Cayley numbers and taking the real part defines a positive definite inner product on the imaginary Cayley numbers. Multiplying two imaginary Cayley numbers and taking the imaginary part defines a multiplication on the imaginary Cayley numbers. The group of automorphisms of the imaginary Cayley numbers which preserves both of these is G_2 and the imaginary Cayley numbers are the fundamental representation V. The inner product and its adjoint give elements of $\mathrm{Hom}(2,0)$ and $\mathrm{Hom}(0,2)$. The multiplication and its adjoints give elements of $\mathrm{Hom}(3,0)$, $\mathrm{Hom}(2,1)$, $\mathrm{Hom}(1,2)$ and $\mathrm{Hom}(0,3)$. The result which this paper is based on is that these elements generate the category of invariant tensors as a linear monoidal category.

Next we define a category $\mathcal{K}$ by taking the free $\mathbb{Z}[\delta]$-linear category on $\mathcal{G}$ and imposing the following relations. These relations are due to Greg Kuperberg. They can be obtained by imposing confluence.

The square relation is given in Figure 10.

Then we have the pentagon relation which is given in [35] and which we do not reproduce here.

These relations say that whenever we see a disk in a diagram with the left hand side of the relation then we replace that disk by the linear

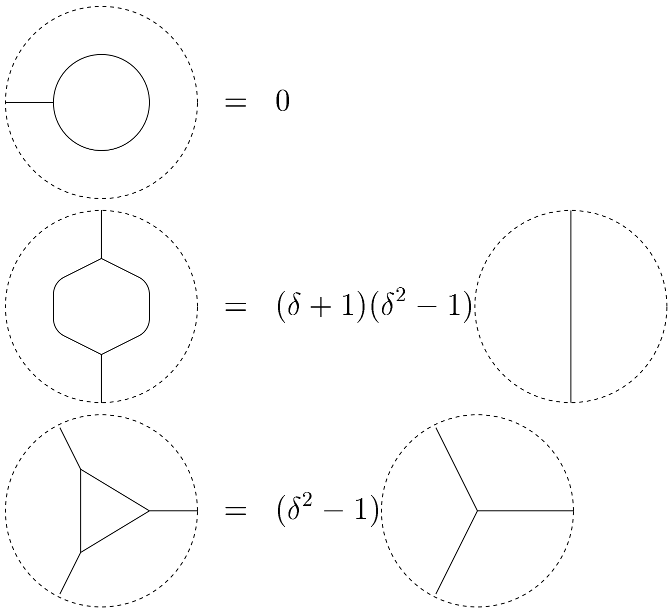


FIGURE 9. Basic relations

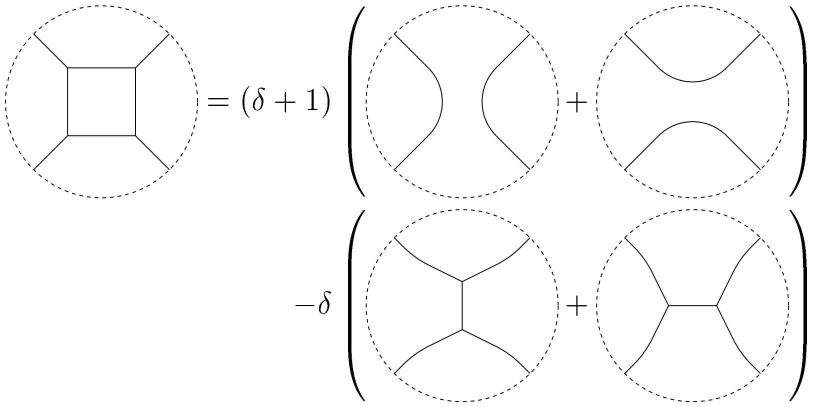


FIGURE 10. Square relation

combination of disks on the right hand side. Thus these relations should be regarded as rewrite rules for morphisms in $\mathcal{G}$.

6.1. **Curvature.** In this section we consider the dual picture. In this picture we replace each trivalent vertex by an equilateral triangle; and we assign length one to each edge. In effect we are considering a cobordism category of polyhedra. A reference for this subsection is [5].

Definition 6.1. *A manifold with polyhedral metric, or polyhedron, is a metric space every point of which has a neighbourhood isometric to some neighbourhood of a two-dimensional cone vertex.*

This neighbourhood may be a plane domain. In fact the points which do not have a plane neighbourhood are a discrete set and these points are called vertices.

Manifolds with boundary are defined similarly. Their boundary points have neighbourhoods isometric to plane angular sectors (of arbitrary angle).

Manifolds of this type may be specified by their development consisting of plane polygons with rules for identifying their sides in pairs.

If G is a polyhedron then we define $\tilde{G}$ to be the minimal metric completion and the boundary $\partial G = \tilde{G} - G$.

Let G be a polyhedron such that $\tilde{G}$ is compact and ∂G consists of a finte number of rectifiable curves. Denote the area of $\tilde{G}$ by F and the length of ∂G by L. The curvature at each point which has a planar neighbourhood is zero. Assume that the curvature K at each vertex satisfies $K \leq 0$.

Theorem 6.1. *Then the area F and the length of the boundary L satisfy the inequality*

$$L^2 \geq 4\pi\chi F$$

where χ is the Euler characteristic of G.

In order to apply this theorem we take the (n, m)-diagram as determining a development of a polyhedron. Take an equilateral triangle with sides of length one for each vertex of the trivalent graph. The edges of the triangle correspond to the edges of the trivalent graph which meet at the vertex. Then each edge of the trivalent graph corresponds to two edges of triangles and these two edges are identified. The polyhedron obtained by these identifications is $\tilde{G}$ and is a finite collection of discs. There is a disc for each component of Γ.

This does not take into account single edges. These should probably be considered as identifying their end points.

From this point of view a (n, m)-diagram is irreducible if and only if the corresponding polyhedron has non-positive curvature. If Γ is connected and has T trivalent vertices and L boundary points then applying the isoperimetric inequality gives

$$\sqrt{3}\pi T \leq L^2$$

This implies that the number of irreducible (n, m)-diagrams is finite.

6.2. **Cut paths.** In this section we introduce the notion of a cut path and show that the category $\mathcal{K}$ has a filtration.

Definition 6.2. *A cut path on a diagram is a path from the left hand edge of the rectangle to the right. This path may cross an edge transversally or it may follow an edge.*

A cut path which crosses a edges and follows b edges is assigned the weight (a, b). Weights are partially ordered by the dominance order. This means that the positive weights are those with both entries positive.

A cut path is minimal if there is no cut path with lower weight. The first result on cut paths is that in an irreducible diagram any two minimal cut paths are connected by a finite sequence of H-moves. This is a combinatorial version of the geometric result that on a manifold with non-positive curvature there is no focusing of geodesics. In particular, any two minimal cut paths have the same weight and so we define the weight of an irreducible diagram to be the weight of any minimal cut path.

Define $I(\lambda, \mu)(n, m)$ to be the subspace of $\mathrm{Hom}(n, m)$ spanned by all (n, m)-diagrams of weight at most (λ, μ). Then this defines an increasing sequence of two-sided ideals in the category $\mathcal{K}$.

This gives a cellular structure on the algebras. Here it is simpler to use the definition of a cellular algebra given in [34] rather than the equivalent original definition in [23].

6.3. **Plactic monoid.** Next we give a reconstruction theorem. This method was introduced in [31] (and [30]) for the A_2 case. Here we follow the same method but for G_2. This method is based on the plactic monoid which was introduced in [41] and further studied in [38].

Definition 6.3. *Let $D(n)$ be the finite set of irreducible $(n, 0)$-diagrams.*

Define the set S to consist of the following seven elements

$$(-2, 1), (-1, 1), (-1, 0), (0, 0), (1, 0), (1, -1), (2, -1)$$

Define a lattice walk of length n to be a sequence of $n+1$ dominant weights

$$\{(\lambda_i, \mu_i) : 0 \le i \le n\}$$

The first conditions are that $(\lambda_0, \mu_0) = (0, 0)$ and $(\lambda_n, \mu_n) = (0, 0)$. For $1 \le i \le n$, define s_i by

$$s_i = (\lambda_i - \lambda_{i-1}, \mu_i - \mu_{i-1})$$

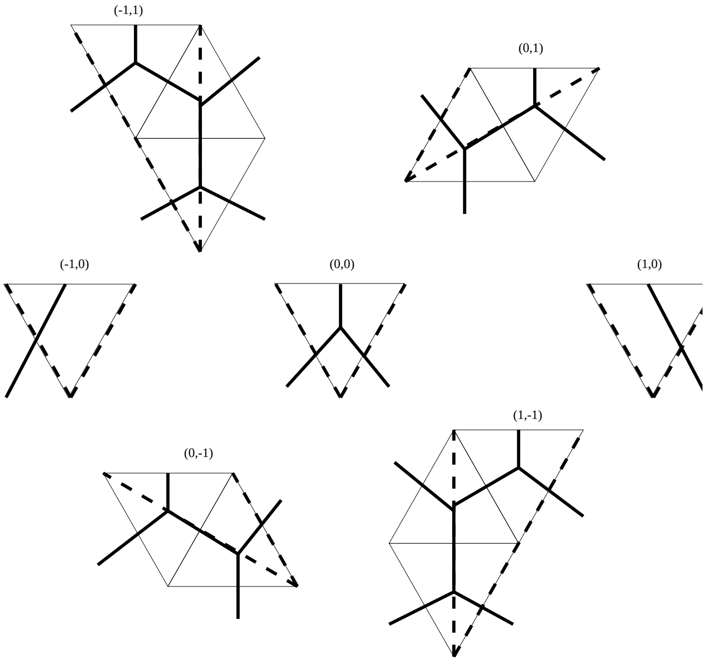


FIGURE 11. Seven steps

Then the conditions on these steps are that $s_i \in S$ for $1 \leq i \leq n$ and that if $\lambda_i = 0$ then $s_i \neq (0,0)$. Denote the set of lattice walks of length n by $W(n)$.

Given an irreducible $(n,0)$-diagram; then the top edge is divided into $n+1$ intervals by the n marked points. Choose a point P_i in each interval, for $0 \leq i \leq n$. Choose a point Q on the bottom edge. Now for $0 \leq i \leq n$ define (λ_i, μ_i) to be the weight of any minimal cut path from Q to P_i. Then the claim is that this is a lattice walk. This is a sequence of dominant weights by construction We have $(\lambda_0, \mu_0) = (0,0)$ because there is a cut path from Q to P_0 which does not meet the trivalent graph. This path essentially just follows the left hand edge of the rectangle. Similarly we have $(\lambda_n \mu_n) = (0,0)$ by following the right hand edge of the rectangle. Thus the main condition to be checked is that $s_i \in S$ for $1 \leq i \leq n$. Once this has been checked it is clear that if $\lambda_i = 0$ then $s_i \neq (0,0)$. This claim can be understood as combinatorial version of the isoperimetric inequality in [2].

The inverse of this map is called the growth algorithm. First define the plactic monoid to be the free monoid on the set S subject to the following relations. This monoid should be related to the plactic monoid discussed in [41].

$$
\begin{aligned}
(1,0)(-1,0) &= (0,0) \\
(1,0)(0,0) &= (1,0) \\
(0,0)(-1,0) &= (-1,0) \\
(1,0)(-1,1) &= (0,1) \\
(1,0)(0,-1) &= (1,-1) \\
(0,1)(-1,0) &= (-1,1) \\
(1,-1)(-1,0) &= (0,-1) \\
(1,-1)(-1,1) &= (0,-1)(0,1) \\
(0,1)(1,-1) &= (0,0)(1,0) \\
(-1,1)(0,-1) &= (-1,0)(0,0) \\
(1,-1)(0,0) &= (0,-1)(1,0) \\
(0,0)(-1,1) &= (-1,0)(0,1) \\
(1,0)(1,-1) &= (1,-1)(1,0) \\
(-1,1)(-1,0) &= (-1,0)(-1,1) \\
(0,0)(0,0) &= (-1,0)(1,0)
\end{aligned}
$$

The first observation is that there is a homomorphism from this monoid to the free abelian group on two generators. This map sends each element of S to the vector which describes the element. In fact this is the group completion of the plactic monoid. This just means that if we consider the above generators and relations as a group presentation then this defines the free abelian group on the two generators $(1,0)$ and $(0,1)$.

The proof of this is as follows. First we see that $(0,0)$ is the identity of the group. Then notice that $(1,0)$ and $(-1,0)$ are inverse. Now use the first few relations to eliminate the generators $(1,-1)$, $(-1,1)$. Then substitute these expressions into the remaining relations to obtain the relations that $(0,-1)$ is inverse to $(0,1)$ and that the two generators $(1,0)$ and $(0,1)$ commute.

One point of view is to consider these as relations on a monoid. Then we consider the set of words of length n which start and end at $(0,0)$ and which can be shortened to the word $(0,0)$ by replacing an occurence of a word on the left hand side by the corresponding word on the right hand side. However it seems more useful to think of this in terms of the theory of regular languages. A language is simply a subset of a free monoid so the lattice walks are a language. The words in this language can be recognised by an algorithm and so this is a regular language. Then we can regard the above relations as a grammar for this language. This means we start with the word $(0, 0)$ and consider the words that can be grown from this by substituting an ocurrence of a word on the right hand side by the corresponding word on the left hand side. It is clear that these are two descriptions of the same set of words.

Next we show that any lattice walk which ends at $(0, 0)$ is reducible. Given a sequence of reductions from a given path to the trivial path, this can be drawn as a diagram. First we have to show that this diagram is irreducible. Next we have to show that if we are given two such sequences then the two diagrams are isotopic. This then shows that we have defined a map from $W(n)$ to $D(n)$.

It now remains to show that these maps are inverse bijections. If we start from a lattice walk and and find a sequence of rewrite rules to reduce it to the trivial walk then this also gives a set of cut paths which recover the original lattice walk. This shows that the composite $W(n) \to D(n) \to W(n)$ is the identity map. In particular the map $W(n) \to D(n)$ is injective and the map $D(n) \to W(n)$ is surjective. The claim that the maps are inverse bijections amounts to saying that we have not left any relations out!

Finally we make some comments on this language from the point of view of complexity theory. First the grammar we have given is not context-free and it can be shown that this language is not context free. It is the intersection of two context-free languages.

The dimensions of the spaces of invariant tensors are

$$1, 0, 1, 1, 4, 10, 35, 120, 455, 1792, 7413, 31780, 140833, \dots$$

These are also the numbers of elements of the sets $W(n)$. Then numerical evidence suggests that this sequence is given asymptoticaly by

$$C7^n n^{-11/2}$$

where the constant C satisfies $8 < C < 9$. If correct this gives an indication of the improvement than be obtained by calculating the values of link invariants or other values of totally contracted tensors using the invariant tensors instead of the direct method of summing over indices. The dimension of the space of tensors is 7^n compare with the above prediction of $C7^n n^{-11/2}$ for the dimension of the space of invariant tensors. This suggests that there is a worthwhile saving but the asymptotic formula is dominated by the term 7^n.

7. Numerology of simple Lie algebras

In this section we let $\mathfrak{g}$ be a simple Lie algebra and consider it as a representation of the automorphism group, G. The automorphism group is considered as an algebraic group and the following is a description of the structure of this algebraic group. Let G_0 be a simple split algebraic group over $\mathbb{Q}$ of adjoint type with Lie algebra $\mathfrak{g}$. Let Γ be the finite group of automorphisms of the Dynkin diagram of G_0. Then Γ acts on G_0 and so the semi-direct product $\Gamma \times G_0$ is defined. This gives the automorphism group G.

The category of finite dimensional representations of G is semi-simple and symmetric monoidal so the exterior and symmetric squares of $\mathfrak{g}$ are defined and are direct sums of highest weight representations. The Lie bracket on $\mathfrak{g}$ gives a G-invariant projection from the exterior square of $\mathfrak{g}$ onto $\mathfrak{g}$ and the Killing form gives a G-invariant projection from the symmetric square of $\mathfrak{g}$ onto the trivial representation.

The starting point for the numerology of simple Lie algebras is that the following two statements hold whatever the type of G_0. The exterior square of $\mathfrak{g}$ is the direct sum of $\mathfrak{g}$ and a second irreducible representation and that the symmetric square of $\mathfrak{g}$ is the direct sum of the trivial representation and at most three other irreducible representations. These statements can be verified by a case by case check.

If we fix G_0 then we can write

$$S^2(\mathfrak{g}) \cong 1 \oplus Y(\alpha) \oplus Y(\beta) \oplus Y(\gamma)$$

This can be generalised. For $p \geq 2$, the tensor product $\mathfrak{g} \otimes \mathfrak{g}^p$ decomposes into the sum of the following seven highest weight representations

$$\mathfrak{g}^p \quad X\mathfrak{g}^{p-1} \quad X\mathfrak{g}^{p-2} \quad \mathfrak{g}^{p-1} \quad Y(\alpha)\mathfrak{g}^{p-1} \quad Y(\beta)\mathfrak{g}^{p-1} \quad Y(\gamma)\mathfrak{g}^{p-1}$$

Now define three numbers $\alpha(i)$ for $i = 1, 2, 3$ by

$$\alpha(i) = \langle \omega(i), \omega(i) + 2\rho \rangle / 2 - \langle \beta, \beta + 2\rho \rangle \tag{17}$$

where β is the highest root of G_0 which is the dominant weight of $\mathfrak{g}$, ρ is half the sum of the positive roots, and $\omega(i)$ is the highest weight of $Y(i)$ for $i = 1, 2, 3$.

The first significance of these parameters is that they give the eigenvalues of the R-matrix acting on $\mathfrak{g} \otimes \mathfrak{g}$. This is a result of Reshetikhin. However the numerology of simple Lie algebras is the observation that a lot of information on the representations of simple Lie algebras can be written in a simple way in terms of these numbers. For example, the quantum dimension of $\mathfrak{g}$ is given by

$$\dim_q(\mathfrak{g}) = -\frac{[1,2,2][2,1,2][2,2,1]}{[1,0,0][0,1,0][0,0,1]}$$

If we fix G_0 and write

$$\Lambda^2(\mathfrak{g}) \cong \mathfrak{g} \oplus X$$

then there is an irreducible representation of $U_q(G)$ corresponding to X and the quantum dimension of this representation is given by

$$\dim_q(X) = \frac{[1,1,2][1,2,1][2,1,1][1,2,2],[2,1,2],[2,2,1]}{[1,0,0][0,1,0][0,0,1][2,0,0][0,2,0][0,0,2]} \frac{[2,2,0],[2,0,2],[0,2,2]}{[1,1,0],[1,0,1],[0,1,1]} \tag{18}$$

Similarly the quantum dimension of $Y(\alpha)$ is given by

$$\dim_q(Y(\alpha)) = -\frac{[2,2,2][2,2,1][2,1,2][1,1,2][1,2,1][1,-2,-2]}{[2,0,0][1,0,0][0,1,0][0,0,1][1,-1,0][1,0,-1]}$$

There is in addition to these representations an infinite sequence of universal representations. These are universal in the sense that they are (virtual) representations of the automorhism group of any semi-simple Lie algebra. The first few terms of this sequence for the exceptional series were studied in [12] and [43]. This sequence Here we define a representation $X_k(\mathfrak{g})$ for any $k \geq 1$ and any semi-simple Lie algebra $\mathfrak{g}$. This sequence is defined by giving the generating function. This follows the generating functions for the sequence of elementary symmetric functions, $\{e_r : r \geq 0\}$, and the sequence of complete symmetric functions, given in [42, Chapter I, §2].

Definition 7.1. *Define a sequence of symmetric functions X_k for $k \geq 0$ by*

$$\sum_{r\geq 0} X_r(-q)^r = \prod_{k\geq 1}\sum_{r\geq 0}(-1)^r e_r q^{kr} = \prod_{k\geq 1}\prod_{i\geq 1}(1 - x_i q^k) \tag{19}$$

Then interpreting a symmetric function as a λ-operation we define $X_k(\mathfrak{g})$ to be the operation X_k applied to the adjoint representation. This defines $X_k(\mathfrak{g})$ as an alternating sum of representations.

It is a straightforward consequence of this definition that

$$\sum_{r\geq 0} \dim(X_r) q^r = \varphi(q)^{\dim(\mathfrak{g})} \tag{20}$$

where $\varphi(q)$ is the formal power series

$$\varphi(q) = \prod_{k\geq 1}(1 - q^k)$$

This definition confirms all the conjectures made in [43]. In particular the polynomials calculated in [43] are just the D'Arcais polynomials with sign changes. These polynomials can be found in [11], in [8, page

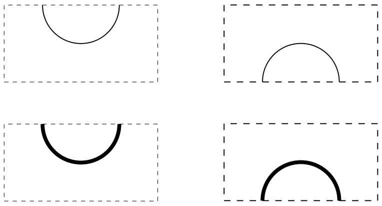


FIGURE 12. Pivotal generators

FIGURE 13. Trivalent generators

159] and they can be found in the On-line Encyclopaedia of Integer Sequences as sequence A008298.

These polynomials are defined by the generating function

$$\sum_{r\geq 0} p_r(D)q^r = \varphi(q)^D = \exp\left(-D\sum_{k\geq 1}\frac{\sigma(k)}{k}q^k\right) \tag{21}$$

where $\sigma(k) = \sum_{d|k} d$

8. A ribbon category

In this section we use the numerology of the previous section to define a ribbon category. The main property of this category is that for each G_0 there is a homomorphism from the coefficient ring to the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ and a functor from the specialisation of the category to the category $\mathrm{Inv}(\mathfrak{g})$.

8.1. **Presentation.** This category is generated as a monoidal category by the following diagrams. The first diagrams determine the pivotal structure. The other generators are the trivalent vertices

Next we define a category $\mathcal{K}$ by taking the free K-linear category on $\mathcal{G}$ and imposing the following relations. These relations are taken as relations for a strict pivotal category. This means that the quotient

$$K\mathcal{G} \to \mathcal{K}$$

is a pivotal functor which satisfies the following relations and is universal with these properties.

These relations say that whenever we see a disk in a diagram with the left hand side of the relation then we replace that disk by the linear

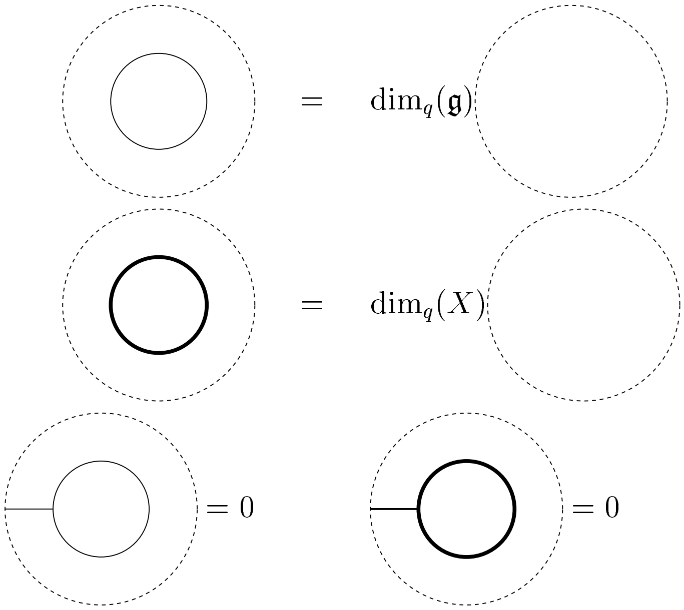


FIGURE 14. Simple relations

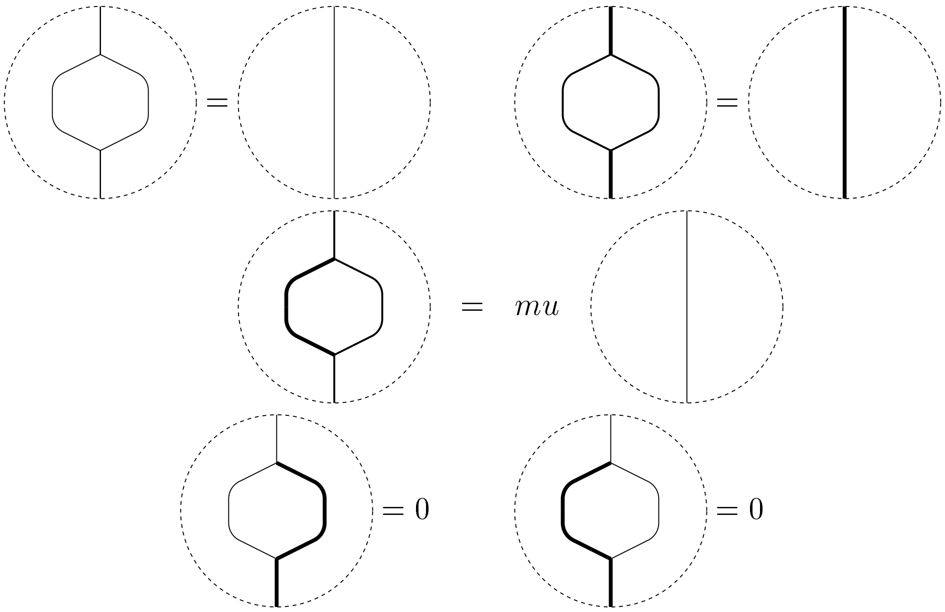


FIGURE 15. Digon relations

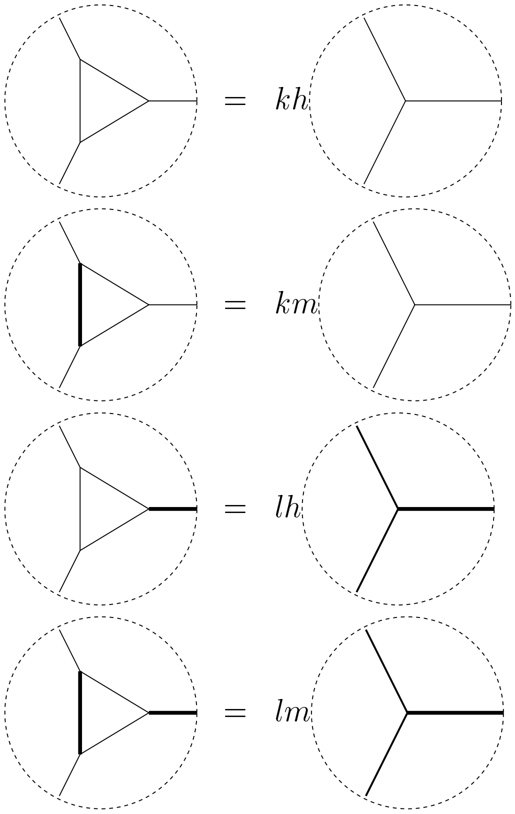
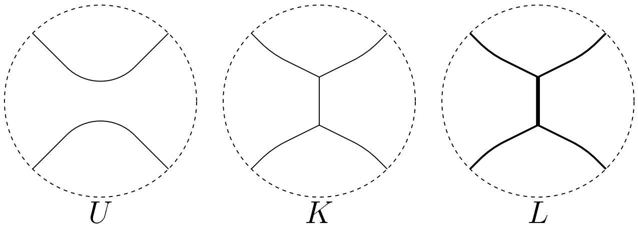


FIGURE 16. Trivalent relations

combination of disks on the right hand side. Thus these relations should be regarded as rewrite rules for morphisms in $\mathcal{G}$.

Next we introduce some notation. The following names some diagrams.



Rotating these diagrams through a quarter of a revolution gives 1, H, M respectively. These diagrams are not linearly independent but satisfy the linear relation

$$(1-U)+\frac{[4,4,4]}{[2,2,2]}(H-K)+k(M-L)=0$$

where the constant k is given by

$$k = \frac{[2,2,2][1,1,0][1,0,1][0,1,1]}{[1,1,1][1,0,0][0,1,0][0,0,1]} \frac{[0,0,2][0,2,0][2,0,0]}{[2,2,0][2,0,2][0,2,2]}$$

Further relations arise from the observation that

$$\mathrm{Hom}(Y(\alpha) \otimes \mathfrak{g}, Y(\beta)) = 0$$

8.2. $6j$**-symbols.** Define the algebra $A(n)$ for $n \geq 0$ to be the endomorphism algebra of n. Then $A(n)$ has elements U_i,K_i,H_i,L_i,M_i for $1 \leq n-1$. These are expected to be generators. This statement is equivalent to the claim that $A(n)$ is a quotient of the group algebra of the n-string braid group $B(n)$.

Furthermore if we choose four elements of $A(2)$ such that adjoining $\{1, K_2\}$ gives a basis then these four elements give a basis of $P_1(i)A(3)P_1(j)$. Similarly if we choose three elements of $A(2)$ such that adjoining $\{1, P_1(j), P_1(k)\}$ gives a basis then these three elements span $K_1 A_3 P_1(Y(i))$.

Define $I(\lambda,\mu)(n,m)$ to be the subspace of $\mathrm{Hom}(n,m)$ spanned by all (n,m)-diagrams of weight at most (λ,μ). Then this defines an increasing sequence of two-sided ideals in the category $\mathcal{K}$.

Here are some of the relations. For each $i > 0$ the elements 1, U_i, K_i, H_i, L_i, M_i, H_i^2 span a six dimensional commutative algebra. The linear relation satisfied by these seven elements is

$$(1 - U_i) + \frac{[4,4,4]}{[2,2,2]}(H_i - K_i) + k(M_i - L_i) = 0$$

where the constant k is given by

$$k = \frac{[2,2,2][1,1,0][1,0,1][0,1,1]}{[1,1,1][1,0,0][0,1,0][0,0,1]} \frac{[0,0,2][0,2,0][2,0,0]}{[2,2,0][2,0,2][0,2,2]}$$

The elements $U_i/\dim_q(g)$,K_i,L_i are orthogonal idempotents. The remaining two string relations are

$$U_i H_i = U_i = H_i U_i$$

$$K_i H_i = khK_i = H_i K_i$$

where the constant kh is given by

$$(22) \quad kh = \frac{[2,2,0][2,0,2][0,2,2]}{[1,1,0][1,0,1][0,1,1][1,1,1]}$$
$$[2\left([1,1,-1] + [1,-1,1] + [-1,1,1]\right) + \left([1,1,3] + [1,3,1] + [3,1,1]\right) - 3[1,1,1]]$$

$$L_i H_i = lL_i = L_i H_i$$

where the constant l is given by

$$l = -\frac{[1,0,0][0,1,0][0,0,1]}{[1,1,1]}\frac{[2,2,0][2,0,2][0,2,2]}{[1,1,0][1,0,1][0,1,1]}$$

$$M_iU_i = muU_i = U_iM_i$$

where the constant mu is given by

$$mu = \frac{[2,2,0][2,0,2][0,2,2][2,1,1][1,2,1][1,1,2]}{[1,1,0][1,0,1][0,1,1][2,0,0][0,2,0][0,0,2]}$$

This leaves

$$M_iK_i = mkK_i = K_iM_i$$

where the constant mk is given by

$$mk = \frac{[2,1,1][1,2,1][1,1,2][1,0,0][0,1,0][0,0,1]}{[2,0,0][0,2,0][0,0,2][1,1,1]}$$

$$M_iL_i = mlL_i = L_iM_i$$

where the constant ml is given by

$$ml = \frac{[2,0,0][0,2,0][0,0,2]}{[1,0,0][0,1,0][0,0,1][1,1,1]} ([2,2,2]+[2,0,0]+[0,2,0]+[0,0,2]) \tag{23}$$

The eigenvalues of H_i and M_i corresponding to the representation $Y(\alpha)$ are respectively

$$\frac{[1,0,0][0,1,1]}{[1,1,1][0,2,2]}$$

$$\frac{[2,0,2][2,2,0][2,1,1][1,0,0][0,1,0][0,0,1]}{[1,0,1][1,1,0][2,2,2][2,0,0][0,2,0][0,0,2]}$$

The other two eigenvalues of H_i and M_i are obtained by symmetry.

This means we can write an equation for H_i^3. The equation for writing $H_iM_i = M_iH_i$ in terms of this basis is given by multiplying the linear relation by H_i and then simplifying using the relations that have been given. This shows that we have given a complete description of the algebra $A(2)$.

There are further relations in $A(3)$. For example, one of the relations corresponding to the triangle is

$$\begin{aligned} P_i(\alpha)H_{i\pm1}P_i(\beta) &= 0 \\ K_iQ_{i\pm1}(\alpha)P_i(\beta) &= 0 \\ P_i(\beta)Q_{i\pm1}(\alpha)K_i &= 0 \end{aligned}$$

and five other similar relations.

The other $6j$-symbols are given by the following. Let $P(\alpha)$ be the idempotent corresponding to the representation $Y(\alpha)$. Let $Q(\alpha)$ be the element obtained by rotating $P(\alpha)$ through a quarter turn. Then the eigenvalue of $Q(\alpha)$ associated with U is

$$-\frac{[1,1,2][2,2,2][1,2,1][-1,2,2]}{[2,0,0][1,0,-1][1,-1,0][1,2,2]}$$

The eigenvalue of $Q(\alpha)$ associated with K is

$$-\frac{[-1,2,2][2,2,2][1,1,2][1,2,1][0,1,1][1,0,0]}{[1,1,1][0,2,2][2,0,0][1,0,-1][1,-1,0][1,2,2]}$$

The eigenvalue of $Q(\alpha)$ associated with L is

$$-\frac{[0,0,1][0,1,0][2,2,2][-1,2,2][1,0,0][0,1,1]}{[1,1,1][2,0,0][0,2,2][1,-1,0][1,0,-1][1,2,2]}$$

The eigenvalue of $Q(\alpha)$ associated with $P(\alpha)$ is

$$\begin{aligned}(24)\quad & \frac{[2,0,0]}{[1,0,0][1,-1,0][1,0,-1][1,2,2]}\times \\ & ([1,2,2]+[2,1,1]-[1,2,1]-[1,1,2]+[1,1,1]-[1,1,0] \\ & -[1,0,1]+2[1,0,0]-[0,1,0]-[0,0,1]+[-1,0,1]+[-1,1,0])\end{aligned}$$

The eigenvalue of $Q(\alpha)$ associated with $P(\beta)$ is

$$\frac{[0,0,1][1,1,2][-1,2,2][2,0,0]}{[1,0,-1][1,-1,0][1,2,2][1,0,0]}$$

The eigenvalue of $Q(\alpha)$ associated with $P(\gamma)$ is

$$\frac{[0,1,0][1,2,1][-1,2,2][2,0,0]}{[1,0,-1][1,-1,0][1,2,2][1,0,0]}$$

The eigenvalues of $Q(\beta)$ and $Q(\gamma)$ are obtained by symmetry.

Finally we define a ribbon structure by defining a homomorphism from the n string braid group $B(n)$ to $A(n)$. This homomorphism together with the pivotal generators defines a ribbon functor from the category of framed tangles to $\mathcal{K}$. The homomorphism is defined on the standard generators of the braid group by

$$T_i \mapsto ti + tuU_i + tkK_i + thH_i + tlL_i + txH_i^2$$

where the constants are given by

$$ti = \frac{(q^{2\alpha}q^{2\beta}q^{2\gamma} + 3q^\alpha q^\beta q^\gamma - q^\alpha q^\beta - q^\gamma q^\alpha + q^\alpha - q^\beta q^\gamma + q^\beta + q^\gamma)}{(q^\alpha q^\beta q^\gamma + 1)^2}$$

$$tu = -\frac{q^\gamma q^\beta q^\alpha (q^\gamma - 1)(q^\beta - 1)(q^\alpha - 1)}{(q^\alpha q^\beta q^\gamma + 1)^2}$$

$$\begin{aligned} th = {} & (q^\alpha q^\beta q^\gamma - 1)(q^{3\alpha} q^{3\beta} q^{3\gamma} + 3q^{2\alpha} q^{2\beta} q^{2\gamma} - q^\gamma q^{2\alpha} q^{2\beta} - q^\beta q^{2\gamma} q^{2\alpha} + \\ & 2q^{2\alpha} q^\beta q^\gamma - q^\alpha q^{2\beta} q^{2\gamma} + 2q^{2\beta} q^\gamma q^\alpha + 2q^{2\gamma} q^\alpha q^\beta - 2q^\alpha q^\beta q^\gamma + q^\alpha q^\beta + q^\gamma q^\alpha + q^\beta q^\gamma) \\ & / q^\gamma q^\beta q^\alpha (q^\alpha q^\beta q^\gamma + 1)^2 \end{aligned}$$

$$\begin{aligned} tk = {} & -(q^{3\alpha} q^{3\beta} q^{3\gamma} + q^{2\gamma} q^{3\alpha} q^{2\beta} + q^{2\alpha} q^{3\beta} q^{2\gamma} + q^{2\alpha} q^{2\beta} q^{3\gamma} + 2q^\gamma q^{2\alpha} q^{2\beta} + 2q^\beta q^{2\gamma} q^{2\alpha} \\ & - 2q^{2\alpha} q^\beta q^\gamma + 2q^\alpha q^{2\beta} q^{2\gamma} - 2q^{2\beta} q^\gamma q^\alpha - 2q^{2\gamma} q^\alpha q^\beta + 7q^\alpha q^\beta q^\gamma \\ & - 2q^\alpha q^\beta - 2q^\gamma q^\alpha + q^\alpha - 2q^\beta q^\gamma + q^\beta + q^\gamma)/(q^\alpha q^\beta q^\gamma + 1)^2 \end{aligned}$$

$$tl = -\frac{(q^\alpha q^\beta q^\gamma + 1)(q^\gamma + 1)(q^\beta + 1)(q^\alpha + 1)}{(q^\beta q^\gamma + 1)(q^\gamma q^\alpha + 1)(q^\alpha q^\beta + 1)}$$

$$tx = \frac{(q^\beta q^\gamma + 1)(q^\gamma q^\alpha + 1)(q^\alpha q^\beta + 1)(q^\alpha q^\beta q^\gamma - 1)^2}{q^\gamma q^\beta q^\alpha (q^\alpha q^\beta q^\gamma + 1)^2}$$

The inverse is given by

$$T_i^{-1} \mapsto tu + tiU_i + thK_i + tkH_i + tlM_i + txH_i^2 \tag{25}$$

$$= \overline{ti} + \overline{tu}U_i + \overline{tk}K_i + \overline{th}H_i + \overline{tl}L_i + \overline{tx}H_i^2 \tag{26}$$

9. Exceptional series

There are three series of simple Lie algebras. The first series will be called the A-series. This consists of the groups $SL(n) \times C_2$. The next series will be called the D-series. This consists of the remaining classical groups namely those which preserve a non-degenerate inner product on a finite dimensional vector space. The third series will be called the E-series. This is a finite series of groups which includes all the exceptional Lie groups. This series consists of the groups of types

$$1, A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8$$

The other homogeneous relation of interest is

$$\alpha(1) + \alpha(2) + 2\alpha(3) = 0$$

which seems to always give the group $A_1 = SL(2)$.

Another interesting line is given by

$$(\alpha, \beta, \gamma) = (-2, m, m + 4)$$

This line is also given by the homogeneous equation

$$2\alpha - \beta + \gamma = 0$$

The Lie algebras on this line with the corresponding values of m are

G_0	A_1	$A_1 \times A_1 \times A_1$	C_3	A_5	D_6	E_7
m	$-2/3$	0	1	2	4	8

Another line is

$$\alpha + \beta + \gamma = 0$$

which corresponds to the super Lie algebras $D(2, 1, ?)$.

This gives a grand total of twenty lines which have six orbits under

9.1. **Exceptional series.** The above numbers are given by

	A	D	E
$\alpha(1)$	$n+1$	$n-4$	$1 + \check{h}/6 = m/2 + 2$
$\alpha(2)$	2	4	$-1 = -1$
$\alpha(3)$	-2	-2	$\check{h}/3 = m + 2$

where for the exceptional series the parameter $\check{h}$ is the dual Coxeter number. The parameteris m and a are alternative parameters. These are related by

$$\check{h} = 3m + 6 \qquad m = (\check{h}/3) - 2 \qquad a = \frac{m+2}{m+4}$$

The values of these parameters for the exceptional Lie groups are given by the following

G_0	1	A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8
$\check{h}$	$6/5$	2	3	4	6	9	12	18	30
m	$-8/5$	$-4/3$	-1	$-2/3$	0	1	2	4	8
a	$1/6$	$1/4$	$1/3$	$2/5$	$1/2$	$3/5$	$2/3$	$3/4$	$5/6$

Here it is a convention to define the dual Coxeter number of the trivial group to be $6/5$. There is a uniform construction of the four exceptional Lie algebras F_4, E_6, E_7, E_8 in terms of the four alternative algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$; this construction is due to Freudenthal and is described in [18], [3], [1]. The parameter m has been chosen to give the dimension of the corresponding alternative algebra in these four cases. The parameter a has the property that $0 < a < 1$ and the denominator is at most six. (This observation is due to Alastair King).

The quantum dimensions of the representations on the exceptional line are calculated in [50].

There is also an interesting involution given by any of the equivalent formulae

$$m \mapsto -m - 8 \qquad \check{h} \mapsto -\check{h} - 12 \qquad a \mapsto 2 - a$$

For each series these parameters satisfy a homogeneous linear relation. These relations are

A	D	E
$\alpha(2)+\alpha(3)=0$	$\alpha(1)+2\alpha(2)=0$	$2\alpha(1)+2\alpha(2)=\alpha(3)$

There is an involution on the nine elements of the exceptional series. This involution is given by

$$\check{h}\to 36/\check{h} \qquad m\to -2m/(m+2) \qquad a\mapsto 1-a$$

This involution also preserves the centre Z and the group Γ. These are given by

G_0	1	A_1	A_2	G_2	D_4	F_4	E_6	E_7	E_8
Z	1	C_2	C_3	1	?	1	C_3	C_2	1
Γ	1	1	C_2	1	S_3	1	C_2	1	1

Each pair of groups related by the involution is a dual reductive pair of subgroups of E_8. This sequence of nine groups and the involution also appears in [40] and [49].

10. Non-reductive groups

In addition to the simple Lie groups there are other specialisations that correspond to Lie groups. There are the cases which correspond to super Lie groups. There is the series of odd symplectic groups and also three other cases.

The parameters for these three cases and the dimensions of the representations are given in the following table

α	g	$Y(1)$	$Y(2)$	$Y(3)$	X
$(1,-3,-5)$	99	3927	945	77	4752
$(5,-1,8)$	190	15504	2640	0	17765
$(1,-4,-7)$	156	10166	1989	90	11934

The parameters for these three cases and the quantum dimensions of the representations are given below

For $(1,-3,-5)$ the dimensions are

$$\dim_q(\mathfrak{g})=\frac{[15][11][9]}{[5][3]} \qquad \dim_q(X)=\frac{[16][15][12][11][9]}{[5][3][2][2]}$$

$$\frac{[14][12][11][10][9]}{[5][4][3][2]} \qquad \frac{[15][14][12][9]}{[4][3][2]} \qquad \frac{[15][14][11]}{[5][3][2]}$$

For $(5,-1,8)$ the dimensions are

$$\dim_q=\frac{[25][19][16]}{[8][5]} \qquad \dim_q(X)=\frac{[26][25][20][19][17][14][11]}{[13][10][7][5][4][2]}$$

$$\frac{[25][24][20[16][11][9]}{[10][8][6][5][3]} \qquad \frac{[27][24][20][19][17][16]}{[9][8][6][5][2]} \qquad 0$$

For $(1,-4,-7)$ the dimensions are

$$\dim_q(\mathfrak{g}) = \frac{[21][16][13]}{[7][4]} \qquad \frac{[22][21][17][16][13][12][9]}{[11][8][7][4][3][2]}$$

$$\frac{[23][20][17][16][14][13]}{[8][7][5][4][2]} \quad \frac{[21][20][17][13][8]}{[8][7][5][4][3]} \quad \frac{[21][20][16][9]}{[8][7][4][3]}$$

Define a Lie algebra to be 2-graded if it is graded by

$$g = g(0) \oplus g(1) \oplus g(2)$$

Then these exceptional cases are all 2-graded. Furthermore in each case $g(0)$ is a simple group and $g(1)$ is an irreducible representation. The types are given by

$$\begin{matrix} [1,-3,-5] & [5,-1,8] & [1,-4,-7] \\ D(6) & E(7) & B(6) \end{matrix}$$

These exceptional cases all ocur as centralisers of unipotent elements in semi-simple Lie groups. In the cases we are interested we take α to be a simple root and let e be a root vector associated to α and f be a root vector associated to $-\alpha$. Then put $h = [e,f]$. The triple e, f, h is a basis of a subalgebra isomorphic to $\mathfrak{sl}(2)$. The element h defines a grading on $\mathfrak{g}$ by

$$\mathfrak{g}(i) = \{g \in \mathfrak{g} | [h,g] = ih\}$$

In the cases we are interested in $\mathfrak{g}(i)$ is non-zero only for the five values $-2 \le i \le 2$. Let $\mathfrak{p}$ be the non-negative part of $\mathfrak{g}$ in this grading and $\mathfrak{p}_+$ the positive part. This means that

$$\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2) \quad \mathfrak{p}_+ = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$$

Then $\mathfrak{p}$ is a subalgebra of the maximal parabolic subalgebra associated to α and $\mathfrak{g}(0)$ is a reductive subalgebra of the Levi subalgebra of this maximal parabolic. Also $\mathfrak{p}$ is a graded subalgebra of $\mathfrak{g}$; and $\mathfrak{p}_+$ is the radical of $\mathfrak{p}$.

For all of the above cases except $(5,-1,8)$ the root α is the highest root. In this case $\mathfrak{g}(\pm 2)$ are one dimensional with basis e and f. Then the radical $\mathfrak{p}_+$ is a Heisenberg algebra and $\mathfrak{p}(0)$ is a subalgebra of $\mathfrak{g}(0)$ of codimension 1.

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