THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2003

REPRESENTATION THEORY

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

1. a) Define the radical of an algebra A. **[4]**

b) Let $A = \mathbb{C}[x]/\langle p(x)\rangle$ for some polynomial $p(x)$. Show that a matrix whose minimal polynomial divides $p(x)$ determines a representation of A. **[7]**

Let $p(x)$ be the polynomial $x^3 - 4x^2 + 5x - 2$ and let A be the algebra $\mathbb{C}[x]/\langle p(x)\rangle$.

c) Find the radical of A. **[7]**

d) Find a representation of A which is both simple and projective. **[7]**

---

2. a) Define the Cartan matrix of an algebra A. **[5]**

b) Give an example of an algebra A whose Cartan matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. **[10]**

c) Give an example of two basic $\mathbb{C}$-algebras with the same Cartan matrix which are not isomorphic. **[10]**

---

3. Let A be the collection of 4×4 matrices with $a_{ij} = 0$ if $i > j$, $a_{11} = a_{44}$ and $a_{22} = a_{33}$.

a) Show that A is a subalgebra of the algebra of 4×4 matrices. **[3]**

b) Find the indecomposable projective A-modules. **[11]**

c) Find the Cartan matrix of A. **[11]**

---

4. For $t \in \mathbb{C}$, let A be the $\mathbb{C}$-algebra generated by u and v with defining relations

$$\begin{aligned} uu &= tu \quad & uvu &= u \\ vv &= tv \quad & vuv &= v \end{aligned}$$

Let U and V be the matrices

$$U = \begin{pmatrix} t & 1 \\ 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 \\ 1 & t \end{pmatrix}$$

a) Show that A has basis $\{1, u, v, uv, vu\}$ by giving the multiplication table. [**3**]

b) Show that $u \mapsto U$ and $v \mapsto V$ defines a representation. [**4**]

c) Calculate the endomorphism algebra of this module and hence show that this representation is indecomposable. [**9**]

d) Show that this representation has a one dimensional invariant subspace if and only if $t^2 = 1$. [**9**]

---

5. a) State the Wedderburn-Artin classification of semi-simple $\mathbb{C}$-algebras. [**3**]

b) Use the Wedderburn-Artin theorem to show that there are three semi-simple $\mathbb{C}$-algebras whose dimension is at most three (up to isomorphism). [**7**]

c) Find five $\mathbb{C}$-algebras of dimension three such that no two are isomorphic. You may assume that any such algebra is a path algebra with relations. [**10**]

d) Prove that no two of these algebras are isomorphic. [**5**]
