ENUMERATION OF NON-POSITIVE PLANAR TRIVALENT GRAPHS

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ABSTRACT. In this paper we construct inverse bijections between two sequences of finite sets. One sequence is defined by planar diagrams and the other by lattice walks. In [Kup96] it is shown that the number of elements in these two sets are equal. This problem and the methods we use are motivated by the representation theory of the exceptional simple Lie algebra G_2. However in this account we have emphasised the combinatorics.

1. INTRODUCTION

The aim of this paper is to give an enumeration of non-positive planar trivalent graphs. Here the graph is embedded in a disk and the number of boundary points is specified. This differs from other enumerations of planar trivalent graphs such as [Tut62] or [GW02] in that neither the number of vertices not the number of edges is specified. So if there were no further conditions then the number of graphs would be infinite. The extra condition that is imposed is that the graph is non-positive. This means that there is no internal face with less than six edges.

This problem arose in the work of Greg Kuperberg on the representation theory of the exceptional simple Lie algebra G_2 in [Kup96]. In particular, one of the results of this paper is that the two sets we consider have the same numbers of elements. This is proved by showing that both sets are a basis of the same vector space.

Here, we give a bijective proof of this result. A bijective proof of the analogous result for A_2 is given in [KK99]. Moreover, the map from diagrams to words given below is constructed by the same method as the analogous map in this reference. However the construction of the inverse map that we give here is new. This construction is based on a diagram model for the crystal graph.

The main result of this paper is the construction of inverse bijections between two sequences of finite sets. The two sets and the bijections are constructed by combinatorial methods and in writing this paper we have emphasised the combinatorial aspects. However both the original problem and the construction of the bijections are motivated by two combinatorial methods in representation theory, namely Littelman

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paths and crystal graphs. So here we put the constructions we have used into context.

In this paper we are studying the invariant theory of the seven dimensional fundamental representation of the exceptional simple Lie algebra of type G_2. The Lie algebra can be constructed as the derivation algebra of the octonions and the representation can then be taken to be the imaginary octonions. The weights and the dominant weights below are the usual weights and dominant weights of this Lie algebra. This representation has the (rare) property that all weight spaces are one dimensional. The set S in (1) consists of the seven weights of this representation; moreover Figure 1 (with the centre point of weight $(0, 0)$) is the weight diagram of this representation.

The two finite sets we consider here each describe a basis for the vector space of invariants in the n-th tensor power of this representation. One basis is given by dominant Littelman paths. The Littelman paths in this example consist of the six straight line paths which are the edges emanating from the origin in Figure 1 together with one path of weight zero. Then the lattice walks are exactly the dominant paths, that is paths which are obtained by concatenations of these seven paths which are dominant at all times. The second method for constructing a basis of invariant tensors is to use crystal graphs. The crystal graph in this example is a directed graph with edges labelled by the simple roots where the vertices are the set S. The crystal graph is obtained from the crystal base. The crystal base and the crystal graph for this representation are given in [KM94]. In this example the vertices are parametrised by the weights and two weights are connected by an edge labelled by a simple root if the difference is the simple root. In this paper we do not make any use of the edges in the crystal graph and so they have been omitted. However the vertices of a crystal graph are also labelled by two dominant weights which we denote H and D where the weight is $H - D$. There is a tensor product construction for crystal graphs and this includes a rule for the labels H and D. The labelling in Figure 3 is designed to reproduce this rule.

2. Lattice walks

In this section we will give the description of the lattice walks that will be related to the diagrams. We will use the terminology of weights which comes from representation theory.

A weight λ is an ordered pair of integers. The set of weights is an abelian group under addition and is partially ordered. The partial order is given by saying $(a_1, b_1) \geq (a_2, b_2)$ if $a_1 \geq a_2$ and $b_1 \geq b_2$. A weight λ such that $\lambda \geq 0$ is called a dominant weight.

Define the set S to be the following set of seven weights

$$S = \{(-2, 1), (0, -1), (-1, 1), (0, 0), (1, 0), (-1, 1), (2, -1)\} \tag{1}$$

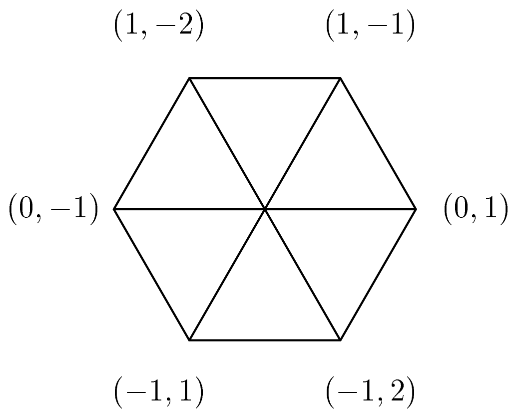


FIGURE 1. Seven steps

Then the main definition of this section is

Definition 2.1. *A lattice walk of length n is a sequence of $n+1$ dominant weights $(\lambda_0, \lambda_1, \dots, \lambda_n)$ such that $(\lambda_i - \lambda_{i-1}) \in S$ for $1 \le i \le n$. The sequence is also required to satisfy the additional condition that, for $1 \le i \le n$, if $\lambda_i = (a, 0)$ for some $a \ge 0$ then $\lambda_{i-1} \ne \lambda_i$.*

If $(\lambda_0, \lambda_1, \dots, \lambda_n)$ is a lattice walk then we say that the walk starts at λ_0 and ends at λ_n. Denote the set of lattice walks of length n which start at λ and end at μ by $W(\lambda, n, \mu)$. Also let the number of elements of $W(\lambda, n, \mu)$ be denoted by $w(\lambda, n, \mu)$.

Next we interpret a lattice walk as a walk in the triangular lattice. First we identify the set of weights with the vertices of the triangular lattice. This is an identification of abelian groups. The identification we use identifies the non-zero weights in S with the regular hexagon of nearest neighbours of the origin in the triangular lattice. This is shown in Figure 1.

Example 2.1. *The following diagram has $w(0, 6, \mu)$ written at position μ.*

$$\begin{array}{cccccccccccc}
 & & & & & & & & & 5 & & \\
 & & & & & 65 & & 40 & & & 9 & \\
 & & 120 & & 176 & & 120 & & 40 & & & 5 \\
35 & 120 & & 180 & & 145 & & 65 & & 15 & & 1
\end{array}$$

It is clear that the sets $W(\lambda, n, \mu)$ can be enumerated. Here we give numbers of the form $w(0, n, \mu)$. Let X and Y be the following Laurent polynomials in the indeterminates x and y:

$$\begin{aligned}
(2)\, X &= 1 + x + x^{-1} + x^{-1}y + xy^{-1} + x^{-2}y + x^2y^{-1} \\
(3)\, Y &= x - x^{-3}y^2 + x^{-6}y^3 - x^{-8}y^3 + x^{-8}y^2 \\
(4)\quad & \quad -x^{-7} + x^{-3}y^{-2} - xy^{-4} + x^4y^{-5} - x^6y^{-5} + x^6y^{-4} - x^4y^{-2}
\end{aligned}$$

Then the numbers $w(0,n,\mu)$ can be computed from the observation that if $\mu=(a,b)$ then $w(0,n,\mu)$ is the coefficient of x^ay^b in the Laurent polynomial YX^n.

The sequence $a(n)=w(0,n,0)$ which is the number of lattice walks of length n which start and end at the origin is of particular interest. This sequence is given in [Slo05] as sequence A059710. Equivalently $a(n)$ is the constant term in the exapnsion of YX^n where X and Y are the Laurent polynomials in (2). It follows from this description that the sequence $a(n)$ is P-recursive which means that it satisfies a finite recurrence relation with polynomial coefficients. Alex Mihailovs has proposed that this sequence satisfies the following recurrence relation

$$\begin{aligned}(n+5)(n+6)a(n) &= 2(n-1)(2n+5)a(n-1)\\ &\quad + (n-1)(19n+18)a(n-2) + 14(n-1)(n-2)a(n-3)\end{aligned}$$

together with the initial conditions $a(0)=1$, $a(1)=0$, $a(2)=1$. The evidence for the proposal is that it gives the first thirty terms of the sequence and gives the correct asymptotics.

3. Diagrams

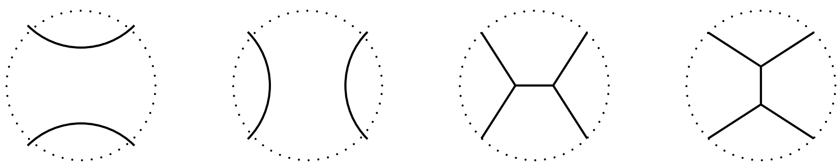
The main definition of this section is the following:

Definition 3.1. *A diagram with n boundary points consists of a disc with n marked points on the boundary together with an embedded graph. The graph has n vertices of valence one which are identified with the marked boundary points by the embedding and all other vertices of the graph have valency three.*

A region is a connected component of the complement of the image of the graph in the disc.

Definition 3.2. *A diagram is non-positive if every region of the disc which is bounded by edges of the graph is bounded by at least six edges of the graph.*

Example 3.1. *There are four non-positive diagrams with four boundary points. These are*



Although we will not make use of this result we first recall the argument from [Kup96] which shows that if we specify the number of boundary points then there are only finitely many non-positive diagrams. This proof depends on the isoperimetric inequality given in [BZ88].

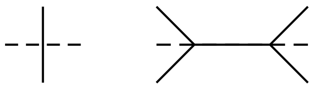
Proposition 3.1. *Let $n \geq 0$. Then there are finitely many non-positive diagrams with n boundary points.*

Proof. Consider a diagram whose graph is connected. Then the dual graph gives a triangulation of the disc. Take each triangle to be a Euclidean equilateral triangle with edge length 1. Then this gives a polyhedral metric on the disc. Now if the planar graph is non-positive then this polyhedral metric has non-positive curvature. Hence the isoperimetric inequality is satisfied. Each triangle has area $\sqrt{3}/2$ and the length of the boundary is n. Hence the isoperimetric inequality gives that there at most $n^2/(\pi\sqrt{3})$ triangles. □

Next we recall a crucial definition from [Kup96].

Definition 3.3. *Assume we are given a diagram. Let A and B be two boundary points which are not marked points. Then a cut path from A to B is a path from A to B such that each component of the intersection with the embedded graph is either an isolated tranverse intersection point or else is an edge of the graph.*

The diagrams for these two cases are:



A cut path which crosses a edges and contains b edges is assigned the weight (a, b). The weight (a, b) is called dominant if $a \geq 0$ and $b \geq 0$. The weights are partially ordered by $(a_1, b_1) < (a_2, b_2)$ if $a_1 + 2b_1 < a_2 + 2b_2$ or if $a_1 + 2b_1 = a_2 + 2b_2$ and $b_1 < b_2$. A cut path is minimal if there is no cut path with the same endpoints and with lower weight.

Definition 3.4. *A triangular diagram is a non-positive diagram together with three points A, X and Y which are in the boundary of the disc but not marked points. Then we require that the edges AX and AY are minimal cut paths.*

Our convention is that we draw a triangular diagram as a graph embedded in the triangle AXY where XY is a horizontal edge and the vertex A is below this edge.

Definition 3.5. *A triangular diagram is reducible if there is a point B inside the triangle (and not on the graph) such that there is a minimal cut path from A to X which passes through B, a minimal cut path from A to Y which also passes through B and such that the the triangular diagram with vertices B, X and Y and edges given by these minimal cut paths from B to X and B to Y is a proper subdiagram.*

A triangular diagram is irreducible if it is not reducible. The length of a triangular diagram is the number of marked points on the edge XY.

4. Bijections

We can now state the main theorem of this paper.

Theorem 4.1. *For all $n \geq 0$ there are inverse bijections between words in S of length n and irreducible triangular diagrams of length n.*

Let $T(n)$ be the set of irreducible triangular diagrams of length n and S^n the set of words in S of length n. Then first we construct a maps $T(n) \to S^n$ for $n \geq 0$. The construction is essentially the same as the construction of the analogous map in [KK99].

Choose a sequence of points in the edge XY, $(X_0, X_1, \dots, X_n)$, with $X = X_0$ and $Y = X_n$ such that no point in the sequence is a marked boundary point and such that for $1 \leq i \leq n$ the interval (X_{i-1}, X_i) contains exactly one marked boundary point. Now for $0 \leq i \leq n$ let λ_i be the weight of a minimal cut path with endpoints A and X_i. Then the claim is that $(\lambda_i - \lambda_{i-1}) \in S$ for $1 \leq i \leq n$. This condition follows from the following:

Proposition 4.1. *The irreducible triangular diagrams of length one are precisely the seven triangular diagrams in Figure 2.*

An isoperimetric inequality for sectors is given in [BH00]. It would be interesting to know if this isoperimetric inequality also holds for polyhedral metrics; and, if so, wether this implies that the number of irreducible triangular diagrams of length one is finite.

Proof. It is straightforward to check that each of the seven diagrams in Figure 2 is an irreducible triangular diagram of length one. It remains to show that these are all such diagrams.

First we observe that if we are given a triangular diagram with one marked point on the top edge and such that the graph has more than one component then the triangular diagram is reducible.

We assume that there is no region bounded by the graph. This means we can now assume that we have a triangular diagram with one marked point on the top edge and whose graph is a tree. Then consider the graph obtained by removing all edges incident to a marked point on the boundary. If this graph is not an interval then the original diagram can be reduced by pruning the tree.

Note that if there are no marked points on the edge AX then the edge AY has one marked point. There is no diagram with one marked point on the top edge and no other marked point. Also there cannot be more than one marked point on the edge AY otherwise the path AXY would have weight less than the weight of the path AY and the

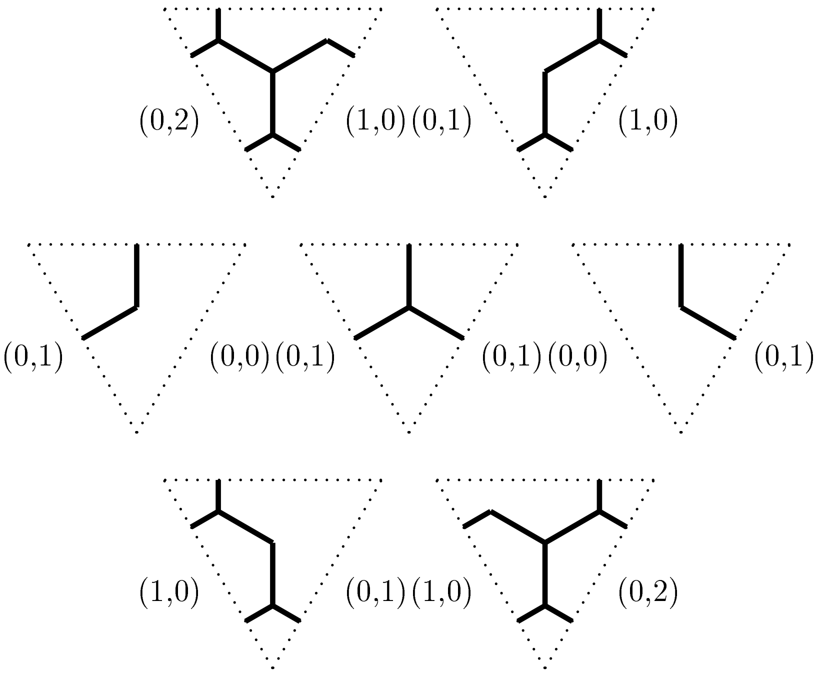


FIGURE 2. Steps as triangles

cut path AY is required to be minimal. If there are two marked points on the boundary then there is only one non-positive diagram.

Then to complete the proof it is sufficient to observe that if you start with a diagram in Figure 2, choose an edge which does not meet the boundary, and add a new edge from a point in this edge to one of the two sides of the triangle then one of the following occurs:

(1) The new diagram is already in Figure 2.
(2) In the new diagram either AX or AY is not a minimal cut path.
(3) The new triangular diagram is reducible.

□

Next we construct a map $S^n \to T(n)$.

Construct a planar graph by taking the subgraph of the square lattice on the vertices (i, j) such that $0 \leq i, j$ and $i + j \leq n$. Then we label the edges of this graph by dominant weights. Label the edge from (i, j) to $(i, j + 1)$ by $D_{i,j}$ and label the edge from (i, j) to $(i + 1, j)$ by $H_{i,j}$. The labelling is constructed by induction on $n - i - j$. If $i + j = n - 1$ then the labels $D_{i,j}$ and $H_{i,j}$ are read off from the sequence of elements of S.

If $D_{i+1,j-1}(k) \geq H_{i,j}(k)$ then put

$$D_{i,j-1}(k) = D_{i+1,j-1}(k) - H_{i,j}(k) \qquad H_{i,j-1} = 0$$

and if $H_{i,j}(k) \geq D_{i+1,j-1}(k)$ then put

$$D_{i,j-1}(k)0 \qquad H_{i,j-1} = H_{i,j}(k) - D_{i+1,j-1}(k)$$

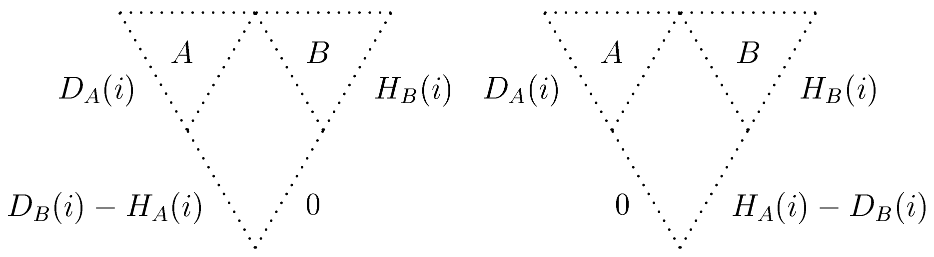


FIGURE 3. Labelling the edges

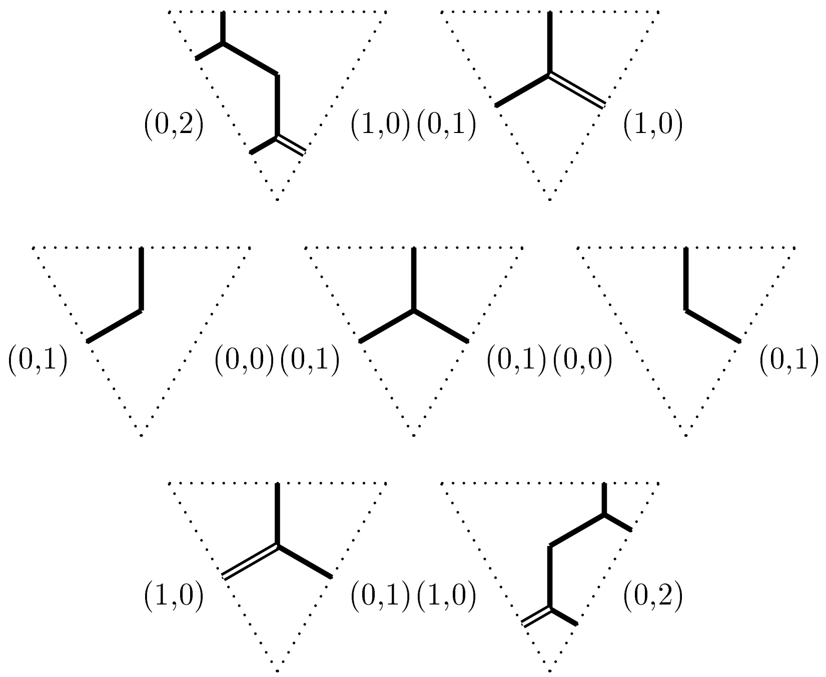


FIGURE 4. Steps as triangles

Now we come to draw the pictures. First we rotate so the lines $i + j$ constant are horizontal. This means that the labelling of the edges by dominant weights is given by the two diagrams in Figure 3.

Now we fill in the triangles and squares to make the diagram. First we introduce a new type of edge. This will be drawn as a double edge. The Definition 3.1 (of a diagram) is modified to allow this new type of edge. The graph is still required to be trivalent but we also allow a vertex to be incident to two of the original type of edge and one of the new type. The Definition 3.3 (of a cut path) is also modified to allow a cut path to intersect any edge transversally. A cut path which crosses a edges of the original type, contains b edges (of the the original type) and crosses c edges of the new type has weight $(a, b + c)$.

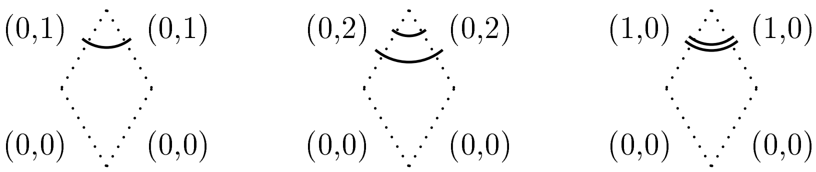
Then we modify the seven triangular diagrams in Figure 2 and redraw them so that the edges are minimal cut paths which do not contain any edge. This gives the seven triangular diagrams in Figure 4.

Then given a word in S we first draw the grid as above. Then we fill in each triangle using the previous diagram. Next we fill in the diamonds. Note that the weights H_A and D_B are both elements of the set

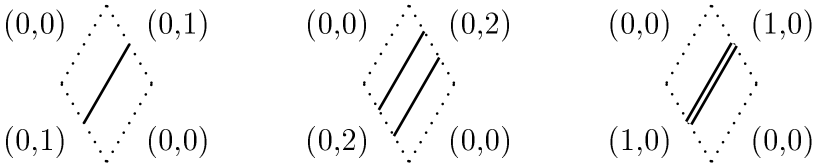
$$\{(0,0),(0,1),(0,2),(1,0)\}$$

This gives sixteen different diamonds. Furthermore note that for each of these sixteen diamonds the other two weights are also elements of this set. This implies that for any word every diamond will have one of these sixteen labellings. Therefore to complete the diagram it is sufficient to know how to fill in a diamond with each of these labellings.

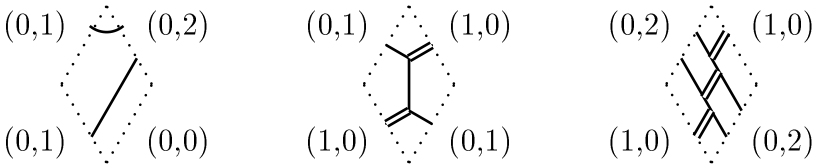
There are four symmetric diamonds which are also the diamonds where the weights on the top two edges are equal and the weights on the bottom two edges are zero. The case in which all four weights are zero is an empty diamond. The other three diamonds are



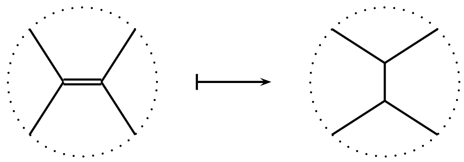
The other twelve diamonds come in pairs and we only give one member of each pair. The other member is obtained by reflection in a vertical line. There are three diagrams in which an oposite pair of edges have weight zero. These give the three diamonds



The remaining three diamonds are



In the resulting diagram each interior vertex is still trivalent and each interior trivalent vertex is either incident to three of the original type of edge or two of the original type of edge and one edge of the new type. Hence each ocurrence of the new type of edge can be removed by the following replacement.



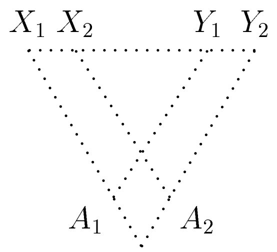


FIGURE 5. Inductive step

The reason we have to introduce the new type of edge and the modified pictures for the elements of S is that if we used the original pictures then we could get a diagram with squares. When working by hand it is straightforward to remove these superfluous squares. Informally, the rule is that whenever we see a ladder with more than one rung, to remove all but one rung.

An important feature of this construction is the following observation:

Lemma 4.1. *Any increasing path which starts and the bottom vertex and follows the construction lines to the top edge of the diagram is a minimal cut path.*

This observation shows that the diagram is non-positive since it shows that we have a minimal cut path from the bottom vertex to the top edge which passes through any internal region. Hence we have constructed a triangular diagram of length n. It remains to check that this is irreducible. This is the statement that the edges of the triangle are the unique minimal cut paths between their endpoints.

Then Lemma 4.1 also shows that if we start with a word construct the diagram and then derive a word that we recover the original word. In particular this shows that for $n \geq 0$, the map $S^n \to T(n)$ is surjective.

Next we show that, for $n \geq 0$, the map $S^n \to T(n)$ is injective. The proof is by induction on the length of the word. The basis of the induction is the case of length one which is proved in Proposition 4.1. Assume the result for words of length n. Let w be a word of length $n+1$. Let w_i be obtained by dropping the final step and w_f be obtained by dropping the first step. Then these are both words oflength n and so by the inductive hypothesis have unique diagrams. This means that in Figure 5 the triangle $A_1X_1Y_1$ can be filled in uniquely using the word w_i and the triangle $A_2X_2Y_2$ can be filled in uniquely using the word w_f. Thus the only part of the diagram that has not been filled in is the lowest diamond between the points A_1 and A_2. Thus the claim is that for each of the sixteen possible diamonds the rule we have given for filling it in is the unique rule that gives a non-positive irreducible triangular diagram. For each of these sixteen diamonds, the claim can be checked using [Kup96, Lemma 6.5].

This proves Theorem 4.1. Then for $n \geq 0$, these bijections restrict to bijections between the set of non-positive diagrams with n boundary points and the set of lattice walks of length n which start and end at the origin.

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