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Fifty Years of the Exact solution of the Two-dimensional Ising Model by Onsager

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Abstract

The exact solution of the two-dimensional Ising model by Onsager in 1944 represents one of the landmarks in theoretical physics. On the occassion of the fifty years of the exact solution, we give a historical review of this model. After briefly discussing the exact solution by Onsager, we point out some of the recent developments in this field. The exact solution by Onsager has inspired several developments in various other fields. Some of these are also briefly mentioned.

I. INTRODUCTION

It is usually said that every theoretical physicist must know (ideally must be taught) the landmarks of theoretical physics. Some of these are: $g-2$ calculation in Quantum Electrodynamics, Onsager's Exact Solution of the Two-Dimensional Ising Model etc. It is our privilage to briefly discuss Onsager's exact solution of the Ising model and its influence in the subsequent developments of statistical mechanics [1]. To appreciate as to why it is regarded as one of the landmarks of theoretical physics, it is worth remembering that for a long time it remained the first and the only (mathematically rigorously) exactly solvable model exhibiting phase transition. Its discovery completely changed the course of developments in Statistical Mechanics and also other areas of physics. Before Onsager's exact solution it was not clear if the formalism of statistical mechanics can handle phase transition. The solution established beyond doubt that phase transitions appear as singularities in the thermodynamic functions and these functions need not have simple discontinuities as hypothesized by Ehrenfest before. Furhtermore, of all the systems in statistical mechanics on which exact calculations have been performed, the 2-dimensional Ising model is not only the most thoroughly investigated but is also the most profound. Its significance was instantly recognised. In this context we would like to recall the letter written by Pauli to Casimir immediately after the World War II. Casimir in his letter had expressed his concern about being cut off for so long from theoretical physics of allied countries. Pauli in his reply said "nothing much of interest has happened except for Onsager's exact solution of the Two-Dimensional Ising Model" [3].

The plan of the article is as follows:In sec.II we first define the Ising model and give an historical account of the various approximate solutions developed leading to the exact solution of Onsager [2]. We shall also briefly mention the relevance of the model. A short sketch of the life and works of Onsager are given in sec.III. In sec.IV we review the exact solution of Onsager [4] and its importance. Some of the open, unsolved problems are also mentioned here. It must be made clear here that our aim is not really to discuss the exact solution since there are several places where an excellent account has already been given. In sec.V we discuss the various developments in this field which have been directly inspired or influenced by Onsager's work.

II. THE MODEL AND ITS BRIEF HISTORY

The Ising model was infact first written down not by Ising but by his thesis supervisor W. Lenz [5] in 1920 who was then working in Rostalk University in Germany. It is somewhat unfortunate that the physics community has given him no credit for this work. This is perhaps because even though Lenz introduced the model, he did not do any calculations using this model. It is perhaps worth recalling that Lenz is instead well known for his work on Runge-Lenz vector in the context of the accidental degeneracy in the hydrogen atom problem. Lenz moved to Hamburg as professor in 1921. One of his first Ph.D. students was Ernest Ising who was born on May 10, 1900. In late 1922, Lenz asked Ising to study his model and the phenomena of ferromagnetism. Ising studied the model and found its exact solution in one dimension and showed that there is no phase transition from para to ferromagnetism [6]. Since then the model is known as Ising model. Let us now briefly state the Ising model.

The Model: It is a lattice model. Consider a d-dimensional periodic lattice (d = 1,2,3,...) having an array of N fixed points. The lattice may be of any type. For example, one could have 3-dimensional cubic or hexagonal lattice. However, we shall be mostly considering the two dimensional square lattice since it is for this that Onsager obtained his exact solution. With each lattice site is associated a spin variable S_i (i = 1,2,...,N) which is a number being +1 or -1. If it is 1 or -1 we call it spin up or spin down. Clearly, a given set of numbers $\{S_i\}$ specify a configuration of the whole system and that there are in all 2^N different configurations.The energy of the system in a given configuration S_i is defined to be

$$E\{S_i\} = -\sum_{<ij>} J_{ij} S_i S_j - B\sum_{i=1}^{N} S_i, \tag{1}$$

where B is the external magnetic field. Usually one assumes that the interaction is isotropic (i.e. $J_{ij} = J > 0$) and that there is only nearest neighbour interaction. For example, for a 2-dimensional square lattice the number of nearest neighbours is 4. Given the energy of the system, the goal is to compute the canonical partition function ($\beta = (kT)^{-1}$, where k is the Boltzmann consatnt and T is the temperature)

$$Q(B,T) = \sum_{S_1}\sum_{S_2}....\sum_{S_N} e^{-\beta E\{S_i\}}, \tag{2}$$

and hence the thermodynamic properties, and to see if the model exhibits any phase transition at a finite nonzero temperature. In other words, in magnetic systems, each molecule has a spin that can orient up or down relative to the direction of the applied external magnetic field B. The question one would like to ask is if this model could lead to spontaneous magnetisation i.e. if below a certain critical temperature T_c, $M(B = 0, T)$ which is essentially same as the order parameter $\overline{L}(B = 0, T)$, is nonzero or not. Here the magnetization $M(B, T)$ is related to the partition function by

$$M(B, T) = +\frac{\partial}{\partial B}[\ln Q(B, T)]. \tag{3}$$

It is worth pointing out here that the Ising model equally well represents a model for (i) Binary Alloys and (ii) Lattice Gas. Binary alloys are mixtures of two types of molecules (for example $\beta-$brass has a cubic structure made out of Zn and Cu atoms) and the question is if below a certain temperature T_c, there is a phase transition with atoms of the same type clustering together. On the other hand, in lattice gas models one considers a mixture of molecules and holes(i.e. empty spaces) on a latice and the question is if below T_c, there is a condensation of molecules into one region of space and holes in the rest of the lattice? Yang and Lee [7] discussed this model and gave a detailed mapping between it and the Ising model. It may be noted here that the lattice gas models are of relevance in the context of gas-liquid and liquid-solid transitions. We thus see that the study of the Ising model is of relevance in a number of phenomena. Infact for $J < 0$, this model could very well represent (i) antiferromagnetic ordering, (ii) superlattice structure in an alloy, and (iii) a solid-like arrangement of molecules with repulsive forces.

A point in order at this stage. Many people tend to dismiss the Ising model as over-simplified representation of intermolecular forces. However the point to note is that the essential features of the cooperative phenomena(i.e. long range order), specially near T_c, do not depend on the details of the intermolecular forces but on the mechanism for propagation of long range order and Ising model offers much hope for an accurate study of this mechanism.

Historical Developments: As mentioned above, Ising was able to obtain the exact solution of the model in 1-dimension [6] and show that there is no phase transition at $T \neq 0$. He then gave some arguments and erroneously concluded that even in two or three dimensions this model will not exhibit any phase transition. This dissuaded several people from working on this model and prompted Heisenberg [8] to introduce more complicated vector interaction between spins known popularly as Heisenberg ferromagnetic interaction. It is interesting to note that Ising seriously believed in his conclusions and felt very frustrated that the model has no usefulness and gave up physics research! Being a Jew, he was prosecuted in Germany and was dismissed from his job in 1933. He managed to run away from Germany and lived in Luxemberg from 1939 to 1947. During all this time he had no contact with physics and lost track of future developments about the model. Only in 1947 when he went to U.S.A. to teach did he realize that his name has become immortal because of the work done by other people!

Almost a decade after Ising's work, Bragg and Williams [9] studied the model in the mean field approximation. They were inspired by an earlier work of Gorsky [10]. The essential assumption here is that the energy of an individual atom in any configuration is determined by the average degree of order prevailing in the entire system rather than by the fluctuating configurations of the neighboring atoms. Clearly, this approximation is exact in the limit $d \to \infty$. One of the serious criticism of this approximation is that it is independent of the dimension d. This approximation predicts a phase transition in the Ising model at finite T for any d which is clearly wrong for d = 1 as one knows from Ising's exact calculation.

Soon afterwords, Bethe [11] improved upon the Bragg-williams approximation by treating somewhat more accurately the interaction between the nearest neighbours. It turns out that this approximation is infact exact for 1-dimension and hence does not predict phase transition at any finite nonzero T while for 2 and higher dimensions, it does predict spontaneous magnetization. Soon afterwords, Peierls [12] wrote a paper entitled "On Ising's Model of Ferromagnetism" in which he gave a simple argument showing that at sufficiently low temperature, Ising model in 2 or 3 dimensions must exhibit spontaneous magnetization. It turned out much later that his argument involved an incorrect step [13] but nevertheless, the conclusion and the general procedure are correct. This was an important step, and the method is used still now in various situations.

The first exact quantitative result for the 2-dimensional Ising model was obtained by Kramers and Wannier [15] when they located the transition temperature by using the symmetry of the two-dimensional lattice to relate the high- and low-T expansion of the partition function. On February 28, 1942 there was a meeting of the New York academy of Sciences in which Wannier gave a talk on this work. At the end of his talk Onsager made a remark announcing that he has been able to obtain the exact solution of the 2-dimensional Ising model for square lattice and without external magnetic field. It is very significant though that he published his results in a journal only two years later. Infact this was always his style. He could see through the math and get the final result but was careful to publish his results only after he has satisfactorily settled all the mathematical questions.

III. ONSAGER–HIS LIFE AND WORKS

Lars Onsager was born at Oslo in 1903. Even though he got his degree in chemical engineering in 1925 he devoted most of his time to studying math and physical sciences. In 1931 he wrote two monumental papers in Physical Review about reciprocal relations in irreversible processes. This essentially started the new branch of thermodynamics − that of irreversible processes. Eventually, for this work, he received Nobel prize in Chemistry in the year 1968. The Nobel citation noted that the publication (two papers in Physical Review of length 22 and 15 pages) was one of the smallests ever to be awarded a Nobel prize! In 1933 he was offered the Sterling Post-Doctoral fellowship at the Chemistry Department of Yale University. But there was one small problem. Onsager had never written his Ph.D. thesis! He was persuaded to write one which he did on the properties of Mathieu functions. This work is recognised as one of the significant contributions to the subject. In 1934 he became the faculty member of the department. From 1944 till his retirement in 1972 he occupied the J. Willard Gibbs chair of Theoretical Chemistry at Yale University. In 1972, he moved to Miami University where he worked in the Center for Theoretical Studies till his death in 1976.

During his entire career, Onsager only published about 60 papers but each was an important one published only after he was fully satisfied with its significance and its mathematical rigour. Many a times he announced his results in conferences as a remark after some talk but published the results much later or never did! We will see one such example below. He would call himself a chemist but truely speaking he was one of the last truely universalist of this century. For more information see Ref. [14].

IV. ONSAGER'S EXACT SOLUTION OF 2-D ISING MODEL

To appreciate Onsager's exact solution, it is worth recalling the work of Kramers and Wannier [15] before Onsager where they showed that the partition function for this problem can be written as the largest eigenvalue of a certain matrix. Onsager's method was similar to these authors except that he emphasized the abstract properties of rather simple operators rather than their explicit representation by unwieldy matrices. It must be said here that his method is highly nontrivial and complicated. Actually his derivation can be easily followed step by step but the over-all plan is quite obscure. Since a simplified version as given by Bruria Kaufman [17] has been discussed at a number of places including textbooks we shall merely quote the result. Onsager showed that the canonical partition function $Q(B=0,T)$ in the limit $N \to \infty$ is given by

$$\lim_{N\to\infty} \ln Q(B=O,T) = \ln(2\cosh(2\beta J)) + \frac{1}{2\pi}\int_0^{\pi} d\phi \ \ln \ \frac{1}{2}(1+\sqrt{1-\kappa^2\sin^2\phi}), \tag{4}$$

where $\kappa \equiv 2\sinh(2\beta J)/\cosh^2(2\beta J)$. From here it followed that the specific heat $C(B=0,T)$ defined by

$$C(B,T) = k\beta^2 \frac{\partial^2}{\partial\beta^2}[\ln Q(B,T)], \tag{5}$$

diverges logarithmically as $T \to T_c$ where T_c is given by

$$\tanh\frac{2J}{kT_c} = \frac{1}{\sqrt{2}} \Longrightarrow \frac{kT_c}{J} = 2.269185. \tag{6}$$

For comparison, kT_c/J is predicted to be 4 and 2.88 in the mean field and Bethe approximations respectively.

Over the years, several alternative simplified methods have been given but all of them are quite involved and lengthy. Bruria Kaufman [17] gave a simplified proof which is based on spinorial representation of the rotation group and it is this proof which is essentially given in K. Huang's book on statistical mechanics. Further, using Onsager's proof, the exact partition function and hence other properties of several other 2-dimensional lattices have been deduced. Finally, Onsager and Bruria Kaufman [20] calculated spin correlation functions.

Further Developments: To justify calling the phenomemon at $T = T_c$ a phase transition, one has to examine the long range order i.e. spontaneous magnetization and show that it is nonzero. This is a very difficult calculation since it has to be done with $B \neq 0$ (but could be small) at the beginning of the calculation and finally putting it equal to 0 at the end of the calculation. On August 23, 1948 there was a talk by Tisza at Cornell University. At the end of the talk, Onsager walked up to the blackboard and coolly announced that he and Bruria Kaufman had solved this problem and they indeed found that the long range order is nonzero. Infact he even wrote down the formula on the blackboard. He repeated his comments at the first postwar IUPAP conference on Statistical Mechanics at Florence in 1949 after a talk by Rushbrook. However, Kaufman and Onsager never published their calculatuions ; it only appeared as a discussion remark [18]. In print the first full calculation was infact published by Yang [19]. Why did Onsager not publish his results? Years later, only in 1969 he gave the reason. In computing the long range order, Onsager was led to a general consideration of Toeplitz matrices but he did not know how to fill out holes in the maths−by the time he did, mathematicians were already there! What Onsager and Yang had obtained was that as T goes to T_c from below then the long range order $\overline{L}(B = 0, T)$ is given by

$$\overline{L}(B = 0, T) = (1 - T/T_c)^{1/8} \tag{7}$$

while it is zero if T approaches T_c from above. It may be noted here that both the mean field and the Bethe approximation predict completely wrong behaviour i.e. they predict that $\overline{L}$ will go like $(1-T/T_c)^{1/2}$. These approximations do not also reproduce the logarithmic divergence of the specific heat. This difference with the approximate theory is rather important. One realizes that the nature of singularity is rather subtle. No matter what approximation one does, short of an exact solution, one never gets the log. This drives home a point that the singularity is due to certain special circumstances that get killed by the approximations. It was several years later, when it could be identified as the effect of fluctuation, as reflected through fluctuation - response theorem. The universality of various phase transitons comes from the nature of fluctuations. It is the renormalization group approach developed by Kadanoff, Wilson and others, that gave the proper framework to handle fluctuations [21].

Unsolved Problems: To appreciate how nontrivial Onsager's exact solution of the 2-dimensional Ising model was, let us remember that till today the exact solution of the 2-dimensional Ising model in nonzero external magnetic field has not been obtained. Further, the 3-dimensional Ising model with or without external magnetic field is also an unsolved problem. Similarly the 2-dimensional Ising model with any additional interaction (as, e.g., nearest and next-to nearest neighbour interactions) is also an unsolved problem. However, special cases like the three spin interaction on a traingular latice can be solved [22].

V. INFLUENCE

Let us go back to the paper and discuss some of the developments where the 1944 paper played a direct role.

(1) **Duality** : Duality between high temperature and low temperature for the Ising partition function was known before the Onsager Solution. However, Onsager used it to the full extent and introduced the idea of star-triangle (ST) transformation that could be used to solve the triangular lattice problem. The existence of duality is important because "it always converts order into disorder and vice versa" [4]. Since the low temperature ordered phase is

characterized by an order parameter, it seems possible to define an analogous quantity for the high temperature phase also. It is only in the eighties that such "disorder" parameters played an important role in conformal invariance. The ST transformation turned out to be important in a different development when attempts were made in the 70's to solve more complex models, like vertex models. The existence of duality and ST guarantees the existence of commuting transfer matrices as expressed through the Yang-Baxter equation. The solvability of these models relies heavily on such commuting matrices [22].

(2) **Transfer matrix** : The major problem was to diagonalize the transfer matrix. The largest eigenvalue determines the thermodynamic behaviour. Onsager could find all the eigenvalues for a finite lattice. This had implications far beyond the original idea of getting only the free energy.

Onsager's result showed that for a square (or any two dimensional) lattice, four different terms are needed if periodic boundary conditions are used. Why four was answered in a clear way by Kasteleyn in 1963 when he gave a general procedure for solving the Ising model using combinatorics ("dimers"). Kasteleyn showed that for a surface of genus g, 4^g "terms" are needed. For open boundary conditions i.e. for strictly planar lattices , g = 0 while, for periodic boundary conditions, g = 1 as it has the geometry of a torus [23]. This also indicates that pushing the same approach to solve a 3-d lattice would require an infinite number of terms.

(3) **Correlation length and correlation**:

Since Onsager could find all the eigenvalues, he could also identify the correlation length. If $\lambda_1 > \lambda_2 > \lambda_3 ...$ are the eigenvalues, then the partition function can be written as

$$Q = \lambda_1^N \left[1 + \left(\frac{\lambda_2}{\lambda_1} \right)^N + ... \right], \tag{8}$$

where N is the number of rows in the direction of transfer. The phase transition occurs when the largest eigenvalue is degenerate. Close to the critical point, one can write $(\lambda_2/\lambda_1)^N \sim e^{-N/\xi}$ where $\xi = -(|\ln \lambda_2/\lambda_1|)^{-1}$. This identifies ξ as a special length scale to characterize the critical behaviour. The degeneracy of the eigenvalues as $T \to T_c$, in turn, implies a diverging length scale at the transition point. In this way, Onsager showed that $\xi \sim| T - T_c |^{-\nu}$ with $\nu = 1$. The renormalization group approach mentioned earlier is based on the idea of a diverging length scale so that the system is scale invariant at the critical point.

Later on, in 1949 Kaufman and Onsager showed that at $T = T_c$, spin correlation $\Gamma(r) \equiv < S(0)S(\vec{r}) >$ deccays as $r^{-1/2}$ for large distance [20]. In 1965, Patashinskii-Pokrovsky suggested the now famous scaling form $\Gamma(r,T) \approx r^{-1/4} D(r/\xi)$ where, apart from the power law prefactor, all distances are scaled by the correlation length at that temperature [24]. Explicit calculations could put the correlation function, in the proper asymptotic limit, in the scaling form but it was also noted that for $T < T_c, \Gamma \sim e^{-r/\xi}/r^2$. This anomalous behaviour in the low temperature phase is a very peculiar property of the Ising model in zero magnetic field. It is in the eighties that this point got clarified through the use of directed polymers [25].

(4) **Boundary effects** : Onsager recognized the importance of boundary conditions. He realized that by changing the couplings in one row to antiferromagnetic coupling ($J \to -J$) would increase the free energy in the ordered state because across this line spins cannot be aligned as in the bulk. This change, in other words, will force an interface in the system. The increase in free energy is then the energy of the interface or the surface tension. Since the interfaces vanishes at the critical point, the surface tension also vanishes at $T = T_c$. The exact solution showed that the surface tension vanishes as $(T_c - T)^{\mu}$ with $\mu = 1$. More general arguments later on proved that $\mu + \nu = 2$ for the two dimensional Ising model. For any other model the right hand side would involve the specific heat exponent also.

(5) **Finite size effects**: Since the partition function was known for finite lattices, the specific heat could also be calculated. It was shown that, at $T = T_c$, the specific heat per spin, C, is finite. No phase transition can take place in a finite system. Furthermore, $C \sim \ln N$ as $N \to \infty$ (for a $N \times \infty$ lattice). It took many years to grasp the significance of this result until the idea of finite size scaling was introduced in 1970. Finite size scaling now plays an important role in analysis of numerical data in wide varities of problems.

A more dramatic result, though cannot be found explicitly, in the Onsager paper, is the finite size behaviour of the free energy. Collecting various terms, from the calculation of the speciofic heat, it was found that for an $m \times n$ lattice

$$\lim_{m\to\infty} \frac{lnZ_{mn}(T_c)}{mn} = \frac{2G}{\pi} + \frac{1}{2}\ln 2 + \frac{\pi}{12}\frac{1}{n} + .., \tag{9}$$

where G is the Catalan constant. This is to be contrasted with Eq. 8 for $T \neq T_c$ where there is an exponential approach to the large lattice (thermodynamic) limit . In 1986, Affleck argued using conformal invariance that in two dimensions, at criticality,

$$\lim_{m\to\infty} \frac{lnZ}{mn} = const + \frac{\Lambda c}{6r}$$

with c the central charge as the only quantity required to identify the critical behaviour ('universality class') in 2 dimensions [26]. A comparision of the two gives c = 1/2 for the Ising model.

(6) **Experiment**: The simplicity and wide applicability of the two dimensional Ising model and its solution sill leave behind a queer sensation that, after all, the model is a bit artificial. So far as the exponents are concerned, they are universal and a wide variety of systems do have Ising exponents. It however remained a challenge to find an experimental system that could exhibit the behaviour one finds from a landmark of theoretical physics. The challenge was met when an alloy was found in 1974 which showed the behaviour of magnetization exactly like the Onsager result with only T_c as the adjustable parameter [27].

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