Surfaces with $p_g = 3$ and $K^2 = 4$ and extension-deformation theory

by
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D.D.

"Щи да каша, пища наша."

– Old Russian proverb.

Summary

In Chapter 2 we give a description of an algorithm suggested by Reid for studying extensions of a variety $C \subset \mathbb{P}^n$ as a hyperplane section of a variety in $\mathbb{P}^{n+1}$.

In Chapter 3 we use this method to study surfaces with numerical invariants $p_g = 3$ and $K^2 = 4$. We find that there are 5 families of such surfaces and produce explicitly the canonical ring for a generic member of each family. In [Ho 1] there is a geometric study of surfaces with these invariants.

Proposition (15.2) is an example of an obstruction to the extension deformation algorithm which appears in degree 4.

In Chapter 4 we write down some one parameter deformations between the families. We conjecture that there are no degenerations, $II \to III_a$ or $II \to III_b$. We draw some geometric conclusions, from the algebraic descriptions, about the branch locus of surfaces of type III, III_a and III_b as double covers of $\mathbb{P}^2$ [Ho1]. It is also shown that a surface of type II is the resolution of a numerical quintic with an elliptic Gorenstein singularity of type $k = 1$.

Introduction

Abstract

The aims of this thesis are 2-fold.

We consider the extension-deformation method of Reid in some detail. In section 6 we explain its equivalence, as a practical tool of calculation, to the cotangent complex as discussed in [LS]. We reproduce the calculation of Pinkham [P] and the result of Griffin [G] as expository examples of this method.

Secondly, we use this method to analyse surfaces with $p_g = 3$ and $K^2 = 4$. We produce explicitly their canonical rings and use this information to draw some conclusions about the geometry of the surfaces and their moduli space.

Extension-deformation Theory

Since the cotangent complex was first introduced many people have made use of it to explore the deformation theory of singularities (e.g. [P], [Sc] and others). In [LS] Lichtenbaum and Schlessinger explain the definitions of the spaces $\mathbb{T}^i$ and their role in producing infinitesimal extensions and deformations of rings. More recently Reid suggested a method to study extensions of a variety considered as a hyperplane section of another variety. This extension-deformation method is equivalent to the calculation of $\mathbb{T}^i$. Griffin [G] used this method to solve the problem of giving an explicit family of numerical quintics, $\{X_t\}$, such that

1) X_t is a smooth quintic in $\mathbb{P}^3$, for $t \neq 0$,

2) X_0 is a smooth numerical quintic whose canonical system has a single base point.

In Chapters 3 and 4 we use the method to classify surfaces with $p_g = 3$ and $K^2 = 4$. In this case, as with the quintics it is possible to find a format of the relations in each ring which survives the extension-deformation process. This should lead to a good understanding of the space of extensions and the moduli space of deformations in each case.

In Proposition (15.2) we give an example of an obstruction occurring very late in the process. There are two formats for the ring in this case which seem only to differ when there are nilpotents in the extension. This means that one of the formats may not be present in the intermediate stages of the calculation but conditions are imposed which force it to return for the final result.

Surfaces of General Type

Much work has been done on the problem of describing surfaces with small K^2 (see for example [R3], [C1], [C2], [Ci], [Ho1], [Ho3]). In particular in [Ho1] there is a geometric description of 3 classes of surfaces with $p_g = 3$ and $K^2 = 4$. Horikawa's approach here is predominantly geometric in nature. There are two main differences between the results here and those in [Ho1]. Due to a mistake on Horikawa's part, surfaces of our type II were omitted from his classification. This mistake was corrected only in a footnote to [Ho1]. Consequently the notation differs and our surfaces of type III correspond to the Horikawa surfaces of type II. Secondly we consider separately surfaces of type III_a and III_b which are indistinguishable using Horikawa's description of their branch loci [Ho1, surfaces of type II.1].

Catanese ([C1], [C2]) has developed an algebraic approach to the study of surfaces with small numerical invariants. The geometric interpretation of this method is related to the study of generic projections to $\mathbb{P}^3$. Using this method Catanese and Debarre [CD] have studied regular surfaces with $p_g = 1$, $K^2 = 2$. The extension-deformation theory of Reid has now been applied in several different cases, and gives some hope of proving the conjecture that the moduli space is irreducible in the torsion zero case of [CD].

CONTENTS

CHAPTER 1 :

Preliminaries.

2. Notation

A *curve* is a projective variety of dimension 1 over $\mathbb{C}$.

A *surface* is a projective variety of dimension 2 over $\mathbb{C}$.

Given a nonsingular surface S, divisors D, D_1, D_2 on S and the corresponding invertible sheaf $\mathcal{O}_S(D)$, we write:

D_1D_2 = intersection number on S.

$h^i(D) = h^i(\mathcal{O}_S(D)) = \dim_{\mathbb{C}} H^i(S,\mathcal{O}_S(D))$ and

$\chi(D) = \chi(\mathcal{O}_S(D)) = h^0(D) - h^1(D) + h^2(D)$.

$|D|$ = the linear system in which D moves,

$r = \dim |D| = h^0(D) - 1$ and

for $h^0(D) \geq 1$, we have $\varphi_{|D|} : S \dashrightarrow \mathbb{P}^r$, the corresponding rational map.

A linear system of degree d and dimension r is denoted by the symbol

g^r_d.

K_S = canonical divisor of S.

$\mathcal{O}_S(K_S) = \bigwedge^2 \Omega^1_S$.

$p_g = p_g(S) = h^0(K_S)$ = geometric genus of S.

$q(S) = h^1(K_S)$ = irregularity of S.

$R(S,D) = R(S,\mathcal{O}_S(D)) = \bigoplus_{n=0}^{\infty} H^0(S,\mathcal{O}_S(nD))$.

$R(S,K_S)$ = canonical ring of S.

Suppose we have a set of elements $\{x_i \in H^0(rK_X) \mid i = 1,...,k\}$. Then we use the notation $S^n(x_i)$ to mean the set of elements $\{x_{i_1}^{r_1}...x_{i_j}^{r_j} \in H^0(nrK_X) \mid \sum_{t=1}^{j} r_t = n, i_s \in \{1,...,k\}\}$.

We are interested mostly in surfaces of *general type* i.e. surfaces S such that dim Proj $R(S,K_S) = 2$.

For the formal definitions of the theory of the cotangent complex see [LS]. Similarly for the theory of weighted projective spaces see [D].

3. Facts and formulas

Let S be a nonsingular surface. We recall the following well known results, for future reference.

Riemann-Roch Formulas (3.1) (see [B])

a) $\chi(D) = 1 - g + \deg(D)$

for a divisor D on a nonsingular curve of genus g.

b) $\chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - DK_S)$

for any divisor D on S.

Adjunction Formula (3.2) (see [B])

$$C^2 + CK_S = 2g - 2$$

for an irreducible curve of genus g on S. Notice that

$$C^2 \equiv CK_S \pmod 2$$

as a useful corollary.

Base Point Free Pencil Trick (3.3) (see [ACGH, p.126])

Let C be a smooth curve, L an invertible sheaf on C, and F a torsion free $\mathcal{O}_C$-module. Suppose that s_1 and s_2 are linearly independent sections of

L and denote by V the subspace of $H^0(C, L)$ they generate. Then the kernel of the cup-product map

$$V \otimes H^0(C, F) \to H^0(C, F \otimes L)$$

is isomorphic to $H^0(C, F \otimes L^{-1}(B))$, where B is the base locus of the pencil spanned by s_1 and s_2.

M. Noether's Theorem (3.4) (see [ACGH, p.117])

If C is a non-hyperelliptic curve, then the homomorphisms

$$\mathrm{Sym}^{\ell} H^0(C, K) \to H^0(C, \ell K)$$

are surjective for $\ell \geq 1$.

The Index Theorem (3.5) (see e.g. [BPVdeV (2.15)])

Let D and E be divisors with rational coefficients on the algebraic surface X. If $D^2 > 0$ and $DE = 0$ then $E^2 \leq 0$ and $E^2 = 0$ if and only if E is rationally equivalent to 0.

Bertini's Theorem (3.6) (see e.g. [Ha] III.10.9.2)

Let X be a nonsingular variety over an algebraically closed field of characteristic 0. Let δ be a linear system on X with no fixed part. Then the general member of δ can only have singularities at the base points.

The Snake Lemma (3.7) (see e.g. [Co2] p.129)

Let R be a ring and consider the following commutative diagram of exact sequences of R-modules:

$$\begin{array}{ccccccccc} & & A & \to & A' & \to & A'' & \to & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \to & B & \to & B' & \to & B'' & & \end{array}$$

Then there is a map, $\Delta : \ker \gamma \to \operatorname{coker} \alpha$, such that

$$\ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma$$

is exact.

4. Structure theorems for Gorenstein rings

Definition (4.1)

Let R be a Noetherian ring and $M \neq 0$ a finite R-module. Then we define

$$\text{grade } M = \inf \{i \mid \text{Ext}^i_A(M, A) \neq 0\}.$$

This definition extends the geometric concept of codimension for an ideal.

Definitions (4.2) (see [Ma,p.142])

An n-dimensional Noetherian local ring R with residue field k is Gorenstein if $\text{Ext}^i_R(k, R) = 0$ for $i \neq n$ and $\simeq k$ for $i = n$.

A Noetherian ring is Gorenstein if its localisation at every maximal ideal is Gorenstein.

For a Gorenstein ring, R and perfect ideal $I \subset R$ we have the equality

$$\text{grade } I = \text{proj dim } (R/I).$$

Let R/I be the canonical ring of a regular surface of general type. Then R/I is Gorenstein [GW] and the codimension of the canonical model of the surface is equal to the grade of I [BE]. In this section we state the structure theorems for ideals of Gorenstein rings of grade 2 and 3.

Proposition (4.1) (see [S])

Let I be a grade 2 ideal in R such that R/I is Gorenstein. Then R/I is a complete intersection.

□

Before stating the theorem for grade 3 ideals we shall say a few words about Pfaffians. It is well-known (e.g. [Co1] p.220) that for every (2n)X(2n) antisymmetric matrix, M, there can be found Pf(M), a polynomial in the entries of M, such that $\text{Pf}(M)^2 = \det M$. We shall usually be interested in the case n = 2 and in this case we have by convention:

$$\mathrm{Pf}\begin{bmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{bmatrix} = -AF + BE - CD.$$

Proposition (4.2) [BE Theorem 4.2]

Let I be a grade 3 ideal in R, such that R/I is Gorenstein. Then I is generated by the diagonal Pfaffians of an antisymmetric (2n+1)x(2n+1) matrix, M. That is the Pfaffians of the antisymmetric 2nx2n matrices formed by removing the ith row and ith column from M, i = 1,...,2n+1.

□

5. Canonical Linear Systems

We collect together the results of several theorems concerning surfaces of general type with given values of p_g and K_S^2 into two lemmas relevant to the particular surfaces which we study.

Lemma (5.1)

Let S be a surface of general type with $p_g = 3$ and $K_S^2 = 4$. Let $C \in |K_S|$ be a general element. Then either

1) $|K_S|$ has no fixed part, g(C) = 5 and deg. base locus ≤ 2

or

2) $|K_S| = |\Gamma| + E$ where Γ has no fixed part and E is a fixed (-2)-cycle.

Proof

We know that $|K_S|$ is not composed of a pencil [Ho 1, XG]. Then let $K_S = \Gamma + E$, where Γ is free and E is fixed. By [Ho 2, Lemma 2]

$$\Gamma^2 \geq 2$$

and by 2-connectedness of K_S either E = 0 or

$$\Gamma E \geq 2.$$

Now $4 = K_S{}^2 \geq \Gamma^2 + \Gamma E \geq 2$. Therefore either $\Gamma^2 = \Gamma E = 2$ and hence $E^2 = -2$ or, alternatively, $\Gamma = K_S$. We consider these 2 cases separately:

1) $K_S = \Gamma$. By the Adjunction Formula (3.2) $g(C) = 5$. Now deg $K_{S|\Gamma} = 4$ and dim $|K_{S|\Gamma}| = 1$. If the degree of the base locus of $|K_S| \geq 3$ there would be a complete linear system of degree ≤ 1 and dimension $= 1$. This contradicts genus $C = 5$, and so deg. base locus ≤ 2 as required.

2) $K_S = \Gamma + E$. We must show that $|\Gamma|$ is base point free. Considering $|K_{S|\Gamma}|$ we see as before that the degree of the base locus ≤ 2 and since deg $E_{|\Gamma} = 2$, $|\Gamma_{|\Gamma}|$, and hence $|\Gamma|$, is base point free.

□

Lemma (5.2)

Let S be as above. Then S is in one of the following mutually exclusive classes:

I. $|K_S|$ is free.

II. $|K_S|$ has one base point.

III. $|K_S|$ has two distinct base points.

III_a. $|K_S| = |\Gamma| + E$ where $|\Gamma|$ is free and E is a fixed (-2)-cycle. $E_{|\Gamma} = P_1 + P_2$ for distinct points P_1 and P_2.

III_b. $K_S = \Gamma + E$ where Γ is free and E is a fixed (-2)-cycle. $E_{|\Gamma} = 2P$.

Proof

From (5.1) we have 2 cases:

1) $|K_S|$ has no fixed part and deg. base locus ≤ 2.

I. deg. base locus $= 0$ and $|K_S|$ is free.

II. deg. base locus $= 1$ and $|K_S|$ has one base point.

III. deg. base locus = 2. Suppose $|K_S|$ does not have distinct base points. Then $|K_{S|C}| = D + 2P$ where D is a g^1_2 and 2P is fixed. By the Adjunction Formula

$$2K_{S|C} = K_C$$

and so (e.g. [ACGH], Appendix B) P is a Weierstrass point and $|2P| = g^1_2$. This contradicts 2P being fixed.

2) $K_S = \Gamma + E$ where $E^2 = -2$. As pointed out in (5.1) $|K_{S|\Gamma}|$ has base locus of degree 2. This gives cases III_a and III_b.

□

CHAPTER 2 :

Extension-Deformation Theory

6. Extension - Deformation Theory

Let X be a canonically embedded surface of general type, and $C \in |K_X|$ a hyperplane section given by the vanishing of $x_0 \in H^0(X, K_X)$. If we can write down the ring

$$R(C, K_{X|C}) = \bigoplus_{n=0}^{\infty} H^0(C, nK_{X|C})$$

explicitly then we can calculate all possibilities for the ring

$$R(X, K_X) = \bigoplus_{n=0}^{\infty} H^0(X, nK_X).$$

We split this section into two parts, one explaining the general strategy behind the above calculation, and one explaining the practical algorithm by which this strategy is carried out in terms of the cotangent complex of [LS]. A more satisfactory theory to accompany this essentially practical approach is nearing completion and is expected to appear in [R2].

General Strategy (6.1)

Suppose C and X are as above, and we are given the ring $R(C, K_{X|C})$, then our aim is to produce the ring $R(X, K_X)$. Let $D = K_{X|C}$, then the strategy (due to Reid) is to produce a sequence of families of rings,

$$\{R(C, D)\}, \{R(2C, D^{(2)})\},...,\{R(nC, D^{(n)})\}, \{R(X, K_X)\}$$

where,

$$R(C, D) = \mathbb{C}[x_1,...,x_k]/I,$$

$$R(\ell C, D^{(\ell)}) = \mathbb{C}[x_0,x_1,...,x_k]/(I^{(\ell)}, x_0^{\ell}),$$

$$R(X, K_X) = \mathbb{C}[x_0,x_1,...,x_k]/(I^{(n)}).$$

Each $R(\ell C, D^{(\ell)})$ is to depend in a linear way on the one before and

we have

$$\{R(\ell C, D^{(\ell)})/(x_0^{\ell-1})\} \subset \{R((\ell-1)C, D^{(\ell-1)})\}.$$

For example, if C is smooth, in computing $R(2C, D^{(2)})$ the theory is essentially that of Kodaira and Spencer [KS].

In particular, the ring $R(X, K_X)$ will be required to satisfy,

$$R(C, D) = R(X, K_X)/(x_0).$$

The following proposition goes some way towards describing the generators and relations of such a ring.

Proposition (6.1) (compare [R2])

Suppose $R = \mathbb{C}[x_1,...,x_k]/I$ and R' is such that

$$R'/(x_0) = R$$

for some $x_0 \in R'$ with $\deg x_0 > 0$. Then

$$R' = \mathbb{C}[x_0,...,x_k]/I'$$

and:

i). If x_0 is a non-zero divisor and $I = (f_1,...,f_r)$, where the f_i are homogeneous, then there are homogeneous $F_1,...,F_r \in I'$ such that $I' = (F_1,...,F_r)$ where F_i reduces mod x_0 to f_i.

ii). With the notation of i); x_0 is a non-zerodivisor of R' if and only if for every syzygy σ_i: $\Sigma l_i f_i \equiv 0 \in \mathbb{C}[x_1,...,x_k]$, there are $L_i \in \mathbb{C}[x_0,...,x_k]$ such that Σ_i: $\Sigma L_i F_i \equiv 0 \in \mathbb{C}[x_0,...,x_k]$ where L_i reduces mod x_0 to l_i.

Proof

We proceed by induction on the degree of $g \in R'$. Firstly $x_0,...,x_k$ generate because given $g \in R'$ we have $g \bmod x_0 \in \mathbb{C}[x_1,...,x_k]$ so we can write $g = g_0(x_1,...,x_k) + x_0 g'$, where $g' \in R'$, $\deg g' < \deg g$.

i) Consider the commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & (x_0) \cap I' & \to & I' & \to & I & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (x_0) & \to & \mathbb{C}[x_0,\ldots,x_k] & \to & \mathbb{C}[x_1,\ldots,x_k] & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & x_0R' & \to & R' & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

We see that $I' \to I$ is surjective by the Snake Lemma (3.7). Thus for all $f \in I$, homogeneous of degree d, there exists $F \in I'$ and $F' \in \mathbb{C}[x_0,\ldots,x_k]$ such that $f = F - x_0F'$ with F homogeneous of degree d.

Choose $F_1,\ldots,F_r$ in this way and let $G \in I'$ be any element. Then $G \longrightarrow \Sigma l_i f_i$ with $l_i \in \mathbb{C}[x_1,\ldots,x_k]$ and thus $G - \Sigma l_i F_i \in (x_0) \cap I'$.

Since x_0 is a non-zerodivisor mod I', $(x_0) \cap I' = x_0I'$ and $G - \Sigma l_i F_i = x_0G'$ for some G' of lower degree than G in I'. Continuing by induction on degree proves i).

ii) Write $F_i \equiv f_i + x_0g_i$ and suppose $\sigma_i : \Sigma l_i f_i \equiv 0$. Then

$$\Sigma l_i F_i \equiv x_0 \Sigma l_i g_i$$

Since $\Sigma l_i F_i \in I'$ and $x_0 \Sigma l_i g_i \in (x_0)$, x_0 a non-zerodivisor implies that $\Sigma l_i g_i \in I'$, and we can write $\Sigma l_i g_i = \Sigma m_i F_i$. This gives us $\Sigma L_i F_i \equiv 0$ where $L_i = l_i - x_0 m_i$.

Conversely, if $g \in \mathbb{C}[x_0,\ldots,x_k]$ and $x_0g = \Sigma l_i F_i$ then $\Sigma l_i f_i \equiv 0$. Thus there exists L_i with $\Sigma L_i F_i \equiv 0$, and $x_0g \equiv \Sigma(l_i - L_i)F_i = x_0 \Sigma m_i F_i \in \mathbb{C}[x_0,\ldots,x_k]$, where $L_i = l_i - x_0 m_i$, therefore $g \in I$.

□

This theorem tells us that the generators, relations and syzygies of

$R' = R(X, K_X)$ occur in the same degrees as those of $R = R(C, D)$. To calculate the relations and syzygies explicitly, we observe that given $f_i \in I$ we are looking for

$$F_i = f_i + x_0 f_i' + x_0^2 f_i'' + \ldots + x_0^{\ell} f_i^{(\ell)} + \ldots,$$

such that $f_i^{(\ell)} \in \mathbb{C}[x_1,\ldots,x_k]$, and the F_i satisfy syzygies as in Proposition (6.1) ii). We get the sequence of rings

$$\{R(C, D)\}, \{R(2C, D^{(2)})\},\ldots,\{R(nC, D^{(n)})\}, \{R(X, K_X)\}$$

by calculating the F_i in stages allowing successively higher powers of x_0.

Practical Details (6.2)

Proposition (6.2) is to be found in a more general form in [LS, Theorem (4.2.5)]. First we make some definitions to explain the terminology found in the diagram $(\mathfrak{E}_\ell)$ in the statement of Proposition (6.2).

Definitions

Suppose we are given

$$R^{(1)} = \mathbb{C}[x_1,\ldots,x_k]/I$$

and

$$R^{(\ell)} = \mathbb{C}[x_0,x_1,\ldots,x_k]/(I^{(\ell)}, x_0^{\ell})$$

such that $R^{(\ell)} \bmod x_0$ is $R^{(1)}$.

1). Define S_R to be the graded $\mathbb{C}[x_1,\ldots,x_k]$-module with grading $[S_R]_d = R^{(1)}{}_{d-1}$ and with multiplication defined by

$$fg = 0 \text{ for all } f, g \in S_R.$$

2). Define $T_{R(\ell)}$ to be the graded $\mathbb{C}[x_0,\ldots,x_k]$-module with grading $[T_{R(\ell)}]_d = [R^{(\ell)}]_d$ and with multiplication defined by

$$\Big(\sum_{i=0}^{\ell-1} x_0^i p_i\Big)\Big(\sum_{i=0}^{\ell-1} x_0^i q_i\Big) = x_0 \sum_{i=0}^{\ell-2} x_0^i \sum_{j+k=i} p_j q_k$$

for all $\sum_{i=0}^{\ell-1} x_0^i p_i$, $\sum_{i=0}^{\ell-1} x_0^i q_i \in T_{R^{(\ell)}}$.

[LS, (4.2.5)] gives the following result in ahighly theoretical setting. The diagram $(\mathfrak{E}_\ell)$ in Proposition (6.2) has been taken from [LS, (4.2.5)] and rewritten in a more pragmatic notation.

Proposition (6.2)

Suppose that we are given $R^{(1)}$ and $R^{(\ell)}$ as above. Then $R^{(\ell+1)}$ exists if and only if there exists a degree preserving module homomorphism β_ℓ in the diagram $(\mathfrak{E}_\ell)$, such that $i_\ell' \circ \beta_\ell = \alpha_\ell'$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma & \xrightarrow{i_\ell'} & (f_1,\ldots,f_r) & \longrightarrow & \mathbb{C}[x_1,\ldots,x_k] & \longrightarrow & R^{(1)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_\ell'} & \swarrow{\scriptstyle \beta_\ell} & \downarrow{\scriptstyle \alpha_\ell} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S_R & \xrightarrow{i_\ell} & T_{R^{(\ell)}} & \xrightarrow{j_\ell} & R^{(\ell)} & \longrightarrow & R^{(1)} & \longrightarrow & 0
\end{array} \qquad (\mathfrak{E}_\ell)$$

The rows of $(\mathfrak{E}_\ell)$ are exact. $(f_1,\ldots,f_r)$ is the free $\mathbb{C}[x_1,\ldots,x_k]$-module generated by $f_1,\ldots,f_r$. $\Sigma$is the $\mathbb{C}[x_1,\ldots,x_k]$-module generated by $\sigma_1,\ldots,\sigma_t$ such that the top row of $(\mathfrak{E}_\ell)$ is exact. The maps $i_\ell: S \to T$ and $j_\ell: T \to R^{(\ell)}$ are given by

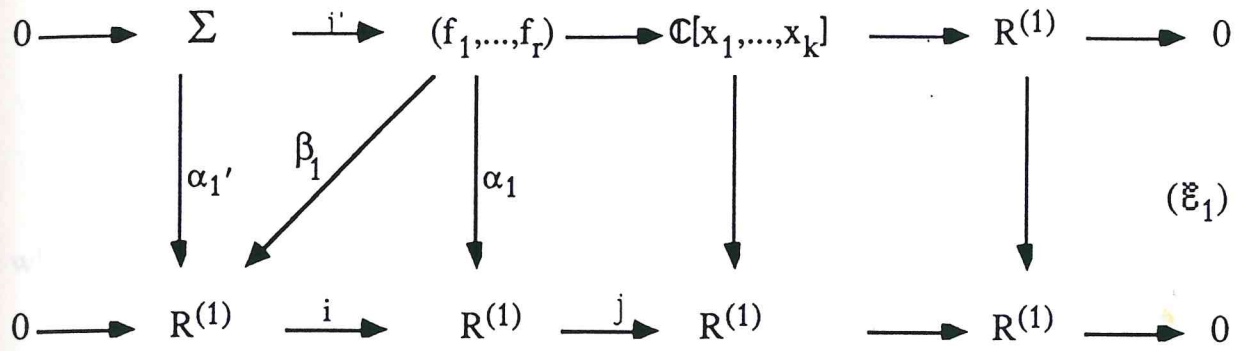
$$i_\ell(p) = x_0^{\ell-1} p \text{ and}$$

$$j_\ell\left(\sum_{i=0}^{\ell-1} x_0^i p_i\right) = x_0 \sum_{i=0}^{\ell-2} x_0^i p_i.$$

The maps α_ℓ and α_ℓ' are defined inductively, indeed they depend on the map $\beta_{\ell-1}$ in the diagram $(\mathfrak{E}_{\ell-1})$.

Step 1 : $(R^{(1)} \to R^{(2)})$.

We have the following diagram:



Here the maps α_1, α_1' and j are all zero and i is the identity map. Thus we must find β_1: $(f_1,...,f_r) \to S_R$ such that deg $\beta_1(f_i)$ = deg f_i -1 and $\Sigma l_i\beta_1(f_i) = 0$ whenever $i'(\sigma_j) = \Sigma l_i f_i$ for some σ_j. The zero map is a solution here and corresponds to the fact that first order extensions are always unobstructed. The construction of β_1 allows us to define,

$$R^{(2)}(\beta_1) = \mathbb{C}[x_0,...,x_k]/(I^{(2)}, x_0^2),$$

where $I^{(2)}$ is generated by $F_1,...,F_r$ given by $F_i = f_i + x_0\beta_1(f_i)$.

Let β: $(f_1,...,f_r) \to S_R$ be a 'general' such map. That is $\{\beta(f_1),...,\beta(f_r)\}$ are generic polynomials satisfying the required conditions.

Step 2 : $(R^{(2)} \to R^{(3)})$.

Suppose we have a ring $R^{(2)}(\beta)$ from step 1. That is, we have a map $\beta : (f_1,...,f_r) \to S_R$ such that $\Sigma l_i\beta(f_i) = 0$ whenever $i'(\sigma_j) = \Sigma l_i f_i$ for some σ_j. $\beta : (f_1,...,f_r) \to S_R$ can then be lifted to $\beta^{\sim} : (f_1,...,f_r) \to \mathbb{C}[x_1,...,x_k]$ where $\Sigma l_i\beta^{\sim}(f_i) \in I$ whenever $i'(\sigma_j) = \Sigma l_i f_i$ for some σ_j.

Then $\alpha_2 : (f_1,...,f_r) \to T_{R^{(2)}}$ is given by $\alpha_2(f_i) = \beta(f_i)$. Consider $\sigma_j \in (\sigma_1,...,\sigma_t)$, where $i'(\sigma_j) = \Sigma l_i f_i$ then step 1 has given the expression,

$$\Sigma l_i\beta^{\sim}(f_i) = \Sigma p_k f_k \in I.$$

We define $\alpha_2'(\sigma_j) = \Sigma p_k \beta(f_k)$, and then β_2 in the diagram $(\mathfrak{E}_2)$ must satisfy:

$$\Sigma l_i \beta_2(f_i) - \Sigma p_k \beta(f_k) = 0.$$

As in step 1 the construction of β_2 allows us to put

$$R^{(3)}(\beta_2) = \mathbb{C}[x_0,\ldots,x_k]/(I^{(3)}, x_0^3),$$

where $I^{(3)}$ is generated $F_1,\ldots,F_r$ given by $F_i = f_i + x_0\beta(f_i) + x_0^2\beta_2(f_i)$.

If there are no maps β_2 making the diagram $(\mathfrak{E}_2)$ commute, the extension is obstructed. In this case, in trying to construct β_2 we are forced to put conditions on β, that is we replace β by the most general β_1 such that $(\mathfrak{E}_2)$ can be made to commute.

Step ℓ : $(R^{(\ell)} \to R^{(\ell+1)})$.

To go from $R^{(\ell)} \to R^{(\ell+1)}$ we repeat the above procedure. The example in the next section shows how this algorithm works in a fairly simple case and how obstructions occur in the calculations.

It is precisely because of the occurrence of obstructions that $\{R^{(\ell)}/(x_0^{\ell-1})\}$ is a subset of, not necessarily equal to $\{R^{(\ell-1)}\}$.

7. Pinkham's Example.

In [P. Section 8] Pinkham computes directly the deformations of the cone over the rational curve, C_4 of degree 4 in $\mathbb{P}^4$, using the theory outlined in section 6. In this section we reproduce his calculation, by way of example, pointing out the similarities with the later calculations in Chapter 4.

Example (7.1)

Let $C_4 \subset \mathbb{P}^4$ be the rational, normal curve defined by:

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{bmatrix} \leq 1$$

where $x_1,\ldots,x_5$ are the coordinates in $\mathbb{P}^4$.

This yields the 6 equations,

$$r_1: x_1x_3 = x_2^2,$$
$$r_2: x_1x_4 = x_2x_3,$$
$$r_3: x_1x_5 = x_2x_4,$$
$$r_4: x_2x_4 = x_3^2,$$
$$r_5: x_2x_5 = x_3x_4,$$
$$r_6: x_3x_5 = x_4^2.$$

There are 8 syzygies between these 6 relations, namely:

$$\sigma_1: x_1r_4 - x_2r_2 + x_3r_1 \equiv 0,$$
$$\sigma_2: x_1r_5 - x_2r_3 + x_4r_1 \equiv 0,$$
$$\sigma_3: x_1r_6 - x_3r_3 + x_4r_2 \equiv 0,$$
$$\sigma_4: x_2r_6 - x_3r_5 + x_4r_4 \equiv 0,$$
$$\sigma_5: x_2r_4 - x_3r_2 + x_4r_1 \equiv 0,$$
$$\sigma_6: x_2r_5 - x_3r_3 + x_4r_1 \equiv 0,$$
$$\sigma_7: x_2r_6 - x_4r_3 + x_5r_2 \equiv 0,$$
$$\sigma_8: x_3r_6 - x_4r_5 + x_5r_4 \equiv 0.$$

Let $R_C = \mathbb{C}[x_1,\ldots,x_5]/I$ where $I = (r_1,\ldots,r_6)$. To write down the first order extensions we must compute $\beta(r_i)$ $i=1,\ldots,6$ such that the syzygies $\sigma_1,\ldots,\sigma_8$ are lifted in the way prescribed in (6.2). Let us suppose that

$$\beta(r_i) = \sum_{j=1}^{5} a_{ij}x_j, i = 1,...,6.$$

Then the ideal $I^{(2)}$ (notation as in (6.2)) will be generated by

$$r_i^{(2)} : r_i + x_0 \sum_{j=1}^{5} a_{ij}x_j,\ i = 1,...,6,$$

where the a_{ij} are subject to restrictions imposed by the syzygies. We shall now calculate the effects of these restrictions one syzygy at a time.

First we make some simplifications using changes of coordinates in $x_1,...,x_5$. For example, if we set

$$x_1' = x_1 + a_{13}x_0,$$

then we can rewrite $r_1^{(2)}$ with the coefficient of x_0x_3 equal to 0. In this way before we begin our calculation we can make the assumptions that $a_{13} = a_{42} = a_{43} = a_{44} = a_{63} = 0$.

Consider σ_1: $x_1r_4 - x_2r_2 + x_3r_1 \equiv 0$. We must have

$$x_1\beta(r_4) - x_2\beta(r_2) + x_3\beta(r_1) \in (r_1,...,r_6),$$

if β is going to make the diagram $(\mathfrak{E}_1)$ commute. That is there exist $p_1,...,p_6 \in \mathbb{C}$ such that:

$$\begin{aligned} &x_1(a_{41}x_1 + a_{45}x_5) - x_2\sum_{j=1}^{5} a_{2j}x_j \\ &+ x_3(a_{11}x_1 + a_{12}x_2 + a_{14}x_4 + a_{15}x_5) \\ &\equiv p_1(x_1x_3 - x_2^2) + p_2(x_1x_4 - x_2x_3) + p_3(x_1x_5 - x_2x_4) \\ &+ p_4(x_2x_4 - x_3^2) + p_5(x_2x_5 - x_3x_4) + p_6(x_3x_5 - x_4^2). \end{aligned}$$

Since this is an identity in $\mathbb{C}[x_1,...,x_5]$, we can equate coefficients. We find the following relations hold between the a_{ij}.

$$a_{15} = 0,$$
$$a_{21} = 0,$$
$$a_{23} = a_{12},$$
$$a_{24} = a_{45},$$
$$a_{25} = a_{14}.$$

Similarly, consider the syzygy

$$\sigma_2: x_1r_5 - x_2r_3 + x_4r_1.$$

As before there must be $p_1,\ldots,p_6 \in \mathbb{C}$ (not the same $p_1,\ldots,p_6$ as before, of course) such that:

$$x_1\sum_{j=1}^{5} a_{5j}x_j - x_2\sum_{j=1}^{5} a_{3j}x_j + x_4(a_{11}x_1 + a_{12}x_2 + a_{14}x_4)$$

$$\equiv p_1(x_1x_3 - x_2^2) + p_2(x_1x_4 - x_2x_3) + p_3(x_1x_5 - x_2x_4)$$

$$+ p_4(x_2x_4 - x_3^2) + p_5(x_2x_5 - x_3x_4) + p_6(x_3x_5 - x_4^2).$$

Equating coefficients we can extend the list of relations between the a_{ij} with those below:

$$a_{14} = 0,$$
$$a_{51} = 0,$$
$$a_{52} = a_{31},$$
$$a_{53} = a_{32},$$
$$a_{54} + a_{11} = a_{33},$$
$$a_{55} + a_{12} = a_{34},$$
$$a_{35} = 0.$$

If we treat the syzygy $\sigma_3: x_1r_6 - x_3r_3 + x_4r_2$ in the same way we get the following polynomials for $\beta(r_i)$ which make the diagram $(\mathfrak{E}_1)$ commute. This is easy, but tedious, to check. Put $a_{11} = a$, $a_{12} = b$, $a_{53} = c$, $a_{54} = d$, then,

$$\begin{aligned}
\beta(r_1) &= ax_1 + bx_2,\\
\beta(r_2) &= ax_2 + bx_3,\\
\beta(r_3) &= cx_2 + (a+d)x_3 + bx_4,\\
\beta(r_4) &= 0,\\
\beta(r_5) &= cx_3 + dx_4,\\
\beta(r_6) &= cx_4 + dx_5.
\end{aligned}$$

Thus we have computed the rings which correspond to first order deformations of the rational, normal quartic in $\mathbb{P}^4$. They depend on 4 free parameters, a, b, c and d.

To find the space of second order deformations we again consider the syzygies one at a time and find β_2: $(r_1,...,r_6) \to R_C$ which makes the diagram $(\mathfrak{E}_2)$ commute. For example, reconsider the syzygy σ_1 and suppose that $\beta_2(r_i) = a_i$, i = 1,...,6. Then we must have:

$$x_1a_4 - x_2a_2 + x_3a_1 - (-a\beta(r_1)) \equiv 0.$$

Substituting for $\beta(r_1)$ and equating coefficients we find that:

$$\begin{aligned}
a_1 &= 0,\\
a_2 &= ab,\\
a_4 &= -a^2.
\end{aligned}$$

Treating σ_2 in the same way we must have:

$$x_1a_5 - x_2a_3 + x_4a_1 - (-c\beta(r_1) - (a+d)\beta(r_2)) \equiv 0.$$

Equating coefficients again gives:

$$\begin{aligned}
(a+d)b &= 0,\\
a_5 &= -ac,\\
a_3 &= bc + a(a+d).
\end{aligned}$$

After considering σ_3 we have the whole story:

$$\begin{aligned}
a_1 &= 0,\\
a_2 &= ab,
\end{aligned}$$

$$a_3 = bc + a(a+d),$$
$$a_4 = -a^2,$$
$$a_5 = -ac,$$
$$a_6 = 0.$$

Moreover, we have $(a+d)b = (a+d)c = (a+d)(a-d) = 0$ as a requirement for first order deformations to extend to second order. It can be seen (e.g. from determinantal presentations given below or by continuing the calculation) that the algorithm terminates here, in the sense that these new relations satisfy $\sigma_1,\ldots,\sigma_5$ after removing the condition on the order of x_0. It is important to notice that in general the algorithm will terminate in $\leq N$ steps where N is the maximum degree of the generators of the module of syzygies ($\sigma_1,..,\sigma_t$) (see section 11). As Pinkham remarks [P], this gives two components of the space of deformations of the cone. If $a+d = 0$, we get the rings given by,

$$R = \mathbb{C}[x_0,\ldots,x_5]/I',$$

where I′ is generated by,

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & x_3+ax_0 & x_4+cx_0 \\ x_2+bx_0 & x_3-ax_0 & x_4 & x_5 \end{bmatrix} \leq 1.$$

If $b = c = a-d = 0$, we get the rings given by,

$$R = \mathbb{C}[x_0,\ldots,x_5]/I',$$

where I′ is generated by,

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & x_3+ax_0 \\ x_2 & x_3-ax_0 & x_4 \\ x_3+ax_0 & x_4 & x_5 \end{bmatrix} \leq 1.$$

Notice that these two rings coincide when

$$a = b = c = d = 0.$$

That is exactly at the ring of the cone over C_4.

8. Griffin's Example and Deformation Theory.

In section 7 we saw how the relations in the ring R_C could be formatted in two different ways. That is, the relations $r_1,\ldots,r_6$ can be generated by either,

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{bmatrix} \leq 1,$$

or,

$$\operatorname{rank}\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{bmatrix} \leq 1.$$

In each case the syzygies in the ring depend only on the shape of the matrix, in the sense that if we replace each x_i by a (reasonably general) linear term $l_i(x_0,x_1,\ldots,x_5)$ the resulting ideal will have an isomorphic resolution as a $\mathbb{C}[x_0,\ldots,x_5]$-module. More precisely the terms in the resolution will have the same number of generators in each degree and the maps will be given by matrices of the same shape. In such cases we say that the given presentation of the relations is *formative* (the ring structure depends only on the format of the equations not on the individual coefficients). The idea is that a formative presentation determines the homological algebra of the ring.

For example, any generic determinantal ring has the formative presentation given by a matrix with a rank condition.

It is sometimes the case that a formative presentation of the relations gives rise to *all* the extensions of a ring. In the notation of Proposition (6.1), if

$$R'/(x_0) = R$$

for some non-zerodivisor $x_0 \in R'$, deg $x_0 = d > 0$, then it is possible that the relations in R' can be placed in the same format as those in R. If this is true for all extensions R' we say that the presentation is *informative* in degree -d (the information about the extensions is completely contained in the format of the equations of R).

In [G], Griffin treated the case of numerically quintic surfaces, X, using the method outlined in section 6. In the case where $|K_X|$ has a single transverse base point he takes a nonsingular curve $C \in |K_X|$ given by $(x_0 = 0)$ and calculates the ring $R(C, K_{X|C})$. Proposition 8.1 is a summary of Griffins results in the above notation. Indeed in [R2] there is an extremely beautiful presentation of the numerical quintics in terms of formative presentations which has a lot in common with the case of Pinkham and the examples given in chapter 4 of this thesis.

Proposition (8.1)

i). Let X and C be as above and let

$$A = \begin{bmatrix} x_1 & x_2 & x_3^2 & z_1 \\ x_2 & x_3 & y & z_2 \end{bmatrix}.$$

Then,

$$R(C, K_{X|C}) = \mathbb{C}[x_1,x_2,x_3,y,z_1,z_2]/I,$$

where I is generated by,

$$\text{rank } A \leq 1,$$

giving $r_1,\ldots,r_6$, and

$$AM(A)^t = 0\,,$$

where

$$M = \begin{bmatrix} Q & a_{11}y^2 & 0 & 0 \\ a_{11}y^2 & a_{12}y^2 & 0 & 0 \\ 0 & 0 & a_{13}y & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

giving r_7, r_8 and r_9. Here $a_{ij} \in \mathbb{C}$ and Q is a weighted, homogeneous quartic $Q(x_i,y)$.

ii). The presentation of the ring given in i) is formative.

iii). The presentation of the ring given in i) is informative.

Proof

i), ii) and iii) are effectively the results of the calculations in [G].

□.

CHAPTER 3:

The canonical linear system.

9. Canonical Curves.

Let X be a surface of general type with $p_g = 3$ and $K_X^2 = 4$. In Lemma (5.2) we saw that X belongs to one of five different classes depending on the behaviour of its canonical linear system $|K_X|$. $|K_X|$ can have 0,1 or 2 base points or a fixed (-2)-cycle. In this section we shall analyse the nature of a general element $C \in |K_X|$. We collect together general facts about the canonical curve into a lemma.

Lemma (9.1)

Let X be a surface as above and $C \in |K_X|$ be a general element.

I. If X is of type I then C is a nonsingular, non-hyperelliptic, genus 5 curve.

II. If X is of type II then C is a nonsingular, non-hyperelliptic, genus 5 curve.

III. If X is of type III then C is a nonsingular, hyperelliptic genus 5 curve.

III_a and III_b. If X is of type III_a or III_b then we can write $C = \Gamma + E$ where E is a (-2)-cycle and Γ is a nonsingular, genus 4 hyperelliptic curve with $\Gamma.E = 2$.

Proof

I. By the Adjunction Formula (3.2) $g(C) = 5$ and by Bertinis Theorem (3.6) C is nonsingular. Suppose C is hyperelliptic and let $D = K_{X|C}$. Then $2D = K_C$ and so $D = g_2^1 + P_1 + P_2$, where P_1 and P_2 are

Weierstrass points [ACGH, p.288, Ex.31,32]. This is true for surfaces of all types, but contradicts the number of base points in Cases I and II.

II. As before g(C) = 5 and C is non-hyperelliptic. Bertini's Theorem (3.6) says that C is nonsingular away from the base point. But from Lemma (5.2) deg base locus = 1 in this case and so C is nonsingular at the base point.

III. g(C) = 5 and $|K_X|_C| = g^1_2 + P_1 + P_2$, where P_1 and P_2 are the distinct base points. Bertinis Theorem (3.6) says that C is nonsingular away from the base points and since deg base locus = 2 C is nonsingular at the base points.

III_a and III_b. By the Adjunction Formula (3.2) g(Γ) = 4. The rest follows from Lemmas (5.1) and (5.2)

□

The eventual aim is to describe the ring

$$R(X, K_X) = \bigoplus_{n=0}^{\infty} H^0(X, nK_X)$$

in terms of generators and relations. We do this in Chapter 4 but first we calculate a ring associated to the canonical curve of X (cf. [Mu, lecture 1]).

In cases I, II and III we consider a nonsingular curve $C \in |K_X|$ and the divisor $D = K_X|_C$ on C, and calculate the graded ring $R(C,D) = \bigoplus_{n=0}^{\infty} H^0(C, nD)$. In cases III_a and III_b we consider a nonsingular curve $\Gamma \in |K_X - E|$, where E is the fixed (-2)-cycle, and an appropriate divisor on Γ. We make heavy use of the Base Point Free Pencil Trick (BPFPT (3.3)).

Case I. $|K_X|$ has no base points.

Theorem (9.2)

Consider $C \in |K_X|$, a nonsingular curve and $D = K_{X|C}$ a divisor on C. Then

$$R(C, D) = \mathbb{C}[x_1, x_2, y_1, y_2] / \langle Q_1, Q_2 \rangle$$

where the generators are deg $(x_i, y_i) = (1, 2)$ and the two polynomials Q_1, Q_2 are weighted homogeneous of degree 4 in x_1, x_2, y_1, y_2.

Proof.

Since deg D = 4, we can calculate, using the Riemann-Roch formula (3.1a), that

$$\dim H^0(D) = 2,$$
$$\dim H^0(2D) = 5 \text{ and}$$
$$\dim H^0(nD) = 4(n-1) \text{ for } n > 2.$$

We aim to verify the information in Table 1 below. The notation is explained in section 2.

Table 1.

degree	generators	new relations
1	x_1, x_2	
2	$S^2(x_i), y_1, y_2$	
3	$S^3(x_i), x_i y_j$	
4	$S^4(x_i), S^2(x_i)y_j, S^2(y_i)$	Q_1, Q_2
≥5	no new generators	no new relations

Let x_1, x_2 be generators of $H^0(C,D)$. Then

$$x_1^2, x_1x_2, x_2^2 \in H^0(C, 2D)$$

are linearly independent (a relation between them would have to factorise, contradicting the independence of x_1 and x_2), so choose y_1, y_2 to complete a basis for $H^0(C, 2D)$.

To show that $S^3(x_i) \cup \{x_iy_j\}_{\substack{i=1,\ldots,3 \\ j=1,2}}$ form a basis of $H^0(C, 3D)$ we use the BPFPT (3.3). Consider the natural map

$$\varphi: H^0(C, D)\otimes H^0(C, 2D) \to H^0(C, 3D).$$

Since $|D|$ is a base point free pencil, by the BPFPT (3.3), $\mathrm{Ker}(\varphi) \cong H^0(C,D)$ and hence dim Ker $\varphi = 2$. As dim $H^0(C, 3D) = 8$, φ must be surjective and $\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ is a basis for $H^0(C, 3D)$.

As C is not hyperelliptic (Lemma 9.1), we can appeal to M. Noether's Theorem (3.4) to see the degree 4 generators. This tells us that the 14 monomials in Table 1 generate this 12 dimensional space and hence we can write down two relations Q_1 and Q_2 between them.

Let

$$\varphi_n: H^0(C, D)\otimes H^0(C, nD) \to H^0(C, (n+1)D).$$

Then by BPFPT
$$\begin{aligned} \dim \mathrm{Im}\, \varphi_n &= [2.(4n-4)] - [4n-8] \\ &= 4n \\ &= h^0(C, (n+1)D). \end{aligned}$$

This shows that the x_i, y_j generate R(C, D) as an algebra, and the only relations are generated by Q_1 and Q_2, proving the theorem.

□

Case II. $|K_X|$ has a single transversal base point.

Theorem (9.3)

Consider $C \in |K_X|$ and $D = K_{X|C}$, then

$$R(C,D) = \mathbb{C}[x_1, x_2, y_1, y_2, z]/I$$

generated by x_i, y_i and z with deg $(x_i, y_j, z) = (1, 2, 3)$ and related by I which is generated by

$$r_1: x_1y_2 - x_2y_1,$$

$$r_2 : x_1z - y_1^2,$$

$$r_3 : x_2z - y_1y_2,$$

$$r_4 : y_1z - x_2A - x_1B,$$

$$r_5 : z^2 - y_2A - y_1B,$$

where

$$A = a_1y_2^2 + a_2x_2^2y_2 + a_3x_2^4, \ a_i \in \mathbb{C},$$

$$B = B(x_1,x_2,y_1,y_2)$$

is a degree 4 weighted homogeneous polynomial in x_1, x_2, y_1, y_2.

Moreover $r_1,\ldots,r_5$ are the diagonal 4X4 Pfaffians of the following matrix [see Section 4].

$$M = \begin{bmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ 0 & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z & -A \\ -x_2 & -y_2 & z & 0 & B \\ -y_1 & -z & A & -B & 0 \end{bmatrix}$$

Proof

Our strategy is to choose sections of $\mathcal{O}_C(D)$ according to their geometrical properties, (in fact, with respect to the order of their vanishing at the base point P). Notice that C is trigonal because we can write $D = D_1 + P$ where P is the base point and $|D_1| = g^1_3$ on C.

We calculate the dimensions in Table 2 by Riemann-Roch. Let u be a generator for $H^0(D-2P)$ and $t : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P)$. Then $ut \in H^0(D-P)$ and we let v be the second element of a basis. Since ut^2 and vt vanish to different order at P they are linearly independent in $H^0(D)$ and so form a basis.

Table 2.

Space	dimension	generators
$H^0(D-2P)$	1	u
$H^0(D-P)$	2	ut, v
$H^0(D)$	2	ut^2, vt
$H^0(D+P)$	3	ut^3, vt^2, w

Similarly we need one extra basis element $w \in H^0(D+P)$ which does not vanish at P.

We are aiming to establish Table 3 where $x_1 = ut^2$, $x_2 = vt$, $y_1 = utw$, $y_2 = vw$ and $z = uw^2$.

Table 3.

n	generators of $H^0(nD)$	new relations
1	x_1, x_2	
2	$S^2(x_i), y_1, y_2$	
3	$S^3(x_i), x_iy_j, z$	r_1
4	generated by monomials in x_i,y_j,z	r_2,r_3
5	generated by monomials in x_i,y_j,z	r_4
6	generated by monomials in x_i,y_j,z	r_5

Notice that ut, $v \in H^0(D-P)$ form a free pencil so we can use the BPFPT (3.3) on the maps

$\varphi_{n,m}$: $H^0(nD+mP) \otimes H^0(D-P) \to H^0((n+1)D+(m-1)P)$

This tells us that Ker $\varphi_{n,m} \cong H^0((n-1)D+(m+1)P)$ so that we can calculate whether $\varphi_{n,m}$ is surjective.

Table 2 tells us that x_1, x_2 generate $H^0(D)$. Consider

$$\varphi_{1,1}: H^0(D+P) \otimes H^0(D-P) \to H^0(2D),$$

then Ker $\varphi_{1,1} \cong H^0(2P)$ which is generated by t^2. This means that dim Im $\varphi_{1,1} = 3x2 - 1 = 5$, and so $\varphi_{1,1}$ is surjective and $H^0(2D)$ is generated by u^2t^4, ut^3v, t^2v^2, wut and wv. That is x_1^2, x_1x_2, x_2^2, y_1 and y_2 as in the notation of Table 3.

To see the generators of $H^0(3D)$ we look at $\varphi_{2,1}$ whose kernel is $H^0(D+2P)$. Now we can see that dim Im $\varphi_{2,1} = 7$ and so Im $\varphi_{2,1} \subset H^0(3D)$ is a codimension 1 subspace generated by the linearly independent monomials u^3t^6, u^2t^5v, ut^4v^2, t^3v^3, wu^2t^3, wut^2v, wtv^2. However $uw^2 \in$

$H^0(3D)$ and is independent of those elements given since it does not vanish at P. Thus we have the generators of $H^0(3D)$ given in table 3. The relation r_1 is given because $x_1y_2 = ut^2vw$ and $x_2y_1 = ut^2vw$ and so

$$r_1 : x_1y_2 = x_2y_1$$

For higher degrees we note first that $\varphi_{n,1}$ is surjective for $n \geq 3$ (by Riemann-Roch for curves (3.1a)

$$h^0(nD+P) \quad = 4n-3$$

and

$$\dim \operatorname{Ker} \varphi_{n,1} = h^0((n-1)D+2P) = 4n-6,$$

so

$$\dim \operatorname{Im} \varphi_{n,1} \quad = 2(4n-3) - (4n-6) = 4n$$

which is $h^0((n+1)D))$. The generators in degree 4 come immediately from this, and the relations r_2 and r_3 again follow from the substitution of the definitions of x_1, x_2, y_1, y_2, z, giving

$$r_2: x_1z = y_1^2,$$

$$r_3: x_2z = y_1y_2.$$

It remains to derive the relations r_4 and r_5. These are inherited from a single relation in $H^0(5D-P)$ which we construct now. The map

$$\varphi_{4,0}: H^0(4D) \otimes H^0(D-P) \to H^0(5D-P)$$

has kernel $H^0(3D+P)$ and so dim Im $\varphi_{4,0} = 15 = h^0(5D-P)$. However we can write down explicitly the monomials in $H^0(5D-P)$ as follows:

There are the 15 linearly independent monomials

u^5t^9, u^4t^8v, $u^3t^7v^2$, $u^2t^6v^3$, ut^5v^4,

t^4v^5, u^4t^6w, u^3t^5vw, $u^2t^4v^2w$, ut^3v^3w,

t^2v^4w, $u^3t^3w^2$, $u^2t^2vw^2$, utv^2w^2, $v^3w^2 \in$ Im $\varphi_{4,0}$

and one more u^2w^3. Therefore we must have a relation

$$R: u^2w^3 = g,$$

where g is a linear combination of the other 15 monomials by surjectivity of $\varphi_{4,0}$.

It follows that in $H^0(5D)$ we must have the relation

$$tR: tu^2w^3 = tg.$$

Rewriting this in terms of x_1, x_2, y_1, y_2 and z gives

$$r_4: y_1z - x_2A - x_1B = 0,$$

where

$$A = a_1y_2^2 + a_2x_2^2y_2 + a_3x_2^4, \ a_i \in \mathbb{C},$$

$B = B(x_1,x_2,y_1,y_2)$ is a polynomial in x_1, x_2, y_1, y_2.

In the same way, in $H^0(6D)$ we have the relations

$$ut^3R = x_1r_4,$$

$$vt^2R = x_2r_4$$

and

$$wR = r_5: z^2 - y_2A - y_1B = 0.$$

That the above relations do indeed suffice to give the ring R(C,D) is a check left to the reader. It can be seen (and follows from the theory of Pfaffian rings) that the relations given satisfy the syzygies $s_1,\ldots s_5$ below.

$$\begin{aligned}
s_1&: & & & x_1r_3 &- x_2r_2 &+ y_1r_1 &\equiv 0,\\
s_2&: & & & -y_1r_3 &+ y_2r_2 &- zr_1 &\equiv 0,\\
s_3&: -x_1r_5 + & y_1r_4 & & &+ zr_2 &- Ar_1 &\equiv 0,\\
s_4&: x_2r_5 - & y_2r_4 &- zr_3 & &&- Br_1 &\equiv 0,\\
s_5&: -y_1r_5 + & zr_4 &+ Ar_3 &+ Br_2 & &&\equiv 0.
\end{aligned}$$

□

Case III. Two distinct base points.

Theorem 9.4

Let X be a surface of type III and $C \in |K_X|$ a nonsingular curve; let $D = K_{X|C}$. Then

$$R(C,D) = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I,$$

where $\deg(x_i, y_j, z_k) = (1, 2, 3)$ and I is the ideal generated by the relations $r_1,\ldots,r_9$ below :

Degree 3 $\quad r_1: x_1^3 - x_2y_1,$

$\quad r_2: x_1y_2 - x_2^3,$

Degree 4 $\quad r_3: y_1y_2 - x_1^2x_2^2,$

$\quad r_4: x_1z_2 - x_2z_1,$

Degree 5 $\quad r_5: y_1z_2 - x_1^2z_1,$

$\quad r_6: x_2^2z_2 - y_2z_1,$

Degree 6 $\quad r_7: -z_1^2 + \lambda y_1^3 + \mu x_2^4y_2 + x_1^2h,$

$\quad r_8: -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h,$

$\quad r_9: -z_2^2 + \lambda x_1^4y_1 + \mu y_2^3 + x_2^2h,$

where $\mu, \lambda \in \mathbb{C}$ and $h = h_C$ is some polynomial of degree 4.

Alternatively, we can write $r_1,\ldots,r_9$ in a quasi-determinantal form as follows :

$r_1,...,r_6$ are given by

$$\text{rank } A \leq 1.$$

where

$$A = \begin{bmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix}.$$

r_7, r_8, r_9 are given by

$$A \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & \lambda y_1 & 0 & 0 \\ 0 & 0 & \mu y_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} A^T = \begin{bmatrix} r_7 & r_8 \\ r_8 & r_9 \end{bmatrix}$$

Proof

The canonical curve C is hyperelliptic in this case, as $|D| = g_2^1 + P_1 + P_2$, where P_1 and P_2 are the base points of $|K_X|$ (P_1 and P_2 are Weierstrass points, as remarked in Lemma (9.1)). There is a standard method of calculating R(D′) for a divisor D′ made up of Weierstrass points on a hyperelliptic curve (see [R2], [G]) and so we shall present the computation with little proof.

Let $u: \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_1+P_2)$, $v: \mathcal{O}_C \hookrightarrow \mathcal{O}_C(P_3+...+P_{12})$ where $P_1,...,P_{12}$ are the Weierstrass points on C. Then we choose $t_1, t_2 \in H^0(g_2^1)$ forming a basis, such that $u^2 = t_1t_2$ and $v^2 = p(t_1,t_2)$, where p is the degree 10 polynomial in t_1,t_2 defining the 10 branch points.

Table 4.

Space	Generators	New names
$H^0(D)$	ut_1, ut_2	x_1, x_2
$H^0(2D)$	$t_1^4, t_1^3t_2, .., t_2^4$	$y_1 = t_1^4$, $y_2 = t_2^4$
$H^0(3D)$	$ut_1^5, ..., ut_2^5$, vt_1, vt_2	$z_1 = vt_1$, $z_2 = vt_2$

Given Table 4, the relations $r_1,...,r_6$ are simply identities following from the definitions of x_1, x_2, y_1, y_2, z_1 and z_2, and the equality $u^2 = t_1t_2$.
For the last 3 relations it is necessary to study the relation $v^2 = p(t_1,t_2)$ in more detail. Since $p(t_1,t_2)$ is degree 10 in t_1 and t_2, using the definitions of y_1 and y_2 we can write

$$p(t_1,t_2) = \lambda y_1^2t_1^2 + \mu y_2^2t_2^2 + t_1t_2h_8(t_1,t_2)$$

where $\lambda, \mu \in \mathbb{C}$ are non-zero and h_8 is of degree 8. Let

$$h_8(t_1, t_2) = a_0t_1^8 + a_1t_1^7t_2 + ... + a_8t_2^8,\ a_i \in \mathbb{C},$$

then we can write $h_8(t_1, t_2) = h(x_1, x_2, y_1, y_2)$, a uniquely determined quartic in R(C, D). Since $z_1^2 = v^2t_1^2$, $z_1z_2 = v^2t_1t_2$ and $z_2^2 = v^2t_2^2$ we get the 3 degree 6 relations in the statement :

$$r_7: -z_1^2 + \lambda y_1^3 + \mu x_2^4y_2 + x_1^2h,$$

$$r_8: -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h,$$

$$r_9: -z_2^2 + \lambda x_1^4y_1 + \mu y_2^3 + x_2^2h.$$

It is easy to check that these relations satisfy the matrix equation given. That is we can put

$$\begin{bmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & \lambda y_1 & 0 & 0 \\ 0 & 0 & \mu y_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & x_1^2 \\ x_2^2 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} r_7 & r_8 \\ r_8 & r_9 \end{bmatrix}$$

The syzygies holding between these 9 relations can be split into two groups. The determinantal ones following purely from the determinantal form of the first 6 relations are as follows :

$$\Sigma_1 : x_1r_3 - y_1r_2 + x_2^2r_1 \equiv 0,$$

$$\Sigma_2 : x_2r_3 - x_1^2r_2 + y_2r_1 \equiv 0,$$

$$\Sigma_3 : x_1r_6 - x_2^2r_4 + z_1r_2 \equiv 0,$$

$$\Sigma_4 : x_1r_5 - y_1r_4 + z_1r_1 \equiv 0,$$

$$\Sigma_5 : x_2r_5 - x_1^2r_4 + z_2r_1 \equiv 0,$$

$$\Sigma_6 : x_2r_6 - y_2r_4 + z_2r_2 \equiv 0,$$

$$\Sigma_7 : y_1r_6 - x_2^2r_5 + z_1r_3 \equiv 0,$$

$$\Sigma_8 : x_1^2 r_6 - y_2 r_5 + z_2 r_3 \equiv 0.$$

The second group depend only on the matrix form of the 9 relations in the following sense. These syzygies can be constructed using the following observation (due to Reid). Consider the following matrix A^*

$$A^* = \begin{bmatrix} -x_2 & x_1 \\ -x_1^2 & y_1 \\ -y_2 & x_2^2 \\ -z_2 & z_1 \end{bmatrix}.$$

Then observe that

$$(A^*)A = \begin{bmatrix} 0 & r_1 & r_2 & r_4 \\ -r_1 & 0 & r_3 & r_5 \\ -r_2 & -r_3 & 0 & r_6 \\ -r_4 & -r_5 & -r_6 & 0 \end{bmatrix}.$$

The equality

$$((A^*)A)MA^T \equiv A^*(AMA^T)$$

can be written as an equality of relations i.e. syzygies.

$$\Sigma_9 : -x_2 r_7 + x_1 r_8 \equiv \lambda y_1^2 r_1 + \mu x_2^2 y_2 r_2 - z_1 r_4,$$

$$\Sigma_{10} : -x_2 r_8 + x_1 r_9 \equiv \lambda x_1^2 y_1 r_1 + \mu y_2^2 r_2 - z_2 r_4,$$

$$\Sigma_{11} : -x_1^2 r_7 + y_1 r_8 \equiv -x_1 h r_1 + \mu x_2^2 y_2 r_3 - z_1 r_5,$$

$$\Sigma_{12} : -x_1^2 r_8 + y_1 r_9 \equiv -x_2 h r_1 + \mu y_2^2 r_3 - z_2 r_5,$$

$$\Sigma_{13} : -y_2 r_7 + x_2^2 r_8 \equiv -x_1 h r_2 - \lambda y_1^2 r_3 - z_1 r_6,$$

$$\Sigma_{14} : -y_2 r_8 + x_2^2 r_9 \equiv -x_2 h r_2 - \lambda x_1^2 y_1 r_3 - z_2 r_6,$$

$$\Sigma_{15} : -z_2 r_7 + z_1 r_8 \equiv -x_1 h r_4 - \lambda y_1^2 r_5 - \mu x_2^2 y_2 r_6,$$

$$\Sigma_{16} : -z_2 r_8 + z_1 r_9 \equiv -x_2 h r_4 - \lambda x_1^2 y_1 r_5 - \mu y_2^2 r_6.$$

$\Sigma_1,\ldots,\Sigma_{16}$ generate the module of syzygies, though the proof of this (and similarly in (9.4_a)) is a calculation best carried out by computer (e.g. [BS]).

□

Cases III_a and III_b $|K_X|$ has a fixed (-2)-cycle.

Theorem (9.4_a)

Let X be a surface with $p_g=3$, $K_X^2=4$ and such that the canonical linear system contains a fixed (-2)-cycle, E. Suppose $C = \Gamma + E \in |K_X|$ as in Lemma (9.1) and put $D = K_{X|C}$, then there are two possibilities:

$$R(C,D) = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I$$

where deg $(x_i, y_i, z_i) = (1, 2, 3)$ and I is generated by $r_1,\ldots,r_6,r_7,r_8,r_9$ as follows:

Degree 3 $\quad r_1: x_1^3 - x_2 y_2,$

$\quad r_2: x_1 y_2 - x_2 y_1,$

Degree 4 $\quad r_3: y_2^2 - x_1^2 y_1,$

$r_4: x_1z_2 - x_2z_1,$

Degree 5 $r_5: y_2z_2 - x_1^2z_1,$

$r_6: y_1z_2 - y_2z_1,$

Degree6

Case a) $r_7: -z_1^2 + \lambda y_1^3 + \mu y_1^2y_2 + x_1^2h,$

$r_8: -z_1z_2 + \lambda y_1^2y_2 + \mu y_1y_2^2 + x_1x_2h,$

$r_9: -z_2^2 + \lambda y_1y_2^2 + \mu y_2^3 + x_2^2h,$

Case b) $r_7: -z_1^2 + \mu y_1^2y_2 + x_1^2h,$

$r_8: -z_1z_2 + \mu y_1y_2^2 + x_1x_2h,$

$r_9: -z_2^2 + \mu y_2^3 + x_2^2h,$

where

$\lambda, \mu \in \mathbb{C},$

$h = \lambda_1x_1^4 + \lambda_2x_1^3x_2 + \lambda_3x_1^2x_2^2 + \lambda_4x_1x_2^3 + \lambda_5y_1^2 + \lambda_6y_1y_2 +$

$\lambda_7y_2^2 + \lambda_8x_1^2y_2, \lambda_i \in \mathbb{C}.$

Alternatively, we can write $r_1,...,r_9$ in the following form :

$r_1,...,r_6$ are given by

$$\operatorname{rank}\begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1$$

and r_7, r_8, r_9 are given by

$$\begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda y_1 + \mu y_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_2 & x_1^2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} r_7 & r_8 \\ r_8 & r_9 \end{bmatrix},$$

In Case b we have $\lambda = 0$.

Proof

Let E be the fixed (-2) - cycle and $|K_X| = |\Gamma| + E$. Then, by Lemma (9.1), we can choose Γ to be a nonsingular genus 4 curve and $K_{X|C} = g_2^1 + P_1 + P_2$ where $E_{|\Gamma} = P_1 + P_2$. The two cases in the statement of the theorem account for the possibilities:

Case a P_1, P_2 distinct points

Case b $P_1 = P_2 = P$.

Notice that $|P_1 + P_2| = g_2^1$ (since $3g_2^1 = |K_\Gamma| = |2K_{X|\Gamma} - P_1 - P_2| = 2g_2^1 + |P_1 + P_2|$), so that in Case b P is a Weierstrass point.

We shall calculate R(C, D) in 3 main steps.

Step 1: we write down the ring $R(\Gamma, Q)$ where Q is a Weierstrass point.

Step 2: we produce from Step 1, $R(\Gamma, 4Q) = R(\Gamma, 2g_2^1)$, because $D = g_2^1 + P_1 + P_2$, and so we have $R(C, D) \subset R(\Gamma, 2g_2^1)$.

Step 3: we write down the condition which characterises R(C, D) as a subring of $R(\Gamma, 2g^1_2)$ and subsequently R(C, D).

Step 1

Let Q be a Weierstrass point on Γ. As in Theorem (9.4),

$$R(\Gamma, Q) = \mathbb{C}[s, t, u]/F_{18},$$

where degree (s, t, u) = (1, 2, 9) and F_{18} is the degree 18 relation given by

$$F_{18} : u^2 = \varphi_{18}(s,t).$$

Step 2

We can pick the generator $t \in g^1_2$ of the form $t = s_1 s_2$ where s_i vanishes at Q_i for $|Q_1 + Q_2| = g^1_2$. We choose $Q_i = P_i$ in Case a and $Q_1 = Q_2 = P$ in Case b. Then we can write down bases for $H^0(4nQ) = H^0(2ng^1_2)$ for small n as in Table 5.

Table 5.

n	generators of $H^0(2ng^1_2)$
1	s^4, s^2t, t^2
2	$s^{2n}t^{4-n}$, n=0,...,4
3	$s^{2n}t^{6-n}$, n=0,...,6, s^3u, stu
4	$s^{2n}t^{8-n}$, n=0,...,8, $s^{2k+1}t^{3-k}u$, k=0,...,3
5	$s^{2n}t^{10-n}$, n=0,...,10, $s^{2k+1}t^{5-k}u$, k=0,...,5
6	$s^{2n}t^{12-n}$, n=0,...,12, $s^{2k+1}t^{7-k}u$, k=0,...,7

Remembering the relation F_{18} the table is self-explanatory and can be computed independently or with reference to [R2].

Step 3

Recall that $D = g_2^1 + P_1 + P_2$ where $|P_1 + P_2| = g_2^1$ and that $2D = 2g_2^1$. Then $R(C,D) \subset R(\Gamma, 2g_2^1)$ and moreover if $f \in H^0(C, D)$ then $f^2 \in R(\Gamma,4g_2^1)$. It follows that

$$f^2(P_1)/s^8 = f^2(P_2)/s^8$$

and so

$$f(P_1)/s^4 = \pm f(P_2)/s^4.$$

By consideration of the dimensions of $H^0(C, nD)$ we must have

$$f(P_1)/s^4 = -f(P_2)/s^4.$$

It can then be seen that $f \in R(\Gamma, 2g_2^1)$ is an element of R(C,D) if and only if

$$f(P_1)/s^{4d} = -f(P_2)/s^{4d},$$

where d = degree of f

Bearing in mind our choice of t we get the following table of generators.

Table 6.

n	generators of $H^0(nD)$
1	$x_1 = s^2t, x_2=t^2$
2	$S^2(x_i), y_1=s^8, y_2=s^6t$
3	$S^3(x_i), x_iy_j, z_1=s^3u, z_2=stu$

Using the definitions in Table 6 the following relations are seen to hold:

$$r_1 : x_1^3 - x_2y_2,$$

$$r_2 : x_1y_2 - x_2y_1,$$

$$r_3: y_2^2 - x_1^2 y_1,$$

$$r_4: x_1 z_2 - x_2 z_1,$$

$$r_5: y_2 z_2 - x_1^2 z_1,$$

$$r_6: y_1 z_2 - y_2 z_1.$$

As usual the degree 6 generators for R(C,D) are made up from lower degrees. However at this stage the relation F_{18} from the ring $R(\Gamma, Q)$ makes its presence felt in R(C,D). Thus we get the 3 degree 6 relations (with a slight abuse of notation),

$$r_7: z_1^2 = s^6 \varphi_{18},$$

$$r_8: z_1 z_2 = s^4 t \varphi_{18},$$

$$r_9: z_2^2 = s^2 t^2 \varphi_{18}.$$

Case a

We can write out $\varphi_{18}(s, t)$ as follows:

$$\varphi_{18} = \lambda s^{18} + \mu s^{16} t + \lambda_1 s^{14} t^2 + \ldots + \lambda_8 t^8.$$

Then r_7, r_8, r_9 can be rewritten,

$$r_7: -z_1^2 + \lambda y_1^3 + \mu y_1^2 y_2 + x_1^2 h,$$

$$r_8: -z_1 z_2 + \lambda y_1^2 y_2 + \mu y_1 y_2^2 + x_1 x_2 h,$$

$$r_9: -z_2^2 + \lambda y_1 y_2^2 + \mu y_2^3 + x_2^2 h,$$

where

$$h = \lambda_1 x_1^4 + \lambda_2 x_1^3 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1 x_2^3 + \lambda_5 y_1^2 +$$

$$\lambda_6 y_1 y_2 + \lambda_7 y_2^2 + \lambda_8 x_1^2 y_2,$$

after renumbering the $\lambda_i \in \mathbb{C}$.

Case b

Previously we chose $t = \tau^2$ for τ vanishing at P. In this case $u = \tau v$ (v vanishes at the Weierstrass points missing P and Q) and so $\varphi_{18} = t\varphi_{16}$. This means that we can write

$$\varphi_{18} = \mu s^{16} t + \lambda_1 s^{14} t^2 + \ldots + \lambda_8 t^8.$$

The form of r_7, r_8, r_9 in Case b now follows from Case a.

Thus we have written

$$R(C,D) = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I,$$

where I is generated as in the statement of the theorem. It is a simple check to see that the determinant given does indeed produce the relations $r_1,\ldots,r_6$.

These relations satisfy the following syzygies $\Sigma_1,\ldots,\Sigma_{16}$:

$$\Sigma_1: x_1 r_3 - y_2 r_2 + y_1 r_1 \equiv 0,$$

$$\Sigma_2: x_2 r_3 - x_1^2 r_2 + y_2 r_1 \equiv 0,$$

$$\Sigma_3: x_1 r_6 - y_1 r_4 + z_1 r_2 \equiv 0,$$

$$\Sigma_4: x_1 r_5 - y_2 r_4 + z_1 r_1 \equiv 0,$$

$$\Sigma_5: x_2 r_5 - x_1^2 r_4 + z_2 r_1 \equiv 0,$$

$$\Sigma_6: x_2 r_6 - y_2 r_4 + z_2 r_2 \equiv 0,$$

$$\Sigma_7: y_2 r_6 - y_1 r_5 + z_1 r_3 \equiv 0,$$

$$\Sigma_8: x_1^2 r_6 - y_2 r_5 + z_2 r_3 \equiv 0,$$

$$\Sigma_9: -x_2 r_7 + x_1 r_8 \equiv (\lambda y_1 + \mu y_2) y_1 r_2 - z_1 r_4,$$

$$\Sigma_{10}: -x_2 r_8 + x_1 r_9 \equiv (\lambda y_1 + \mu y_2) y_2 r_2 - z_2 r_4,$$

$$\Sigma_{11}: -x_1^2 r_7 + y_2 r_8 \equiv -x_1 h r_1 + (\lambda y_1 + \mu y_2) y_1 r_3 - z_1 r_5,$$

$$\Sigma_{12}: -x_1^2 r_8 + y_2 r_9 \equiv -x_2 h r_1 + (\lambda y_1 + \mu y_2) y_2 r_3 - z_2 r_5,$$

$$\Sigma_{13}: -y_2 r_7 + y_1 r_8 \equiv -x_1 h r_2 - z_1 r_6,$$

$$\Sigma_{14}: -y_2 r_8 + y_1 r_9 \equiv -x_2 h r_2 - z_2 r_6,$$

$$\Sigma_{15}: -z_2 r_7 + z_1 r_8 \equiv -x_1 h r_4 - (\lambda y_1 + \mu y_2) y_1 r_6,$$

$$\Sigma_{16}: -z_2 r_8 + z_1 r_9 \equiv -x_2 h r_4 - (\lambda y_1 + \mu y_2) y_2 r_6.$$

□

Remark (9.5)

The rings in Theorems (9.4) and (9.4_a) have been presented in a format depending on certain matrices. That is in each case the relations are generated by

$$\text{rank } A \leq 1$$

and

$$AMA^T = 0$$

for a given 2×4 matrix A and a symmetric matrix M.

Given a polynomial ring P we make the observation that there is a variety defined by

$$\text{rank} \begin{bmatrix} l_{11} & \dots & l_{14} \\ l_{21} & \dots & l_{24} \end{bmatrix} \leq 1$$

and

$$\begin{bmatrix} R_7 & R_8 \\ R_8' & R_9 \end{bmatrix} : \begin{bmatrix} l_{11} & \cdots & l_{14} \\ l_{21} & \cdots & l_{24} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ \vdots & \vdots \\ A_{14} & A_{24} \end{bmatrix} = 0$$

for $l_{ij}, A_{ij} \in P$.

If $I \subset P$ is generated by the matrix conditions above we can write out the generators explicitly as follows:

$R_{ij} : l_{1i}l_{2j} - l_{1j}l_{2i}$, $1 \leq i < j \leq 4$, gives the 6 determinantal relations

$$R_7 : \sum_{i=1}^{4} l_{1i}A_{1i},$$

$$R_8 : \sum_{i=1}^{4} l_{1i}A_{2i},$$

$$R_8' : \sum_{i=1}^{4} l_{2i}A_{1i},$$

$$R_9 : \sum_{i=1}^{4} l_{2i}A_{2i},$$

with the condition that $R_8 \equiv R_8'$ in P. We make the obvious convention that $R_{ij} = -R_{ji}$ for $i > j$.

The point is that these 9 relations will then satisfy the syzygies below:

$S_{ijk} : l_{1i}R_{jk} - l_{1j}R_{ik} + l_{1k}R_{ij}$, $1 \leq i < j < k \leq 4$,

$T_{ijk} : l_{2i}R_{jk} - l_{2j}R_{ik} + l_{2k}R_{ij}$, $1 \leq i < j < k \leq 4$,

give the 8 determinantal syzygies, and

$$U_j : -l_{2j}R_7 + l_{1j}R_8 \equiv \sum_{i \neq j} R_{ij}A_{1j},\ 1 \leq j \leq 4,$$

$$U_j' : -l_{2j}R_8 + l_{1j}R_9 \equiv \sum_{i \neq j} R_{ij}A_{2j},\ 1 \leq j \leq 4,$$

give the remaining 8. As remarked in the proof of Theorem (9.4) these last 8 syzygies are constructed from the identity

$$((A^*)A)MA^T \equiv A^*(AMA^T)$$

where

$$A^* = \begin{bmatrix} -l_{21} & l_{11} \\ \vdots & \vdots \\ -l_{24} & l_{14} \end{bmatrix}.$$

The rings of Theorem (9.4) and (9.4_a) are now of this form if we put the product

$$MA^T = \begin{bmatrix} A_{11} & A_{12} \\ \vdots & \vdots \\ A_{14} & A_{24} \end{bmatrix}.$$

10. Deformations of canonical curves.

In section 16 we calculate the number of moduli for surfaces of each of the 5 types and in some cases write down concretely a degeneration between surfaces of two different types. In this section we do the same thing for the canonical curves of the surfaces. These calculations serve as something of a model for the larger (surface) ones.

Theorem (10.1)

In the following diagram

$$I \to II \to III \to III_a \to III_b,$$

for any given canonical curve of a surface of type B, $\mathfrak{C}$, there can be found a family $\{\mathfrak{C}\}_t$ of curves, such that $\mathfrak{C}_t$ is the canonical curve of a surface of type A for $t \neq 0$, and $\mathfrak{C}_0 = \mathfrak{C}$ is the canonical curve of a surface of type B, whenever there is an arrow from A to B.

Proof.

In each case we simply write down a family of rings, R_t, which are rings of the type calculated for curves of type A when $t \neq 0$ and a general ring of type B when $t=0$.

In order for the corresponding deformation to be flat, the syzygies in R_t reduced modulo t, must be equal to the syzygies in R_0.

Type I → Type II

Consider the ring,

$$R_t = \mathbb{C}[x_1,x_2,y_1,y_2,z]/I_t,$$

where I_t is generated by $r_1^t,\ldots,r_5^t$ given by the 4X4 Pfaffians of the following matrix (see Section 4):

$$M_t = \begin{bmatrix} 0 & t & x_1 & x_2 & y_1 \\ t & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z & -A \\ -x_2 & -y_2 & z & 0 & B \\ -y_1 & -z & A & -B & 0 \end{bmatrix}.$$

Here A and B are as in Theorem (9.3).

First notice that this guarantees that R_t mod t is the ring of a curve of type II. Let us write down the relations $r_1^t,\ldots,r_5^t$.

$$r_1^t : x_1y_2 - x_2y_1 + tz,$$

$$r_2^t : x_1z - y_1^2 + tA,$$

$$r_3^t : x_2z - y_1y_2 - tB,$$

$$r_4^t : y_1z - x_2A - x_1B,$$

$$r_5^t : z^2 - y_2A - y_1B.$$

Now, for $t\neq 0$, r_1^t allows us to eliminate z from R_t. If we do this systematically in the rest of the equations we get two degree 4 relations,

$$r_2^t : -x_1(x_1y_2 - x_2y_1)/t - y_1^2 + tA,$$

$$r_3^t : -x_2(x_1y_2 - x_2y_1)/t - y_1y_2 - tB,$$

and find that these span I_t. In particular

$$r_4^t = x_1 r_3^t/t - x_2 r_2^t/t,$$

$$r_5^t = y_1 r_3^t/t - y_2 r_2^t/t.$$

So we see that for $t \neq 0$, $R_t = \mathbb{C}[x_1,x_2,y_1,y_2]/\langle Q_1,Q_2\rangle$, as required.

Type II → Type III

In this case there are several *ad hoc* ways of writing down families of rings which satisfy our requirements. We give one more or less at random.

Let $R_t = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I_t$, where I_t is generated by

$$r_1^t : x_1^3 - x_2y_1 + tz_1,$$

$$r_2^t : x_2^3 - x_1y_2 - t^2\lambda x_2y_1,$$

$$r_3^t : y_1y_2 - x_1^2x_2^2 - tx_2z_2,$$

$$r_4^t : x_1z_2 - x_2z_1 + t\lambda y_1^2,$$

$$r_5^t : x_1^2z_1 - y_1z_2 + t\mu x_2y_2^2 + tx_1h,$$

$$r_6^t : x_2^2z_2 - y_2z_1 + t\lambda x_1^2x_2y_1,$$

$$r_7^t : z_1^2 - \lambda y_1^3 - \mu x_1x_2y_2^2 - x_1^2h,$$

$$r_8^t : -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h,$$

$$r_9^t : z_2^2 - \lambda x_1^4 y_1 - \mu y_2^3 - x_2^2 h + t^2 \lambda h y_1.$$

Again the notation used here corresponds to that in section 9. Using r_1^t to eliminate z_1 when $t \neq 0$ we find that I_t can be generated by the 4X4 Pfaffians of the following matrix.

$$M_t = \begin{bmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ 0 & 0 & x_2^2 - t^2\lambda y_1 & y_2 & x_1^2 x_2 + tz \\ -x_1 & -x_2^2 + t^2\lambda y_1 & 0 & -x_1^2 x_2 + tz & t^2 \mu y_2^2 \\ -x_2 & -y_2 & x_1^2 x_2 - tz & 0 & -t^2 h + x_1^4 \\ -y_1 & -x_1^2 x_2 - tz & -t^2 \mu y_2^2 & t^2 h - x_1^4 & 0 \end{bmatrix},$$

where we are writing z for z_2. M_t gives us the relations r_2^t, r_3^t, r_4^t, r_5^t, r_9^t and we can write

$$r_6^t = -x_2 r_3^t/t - x_1^2 r_2^t/t,$$

$$r_7^t = -x_1 r_5^t/t - y_1 r_4^t/t,$$

$$r_8^t = x_1^2 r_4^t/t + x_2 r_5^t/t.$$

Thus R_t is a ring of type II as required.

Type III → Type III_a

Consider the ring,

$$R_t = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I_t$$

where I_t is generated by

$$\text{rank } M = \text{rank} \begin{bmatrix} x_1 & y_1 & y_2 & z_1 \\ x_2 & x_1^2 & y_1 + ty_2 & z_2 \end{bmatrix} \leq 1,$$

and $MPM^T = 0$ for some symmetric P.

For $t = 0$ this is clearly a ring of type III_a. For $t \neq 0$ we can perform a sequence of basis changes and row and column operations on the matrix M until we are left with the presentation given in Theorem (9.4) for a ring of type III.

Type $III_a \rightarrow$ Type III_b

The presentation of the rings given in Theorem (9.4_a) makes the deformation clear in this case. Consider the rings

$$R_\lambda = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I_\lambda,$$

where I_λ is generated by $r_{1,\lambda},\ldots,r_{9,\lambda}$:

$r_{1,\lambda},\ldots,r_{6,\lambda}$ given by

$$\text{rank} \begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1,$$

$r_{7,\lambda}$, $r_{8,\lambda}$, $r_{9,\lambda}$ given by

$$r_7: z_1^2 - \lambda y_1^3 - \mu y_1^2 y_2 - x_1^2 h,$$

$$r_8: z_1 z_2 - \lambda y_1^2 y_2 - \mu y_1 y_2^2 - x_1 x_2 h,$$

$$r_9: z_2^2 - \lambda y_1 y_2^2 - \mu y_2^3 - x_2^2 h.$$

Then R_λ is a ring of type III_a and R_0 is a ring of type III_b as required.

□

The degeneration II → III given is not a structured one. However we can write down a degeneration I → III in a structured way (10.2). We hope to find a 2-parameter degeneration I → III factoring through II at a later date but as yet this remains undiscovered.

Proposition (10.2)

The degeneration of type I curves to type III curves can be expressed by the ring

$$\mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I_t$$

where degree $(x_i, y_i, z_i) = (1, 2, 3)$ and I_t is generated by 15 relations $\{r_{ij}\}_{\substack{1\le i\le j-1.\\ 2\le j\le 6}}$ r_{ij} is derived from an antisymmetric 6X6 matrix, M_t, by deleting the ith and jth rows and columns and taking the Pfaffian of the 4X4 antisymmetric matrix remaining (see section 4). The matrix M_t is shown below:

$$\begin{bmatrix}
0 & t & z_1 & x_2^2 & y_1 & x_1 \\
. & 0 & z_2 & y_2 & x_1^2 & x_2 \\
. & . & 0 & -\lambda x_1 y_1^2 + \lambda x_1^2 x_2 y_1 - x_2 h & -\mu x_2 y_2^2 + \mu x_1 x_2^2 y_2 - x_1 h & \lambda y_1^2 + \mu y_2^2 \\
. & . & . & 0 & -x_2 z_2 + x_1 z_1 + th & z_2 + t\lambda x_2 y_1 \\
. & . & . & . & 0 & z_1 - t\mu x_1 y_2 \\
. & . & . & . & . & 0
\end{bmatrix}$$

Proof

Let

$$R(C, K_{X|C}) = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I$$

be as in section (9.4).

It suffices to show that the ideal generated by $r_{12},\ldots,r_{56}$ is equal to I when $t = 0$ but when $t \neq 0$ we can eliminate z_1 and z_2 and be left with two quartics. Below is a list of the 15 relations generated from M_t.

$$r_{56} : t(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h) + y_2 z_1 - x_2^2 z_2,$$

$$r_{46} : t(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h) + x_1^2 z_1 - y_1 z_2,$$

$$r_{36} : t(x_2 z_2 - x_1 z_1 - th) + x_1^2 x_2^2 - y_1 y_2,$$

$$r_{26} : z_1(x_2 z_2 - x_1 z_1 - th) - x_2^2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h)$$

$$+ y_1(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h),$$

$$r_{16} : z_2(x_2 z_2 - x_1 z_1 - th) - y_2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h)$$

$$+ x_1^2(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h),$$

$$r_{45} : -t(\lambda y_1^2 + \mu y_2^2) + x_2 z_1 - x_1 z_2,$$

$$r_{35} : -t(z_2 + t\lambda x_2 y_1) + x_2^3 - x_1 y_2,$$

$$r_{25} : -z_1(z_2 + t\lambda x_2 y_1) + x_2^2(\lambda y_1^2 + \mu y_2^2) + x_1(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h),$$

$$r_{15} : -z_2(z_2 + t\lambda x_2 y_1) + y_2(\lambda y_1^2 + \mu y_2^2) + x_2(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h),$$

$$r_{34} : -t(z_1 - t\mu x_1 y_2) + x_2 y_1 - x_1^3,$$

$r_{24} : -z_1(z_1 - t\mu x_1 y_2) + y_1(\lambda y_1^2 + \mu y_2^2) + x_1(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h),$

$r_{14} : -z_2(z_1 - t\mu x_1 y_2) + x_1^2(\lambda y_1^2 + \mu y_2^2) + x_2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h),$

$r_{23} : -x_2^2(z_1 - t\mu x_1 y_2) + y_1(z_2 + t\lambda x_2 y_1) + x_1(x_2 z_2 - x_1 z_1 - th),$

$r_{13} : -y_2(z_1 - t\mu x_1 y_2) + x_1^2(z_2 + t\lambda x_2 y_1) + x_2(x_2 z_2 - x_1 z_1 - th),$

$r_{12} : (\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 h)(z_1 - t\mu x_1 y_2) - (\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h)(z_2 +$

$t\lambda x_2 y_1) + (\lambda y_1^2 + \mu y_2^2)(x_2 z_2 - x_1 z_1 - th).$

We make the following definitions:

$r_1^t : -r_{34},$

$r_2^t : r_{35},$

$r_3^t : -r_{45},$

$r_4^t : -r_{36},$

$r_5^t : -r_{46},$

$r_6^t : -r_{56}.$

$r_7^t : -r_{24} - \mu x_2 y_2 r_{35} - \mu y_2 r_{36},$

$r_8^t : -r_{25} + \lambda x_2 y_1 r_{34},$

$$r_9^t : -r_{15} + \lambda x_1 y_1 r_{34} - r_{36}.$$

It is now easy to check that $r_i^0 = r_i$ and that those r_{ij} which we have not used in the above definitions are generated by the others. The Proposition follows easily.

□

CHAPTER 4:

Surfaces.

11. Form of extension calculations

As can be seen in section 7 extension-deformation calculations, when worked through fully, are long even in simple cases (such as Pinkham's example). In this chapter we work through the calculations extending the relatively complicated rings given in section 9. This makes the proofs of Theorems (13.1), (14.1) and (15.1) extremely long. For this reason we have kept to a strict scheme when making the statements and proofs of these theorems and to aid the reader we present this scheme below.

Scheme for Theorems in Chapter 4.

Statement of theorem is of the form

Let X be a surface of type I (II, III, III_a, III_b). Then the presentation for R(C, D) in Section 9 is formative and informative in degree -1 (see section 8). In fact

$$R(X, K_X) = \mathbb{C}[x_0,...]/\tilde{I}$$

where $\tilde{I}$ is generated by $\tilde{r}_1,...,\tilde{r}_n$ as follows:

Description of $\tilde{r}_1,...,\tilde{r}_n$

Proof of theorem is of the form

Reproduction of R(C, D)

The first item in the proof is a reproduction of the format of the ring R = R(C, D) for $C \in |K_X|$, as given in Section 9. We then observe that this format is formative. To prove the theorem we need to extend this ring. That is, if R is related by $r_1,...,r_n$ with syzygies $\Sigma_1,...,\Sigma_m$, we want to write

$$\tilde{r}_i : r_i = a_i x_0,$$

such that the $\tilde{r}_i$ satisfy syzygies $\tilde{\Sigma}_j$ which reduce mod x_0 to Σ_j.

Step 0 : Simplifications

It is possible to make simplifying coordinate changes to reduce the possible choices of the a_i. This has the effect of shortening the calculations considerably.

Step 1 : $R \to R^{(2)}$

We make the extension $R \to R^{(2)}$ where $R^{(2)}/(x_0) = R$ as outlined in Section 6. This involves taking the syzygies one at a time and finding the $r_i^{(2)}$ $(i = 1,...,n)$ which will lift them. It is usual that after lifting just 5 or 6 syzygies the expressions for $r_i^{(2)}$ $(i = 1,...,n)$ can be written in the same formative presentation as the r_i $(i = 1,...,n)$. This means that the other syzygies will automatically be lifted by these expressions.

Step 2 : $R^{(2)} \to R^{(3)}$

We make the extension $R^{(1)} \to R^{(2)}$ where $R^{(2)}/(x_0) = R$. Again we put the relations in the ring $R^{(2)}$ into the format of the relations in R, if this is possible.

Step ℓ : $R^{(\ell)} \to R^{(\ell+1)}$

Having made the extension up to $R^{(\ell)}$ we extend to $R^{(\ell+1)}$. The algorithm will certainly have terminated in n steps, where n is the maximum degree of the generators of the module of syzygies. In fact keeping a formative presentation will guarantee that it terminates in many fewer steps than this.

12. Surfaces of type I.

Theorem (12.1)

Let X be a surface of general type with $p_g=3$, $K_X^2=4$ and such that the canonical linear system , $|K_X|$ has no base points. Then

$$R(X, K_X) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2] /<R_1^{(4)},R_2^{(4)}>$$

where degree $(x_i, y_j) = (1, 2)$ and $R_1^{(4)},R_2^{(4)}$ are weighted homogeneous polynomials of degree 4. Hence X is the complete intersection $X_{4,4} \subset \mathbb{P}(1^3,2^2)$.

Remark (12.2)

The generalised hyperplane section principle (Proposition (6.1)) tells us that in each degree there are the same number of generators and relations in $R(X, K_X)$ as there are in R(C, D). This means that $R(X, K_X)$ is a Gorenstein grade 2 ring and the theorem follows immediately from this (Proposition (4.1)). However since the result is such an easy one the proof given below has been included to to illustrate the extension-deformation method of Section 6 in a transparent way.

Proof of (12.1)

Let $C \in |K_X|$ be a nonsingular curve, given by the vanishing of $x_0 \in H^0(X,K_X)$. In Theorem (9.2) we found that:

$$R(C,D) = \mathbb{C}[x_1,x_2,y_1,y_2]/<Q_1, Q_2> ,$$

where $Q_1(x_1,x_2,y_1,y_2)$, $Q_2(x_1,x_2,y_1,y_2)$ are weighted homogeneous degree 4 polynomials.

Step 0 : Simplifications

As the calculation is very simple we do not need to simplify by making coordinate changes.

Step 1 : $R \to R^{(2)}$

We make an extension of the ring R(C,D) to one $R(2C,D^{(2)})$. That is we must find the most general $\beta : \langle Q_1, Q_2 \rangle \to R(C, D)$ such that $\beta(Q_1)$ and $\beta(Q_2)$ are degree 3 polynomials and

$$Q_1\beta(Q_2) - Q_2\beta(Q_1) \in \langle Q_1, Q_2 \rangle.$$

Of course this equation holds for arbitrary degree 3 polynomials so we can write,

$$R(2C, D^{(2)}) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2]/(I^{(2)}, x_0^2),$$

where $I^{(2)}$ is generated by

$$r_1^{(2)}: Q_1 - x_0\beta(Q_1),$$

$$r_2^{(2)}: Q_2 - x_0\beta(Q_2).$$

Step 2 : $R^{(2)} \to R^{(3)}$

Similarly for the second order extension we find $\beta_2: \langle Q_1, Q_2 \rangle \to R(C, D)$ such that $\beta_2(Q_1)$ and $\beta_2(Q_2)$ are degree 2 polynomials and

$$Q_1\beta_2(Q_2) - Q_2\beta_2(Q_1) \in \langle Q_1, Q_2 \rangle.$$

Again this is true for any choice of degree 2 polynomials.

Step ℓ : $R^{(\ell)} \to R^{(\ell+1)}$

Repeating the process twice more gives us $R^{(5)}$ of the form,

$$R^{(5)} = \mathbb{C}[x_0,x_1,x_2,y_1,y_2]/(R_1^{(4)},R_2^{(4)},x_0^5),$$

where $R_1^{(4)}$ and $R_2^{(4)}$ are weighted degree 4 polynomials.

This yields the theorem immediately.

□

13).Surfaces of type II.

Theorem (13.1)

Let X be a surface of general type with $p_g=3$, $K_X^2=4$ and such that the canonical linear system has one base point. Then the Pfaffian presentation of the ring R(C, D) calculated in Theorem (9.3) is formative and informative in degree -1. The canonical ring of a surface of type II is of the form:

$$R(X, K_X) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z] / \tilde{I}$$

where $\tilde{I}$ is generated by $\tilde{r}_1,\ldots,\tilde{r}_5$ corresponding to the 4x4 Pfaffians of the matrix M [cf. curve case (9.3)],

$$M = \begin{bmatrix} 0 & 0 & x_1 & x_2 & y_1 + \lambda_5 x_0^2 \\ 0 & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z + x_0 f_1 & -A + x_0 f_2 \\ -x_2 & -y_2 & z - x_0 f_1 & 0 & B + x_0 f_3 \\ -y_1 - \lambda_5 x_0^2 & -z & A - x_0 f_2 & -B - x_0 f_3 & 0 \end{bmatrix}.$$

Here A and B are as in Theorem (9.3), $\lambda_5 \in \mathbb{C}$ and f_1, f_2, f_3 are all polynomials with coefficients in $\mathbb{C}$ written out in full below.

$$f_1 = t_3x_1^2 + u_2x_0^2,$$

$$f_2 = -t_4x_1^3 - t_5x_1^2x_2 - t_6x_1x_2^2 - t_7x_2^3 - t_8x_1y_2 - t_9x_2y_2 - t_{10}z - v_1x_0^3 - u_3x_0^2x_1 - q_6x_0x_1^2 - q_7x_0x_1x_2 - q_8x_0x_2^2 - q_{10}x_0y_2,$$

$$f_3 = t_1x_1y_2 + t_2x_2y_2 + t_{11}x_1^3 + vx_0^3 + u_1x_0^2x_1 + q_{11}x_0x_1^2 + q_2x_0y_2 + q_4x_0y_1.$$

The relations thus given are precisely

$$\tilde{r}_1 : x_1y_2 - x_2y_1,$$

$$\tilde{r}_2 : x_1z - y_1^2 - \lambda_5x_0^2y_1,$$

$$\tilde{r}_3 : x_2z - y_1y_2 - \lambda_5x_0^2y_2,$$

$$\tilde{r}_4 : (y_1 + \lambda_5x_0^2)(z - x_0f_1) - x_2(A - x_0f_2) - x_1(B + x_0f_3),$$

$$\tilde{r}_5 : z(z - x_0f_1) - y_2(A - x_0f_2) - y_1(B + x_0f_3).$$

These relations satisfy the syzygies, $\tilde{s}_1,\ldots,\tilde{s}_5$ below given by:

$$\tilde{s}_1 : (y_1 + \lambda_5x_0^2)\tilde{r}_1 - x_2\tilde{r}_2 + x_1\tilde{r}_3 \equiv 0$$

$$\tilde{s}_2 : z\tilde{r}_1 - y_2\tilde{r}_2 + y_1\tilde{r}_3 \equiv 0$$

$$\tilde{s}_3 : (A - x_0f_2)\tilde{r}_1 - (z - x_0f_1)\tilde{r}_2 + y_1\tilde{r}_4 - x_1\tilde{r}_5 \equiv 0$$

$$\tilde{s}_4 : (B + x_0f_3)\tilde{r}_1 + (z - x_0f_1)\tilde{r}_3 - y_2\tilde{r}_4 + x_2\tilde{r}_5 \equiv 0$$

$$\tilde{s}_5 : -(B + x_0f_3)\tilde{r}_2 - (A - x_0f_2)\tilde{r}_3 + z\tilde{r}_4 - (y_1 + \lambda_5x_0^2)\tilde{r}_5 \equiv 0$$

Remark (13.2)

By the general hyperplane section principle (Proposition (6.1)) we know the number of generators and relations in each degree for the ring $R(X, K_X)$. Observing that the ring must be Gorenstein and codimension 3 the structure theorems in section 4 tell us that the ideal of relations is generated by Pfaffians. We prove this explicitly by using the extension-deformation algorithm.

Proof of (13.1)

Let $C \in |K_X|$ be the general nonsingular curve given by the vanishing of $x_0 \in H^0(X, K_X)$, and $D = K_{X|C}$. In Theorem (9.3) we showed that,

$$R(C,D) = \bigoplus_{n=0}^{\infty} H^0(C,nD) = \mathbb{C}[x_1, x_2, y_1, y_2, z]/I,$$

where deg $(x_i, y_j, z) = (1, 2, 3)$ and I is generated by

r_1: $x_1y_2 - x_2 y_1$,

r_2 : $x_1z - y_1^2$,

r_3 : $x_2z - y_1y_2$,

r_4 : $y_1z - x_2A - x_1B$,

r_5 : $z^2 - y_2A - y_1B$,

where

$$A = a_1y_2^2 + a_2x_2^2y_2 + a_3 x_2^4 , a_i \in \mathbb{C}$$

and

$B = B(x_1,x_2,y_1,y_2)$ is a general degree 4 polynomial in the weighted coordinates x_1, x_2, y_1, y_2.

We have already written these down in Theorem (9.3) as the diagonal 4x4 Pfaffians of the following matrix,

$$\begin{bmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ 0 & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z & -A \\ -x_2 & -y_2 & z & 0 & B \\ -y_1 & -z & A & -B & 0 \end{bmatrix}$$

The syzygies between the relations are,

$$s_1: x_1r_3 - x_2r_2 + y_1r_1 \equiv 0,$$

$$s_2 : y_1r_3 - y_2r_2 + zr_1 \equiv 0,$$

$$s_3 : x_1r_5 - y_1r_4 - zr_2 + Ar_1 \equiv 0,$$

$$s_4 : x_2r_5 - y_2r_4 - zr_3 - Br_1 \equiv 0,$$

$$s_5 : y_1r_5 - zr_4 - Ar_3 - Br_2 \equiv 0.$$

It is easy to notice here that the Pfaffian format is formative (section 8). Having constructed R(C,D) we use the algorithm in Proposition (6.2) to calculate the ring

$$R(X,K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/\tilde{I},$$

where $\tilde{I}$ is generated by,

$$\tilde{r}_1 : x_1y_2 - x_2 y_1 - \alpha x_0,$$

$$\tilde{r}_2 : x_1z - y_1^2 - \beta_1 x_0,$$

$$\tilde{r}_3 : x_2z - y_1y_2 - \beta_2 x_0,$$

$$\tilde{r}_4 : y_1z - x_2A - x_1B - \gamma x_0,$$

$$\tilde{r}_5 : z^2 - y_2A - y_1B - \delta x_0.$$

Here deg $(\alpha, \beta_i, \gamma, \delta) = (2, 3, 4, 5)$, and the $\tilde{r}_i$ satisfy syzygies $\tilde{s}_i$ which reduce mod x_0 to s_i.

Step 0 : Simplifications.

We make some coordinate changes. For example consider

$$\tilde{r}_1 : x_1y_2 - x_2 y_1 - \alpha x_0.$$

By judicious changes of coordinates in x_1, x_2, y_1 and y_2 we can ensure that

$$\tilde{r}_1 : x_1y_2 - x_2 y_1 - \hat{\alpha} x_0^3$$

where $\hat{\alpha} \in \mathbb{C}$.

Similarly, looking at

$$\tilde{r}_2 : x_1z - y_1^2 - \beta_1 x_0$$

if $\beta_1 = \beta x_2y_1 + b_1'$ then using $\tilde{r}_1$ we can write

$$\beta_1 = \beta x_1y_2 - \hat{\alpha}\beta\, x_0^3 + b_1'$$

so we may assume that β_1 has no terms in x_2y_1. By change of coordinates in z we may also assume that β_1 has no monomials in which x_1 appears. So, we have arranged that

$$\tilde{r}_2 : \quad x_1z - y_1^2 - \lambda_1 x_0x_2^3 - \lambda_2 x_0x_2y_1 - \lambda_3 x_0z - \lambda_4 x_0^2x_2^2 -$$

$$\lambda_5 x_0^2y_1 - \lambda_6 x_0^2y_2 - \lambda_7 x_0^3x_2 - \lambda_8 x_0^4 \,, \lambda_i \in \mathbb{C}.$$

By using $\tilde{r}_1$ we can assume that β_2 has no monomial in x_2y_1 and using all the relations we can assume that x_2y_1, y_1^2, y_1y_2 and y_1z do not appear in γ or δ.

Step 1 : $R \to R^{(2)}$

We begin our construction of

$$R(2C,D^{(2)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/(x_0^2, I^{(2)})$$

where $I^{(2)}$ is generated by,

$$r_1^{(2)} : x_1y_2 - x_2 y_1 - \alpha' x_0,$$

$$r_2^{(2)} : x_1z - y_1^2 - \beta'_1 x_0,$$

$$r_3^{(2)} : x_2z - y_1y_2 - \beta'_2 x_0,$$

$$r_4^{(2)} : y_1z - x_2A - x_1B - \gamma' x_0,$$

$$r_5^{(2)} : z^2 - y_2A - y_1B - \delta' x_0,$$

where deg $(\alpha'(x_1,x_2,y_1,y_2), \beta'_i(x_1,x_2,y_1,y_2,z), \gamma'(x_1,x_2,y_1,y_2,z), \delta'(x_1,x_2,y_1,y_2,z)) = (2, 3, 4, 5)$. These satisfy the syzygies $s_1^{(2)},\ldots,s_5^{(2)}$ such that $s_i^{(2)}$ reduced mod x_0 is s_i. By Step 0 we notice that:

$$r_1^{(2)} : x_1y_2 - x_2y_1,$$

$$r_2^{(2)} : x_1z - y_1^2 - (\lambda_1 x_0x_2^3 + \lambda_2 x_0x_2y_2 + \lambda_3 x_0z).$$

Let us look at the syzygy s_1 which must lift to a syzygy, which we call $s_1^{(2)}$. This means that

$$x_1 r_3^{(2)} - x_2 r_2^{(2)} + y_1 r_1^{(2)} = x_0 S$$

where $S \in I^{(2)}$, that is,

$$x_1 \beta'_2 - \lambda_1 x_2^4 - \lambda_2 x_2^2 y_2 - \lambda_3 x_2 z \in I$$

is of degree 4. Equating the left-hand side of this with a general degree 4 element of I we have

$$\begin{aligned} x_1\beta'_2 - \lambda_1 x_2^4 - \lambda_2 x_2^2 y_2 - \lambda_3 x_2 z &= p_1 r_3 + p_2 r_2 + x_1 p_3 r_1 + x_2 p_4 r_1 \\ &= p_1 x_2 z - p_1 y_1 y_2 + p_2 x_1 z - p_2 y_1^2 \\ &\quad + p_3 x_1^2 y_2 - p_3 x_1 x_2 y_1 + p_4 x_2 x_1 y_2 \\ &\quad - p_4 x_2^2 y_1. \end{aligned}$$

Comparing coefficients it is easy to see that $\lambda_1=\lambda_2=\lambda_3=\beta'_2=0$.

We have shown that in $R(2C,D^{(2)})$ we are forced to put $\alpha'=\beta'_1=\beta'_2=0$ in order for the syzygy s_1 to lift to a syzygy in $R(2C,D^{(2)})$. In a similar way consider the syzygy s_3 which lifts to a syzygy $s_3^{(2)}$. This tells us that

$$x_1\delta - y_1\gamma \in I$$

is of degree 6. That is

$$x_1\delta - y_1\gamma = t_1r_5 + t_2x_1r_4 + t_3r_4 + t_4x_1^2r_3 + t_5x_1x_2r_3 +$$

$$t_6x_2^2r_3 + t_7y_1r_3 + t_8y_2r_3 + t_9x_1^2r_2 + t_{10}x_1x_2r_2$$

$$+ t_{11}x_2^2r_2 + t_{12}y_1r_2 + t_{13}y_2r_2 + t_{14}x_1^3r_1 +$$

$$t_{15}x_1^2x_2r_1 + t_{16}x_1x_2^2r_1 + t_{17}x_2^3r_1 + t_{18}x_1y_1r_1 +$$

$$t_{19}x_1y_2r_1 + t_{20}x_2y_1r_1 + t_{21}x_2y_2r_1 + t_{22}zr_1.$$

We can assume that

$$\delta=\delta' + \delta'',$$

$$\gamma=\gamma' + \gamma'',$$

where

$$x_1\delta' = y_1\gamma'.$$

So there is some $f \in \mathbb{C}[x_1,x_2,y_1,y_2,z]$ of degree 3, such that $\delta' = y_1f$ and $\gamma' = x_1f$. Given our restrictions on δ we can assume that $f = \varphi x_1^3$, $\varphi \in \mathbb{C}$. So we equate coefficients in,

$$x_1\delta'' - y_1\gamma'' = t_1r_5 + t_2x_1r_4 + t_3r_4 + t_4x_1^2r_3 + t_5x_1x_2r_3 +$$

$$t_6x_2^2r_3 + t_7y_1r_3 + t_8y_2r_3 + t_9x_1^2r_2 + t_{10}x_1x_2r_2$$

$$+ t_{11}x_2^2r_2 + t_{12}y_1r_2 + t_{13}y_2r_2 + t_{14}x_1^3r_1 +$$

$$t_{15}x_1^2x_2r_1 + t_{16}x_1x_2^2r_1 + t_{17}x_2^3r_1 + t_{18}x_1y_1r_1 +$$

$$t_{19}x_1y_2r_1 + t_{20}x_2y_1r_1 + t_{21}x_2y_2r_1 + t_{22}zr_1$$

and discover that $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, $t_6 = 0$, $t_7 = 0$, $t_8 = 0$, $t_{10} = 0$, $t_{11} = 0$, $t_{12} = 0$, $t_{13} = 0$, $t_{18} = 0$ and $t_{20} = 0$.

Also

$$\gamma = \varphi x_1^4 + t_4 x_1^2 y_2 + t_5 x_1 x_2 y_2 + t_9 x_1^2 y_1 + t_{14} x_1^3 x_2 + t_{15} x_1^2 x_2^2 + t_{16} x_1 x_2^3 + t_{17} x_2^4 + t_{19} x_1 x_2 y_2 + t_{21} x_2^2 y_2 + t_{22} x_2 z,$$

$$\delta = \varphi x_1^3 y_1 + t_4 x_1 x_2 z + t_5 x_2^2 z + t_9 x_1^2 z + t_{14} x_1^3 y_2 + t_{15} x_1^2 x_2 y_2 + t_{16} x_1 x_2^2 y_2 + t_{17} x_2^3 y_2 + t_{19} x_1 y_2 z + t_{21} x_2 y_2^2 + t_{22} y_2 z.$$

It is now a relatively simple matter to check that this γ and δ do in fact give a lift of s_4 and s_5 to $s_4^{(2)}$ and $s_5^{(2)}$ respectively:

$$s_4^{(2)} : x_2\delta - y_2\gamma \in I,$$

$$s_5^{(2)} : y_1\delta - z\gamma \in I.$$

Thus we have computed

$$R(2C,D^{(2)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/(x_0^2, I^{(2)})$$

where, after relabelling the coefficients in γ and δ, $I^{(2)}$ is generated by

$$r_1^{(2)} : x_1y_2 - x_2y_1,$$

$$r_2^{(2)} : x_1z - y_1^2,$$

$$r_3^{(2)} : x_2z - y_1y_2,$$

$$r_4^{(2)}: \; y_1z - x_2A - x_1B - t_1x_0x_1^2y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1^2y_1 -$$

$$t_4x_0x_1^3x_2 - t_5x_0x_1^2x_2^2 - t_6x_0x_1x_2^3 - t_7x_0x_2^4 - t_8x_0x_1x_2y_2$$

$$- t_9x_0x_2^2y_2 - t_{10}x_0x_2z - t_{11}x_0x_1^4,$$

$$r_5^{(2)}: \; z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_2^2z - t_3x_0x_1^2z -$$

$$t_4x_0x_1^3y_2 - t_5x_0x_1^2x_2y_2 - t_6x_0x_1x_2y_2 - t_7x_0x_2^3y_2 -$$

$$t_8x_0x_1y_2^2 - t_9x_0x_2y_2^2 - t_{10}x_0y_2z - t_{11}x_0x_1^3y_1.$$

These relations satisfy the following syzygies

$$s_1^{(2)}: \; x_1r_3^{(2)} - x_2r_2^{(2)} + y_1r_1^{(2)} \equiv 0,$$

$$s_2^{(2)}: \; y_1r_3^{(2)} - y_2r_2^{(2)} + zr_1^{(2)} \equiv 0,$$

$$s_3^{(2)}: \; x_1r_5^{(2)} - y_1r_4^{(2)} - zr_2^{(2)} + Ar_1^{(2)} - t_1x_0x_1^2r_3^{(2)} - t_2x_0x_1x_2r_3^{(2)}$$

$$- t_3x_0x_1^2r_2^{(2)} - t_4x_0x_1^3r_1^{(2)} - t_5x_0x_1^2x_2r_1^{(2)} - t_6x_0x_1x_2^2r_1^{(2)} -$$

$$t_7x_0x_2^3r_1^{(2)} - t_8x_0x_1y_2r_1^{(2)} - t_9x_0x_2y_2r_1^{(2)} - t_{10}x_0zr_1^{(2)}$$

$$\equiv 0,$$

$$s_4^{(2)}: \; x_2r_5^{(2)} - y_2r_4^{(2)} - zr_3^{(2)} + Br_1^{(2)} + t_1x_0x_1y_2r_1^{(2)} -$$

$$t_1x_0x_1x_2r_3^{(2)} + t_2x_0x_2y_2r_1^{(2)} - t_2x_0x_2^2r_3^{(2)} - t_3x_0x_1^2r_3^{(2)} +$$

$$t_{11}x_0x_1^3r_1^{(2)} \equiv 0,$$

$$s_5^{(2)}: \; y_1r_5^{(2)} - zr_4^{(2)} - Ar_3^{(2)} - Br_2^{(2)} + t_1x_0x_1zr_1^{(2)} + t_2x_0x_2zr_1^{(2)}$$

$$+ t_4x_0x_1^3r_3^{(2)} + t_5x_0x_1^2x_2r_3^{(2)} + t_6x_0x_1x_2^2r_3^{(2)} + t_7x_0x_2^3r_3^{(2)} +$$

$$t_8x_0x_1y_2r_3^{(2)} + t_9x_0x_2y_2r_3^{(2)} + t_{10}x_0zr_3^{(2)} + t_{11}x_0x_1^3r_2^{(2)}$$

$$\equiv 0.$$

Notice that these relations can be written in the Pfaffian form as previously. The following matrix yields $r_1^{(2)},\ldots,r_5^{(2)}$ as its diagonal 4x4 Pfaffians:

$$M^{(2)} = \begin{bmatrix} 0 & 0 & x_1 & x_2 & y_1 \\ 0 & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z + x_0p_1 & -A + x_0p_2 \\ -x_2 & -y_2 & z - x_0p_1 & 0 & B + x_0p_3 \\ -y_1 & -z & A - x_0p_2 & -B - x_0p_3 & 0 \end{bmatrix},$$

where

$$p_1 = t_3x_1^2,$$

$$p_2 = -t_4x_1^3 - t_5x_1^2x_2 - t_6x_1x_2^2 - t_7x_2^3 - t_8x_1y_2 - t_9x_2y_2 - t_{10}z,$$

$$p_3 = t_1x_1y_2 + t_{11}x_1^3.$$

Step 2 : $R^{(2)} \to R^{(3)}$

Using this ring we extend $R(2C,D^{(2)})$ to

$$R(3C,D^{(3)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/(x_0^3, I^{(3)}),$$

where $I^{(3)}$ is generated by,

$$r_1^{(3)} : x_1y_2 - x_2y_1,$$

$$r_2^{(3)} : x_1z - y_1^2 - \lambda_4x_0^2x_2^2 - \lambda_5x_0^2y_1 - \lambda_6x_0^2y_2,$$

$$r_3^{(3)} : x_2z - y_1y_2 - \beta''_2x_0^2,$$

$$r_4^{(3)} : y_1z - x_2A - x_1B - t_1x_0x_1^2y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1^2y_1 - t_4x_0x_1^3x_2 - t_5x_0x_1^2x_2^2 - t_6x_0x_1x_2^3 - t_7x_0x_2^4 - t_8x_0x_1x_2y_2 - t_9x_0x_2^2y_2 - t_{10}x_0x_2z - t_{11}x_0x_1^4 - \gamma''x_0^2,$$

$$r_5^{(3)} : z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_2^2z - t_3x_0x_1^2z - t_4x_0x_1^3y_2 - t_5x_0x_1^2x_2y_2 - t_6x_0x_1x_2y_2 - t_7x_0x_2^3y_2 - t_8x_0x_1y_2^2 - t_9x_0x_2y_2^2 - t_{10}x_0y_2z - t_{11}x_0x_1^3y_1 - \delta''x_0^2,$$

satisfying $s_i^{(3)}$ which reduce mod x_0^2 to $s_i^{(2)}$, i = 1,...,5.

If we consider $s_1^{(3)}$ we find that β''_1 ($= \lambda_4 x_0^2 x_2^2 + \lambda_5 x_0^2 y_1 + \lambda_6 x_0^2 y_2$) and β''_2 must satisfy

$$x_1\beta''_2 - x_2\beta''_1 \in I$$

which is of degree 3. Hence it can easily be seen that $\lambda_4 = \lambda_6 = 0$ and $\beta''_2 = \lambda_5 y_2$. Similarly s_2 gives

$$x_1\beta''_2 - x_2\beta''_1 \in I$$

but

$$x_1\beta''_2 - x_2\beta''_1 = \lambda_5 y_1 y_2 - \lambda_5 y_1 y_2 = 0,$$

as required.

We must also check that s_3, s_4 and s_5 can be lifted using these values and calculate γ'' and δ''. s_3 renders

$$x_1\delta'' - y_1\gamma'' + \lambda_5 z y_1 \in I$$

of degree 5. That is

$$\begin{aligned} x_1\delta'' - y_1\gamma'' + \lambda_5 z y_1 = \; & q_1 r_4 + q_2 x_1 r_3 + q_3 x_2 r_3 + q_4 x_1 r_2 \\ & + q_5 x_2 r_2 + q_6 x_1^2 r_1 + q_7 x_1 x_2 r_1 + \\ & q_8 x_2^2 r_1 + q_9 y_1 r_1 + q_{10} y_2 r_1. \end{aligned}$$

Equating coefficients we have

$$q_1 = q_3 = q_5 = q_9 = 0,$$

$$\gamma'' = -\lambda_5 z + q_2 x_1 y_2 + q_4 x_1 y_1 + q_6 x_1^2 x_2 + q_7 x_1 x_2^2 + q_8 x_2^3 + q_{10} x_2 y_2 + q_{11} x_1^3,$$

$$\delta'' = q_2 x_2 z + q_4 x_1 z + q_6 x_1^2 y_2 + q_7 x_1 x_2 y_2 + q_8 x_2^2 y_2 + q_{10} y_2^2 + q_{11} x_1^2 y_1.$$

Substituting these values into similar equations derived from s_4 and s_5 shows that we have indeed constructed

$$R(3C,D^{(3)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z] / (x_0^3, I^{(3)}),$$

where $I^{(3)}$ is generated by,

$r_1^{(3)}$: $x_1 y_2 - x_2 y_1$,

$r_2^{(3)}$: $x_1 z - y_1^2 - \lambda_5 x_0^2 y_1$,

$r_3^{(3)}$: $x_2 z - y_1 y_2 - \lambda_5 x_0^2 y_2$,

$r_4^{(3)}$: $(y_1 - \lambda_5 x_0^2)(z - t_3 x_0 x_1^2) - x_2(A - x_0(-t_4 x_1^3 - t_5 x_1^2 x_2 - t_6 x_1 x_2^2 - t_7 x_2^3 - t_8 x_1 y_2 - t_9 x_2 y_2 - t_{10} z - q_6 x_0 x_1^2 - q_7 x_0 x_1 x_2 - q_8 x_0 x_2^2 - q_{10} x_0 y_2)) - x_1(B + x_0(t_1 x_1 y_2 + t_2 x_2 y_2 + t_{11} x_1^3 + q_{11} x_0 x_1^2 + q_2 x_0 y_2 + q_4 x_0 y_1))$,

$$r_5^{(3)}: z(z - t_3x_0x_1^2) - y_2(A - x_0(-t_4x_1^3 - t_5x_1^2x_2 - t_6x_1x_2^2 - t_7x_2^3 - t_8x_1y_2 -$$

$$t_9x_2y_2 - t_{10}z - q_6x_0x_1^2 - q_7x_0x_1x_2 - q_8x_0x_2^2 - q_{10}x_0y_2)) -$$

$$y_1(B + x_0(t_1x_1y_2 + t_2x_2y_2 + t_{11}x_1^3 + q_{11}x_0x_1^2 + q_2x_0y_2 + q_4x_0y_1)).$$

It can be seen quite clearly that these relations can be given in Pfaffian form as before. The matrix, $M^{(3)}$, can be formed by picking the entries $(M^{(3)})_{ij}$ equal to the entries M_{ij} reduced mod x_0^3. The reader will by now, no doubt, have understood both the mechanics of the calculation and be able to guess the solution at the end of each step. We show the rest of the calculation only for the sake of completeness.

Step 3 : $R^{(3)} \to R^{(4)}$

We now extend to

$$R(4C,D^{(4)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z] / (x_0^4, I^{(4)}).$$

By consideration of s_1 we see easily that

$$r_1^{(4)}: x_1y_2 - x_2y_1,$$

$$r_2^{(4)}: x_1z - y_1^2 - \lambda_5x_0^2y_1,$$

$$r_3^{(4)}: x_2z - y_1y_2 - \lambda_5x_0^2y_2,$$

To compute γ''' and δ''' consider the lift of s_3,

$$s_3 : x_1\delta''' - y_1\gamma''' - t_1\lambda_5 x_1^2 y_2 - t_2\lambda_5 x_1 x_2 y_2 - t_3\lambda_5 x_1^2 y_1 \in I.$$

This is a degree 4 element of I.

Notice that we can write

$$\gamma''' = u_1 x_1^2 + \bar{\gamma}'''$$

$$\delta''' = u_1 x_1 y_1 + \bar{\delta}'''$$

Then a similar calculation to the previous ones shows us that

$$\bar{\gamma}''' = u_2 y_1 + u_3 x_1 x_2 - t_3\lambda_5 x_1^2$$

$$\bar{\delta}''' = u_2 z + (u_3 + t_1\lambda_5) x_1 y_2 - t_2\lambda_5 x_2 y_2$$

It is again a simple task to check that these new relations do allow us to lift all the syzygies as required . In fact we get

$$R(4C,D^{(4)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z] / (x_0^4, I^{(4)}),$$

where $I^{(4)}$ is generated by

$$r_1^{(4)}: x_1 y_2 - x_2 y_1,$$

$$r_2^{(4)}: x_1 z - y_1^2 - \lambda_5 x_0^2 y_1,$$

$$r_3^{(4)}: x_2 z - y_1 y_2 - \lambda_5 x_0^2 y_2,$$

$$r_4^{(4)}: y_1z - x_2A - x_1B - t_1x_0x_1^2y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1^2y_1 - t_4x_0x_1^3x_2 -$$

$$t_5x_0x_1^2x_2^2 - t_6x_0x_1x_2^3 - t_7x_0x_2^4 - t_8x_0x_1x_2y_2 - t_9x_0x_2^2y_2 - t_{10}x_0x_2z -$$

$$t_{11}x_0x_1^4 - \lambda_5x_0^2z - q_2x_0^2x_1y_2 - q_4x_0^2x_1y_1 - q_6x_0^2x_1^2x_2 - q_7x_0^2x_1x_2^2 -$$

$$q_8x_0^2x_2^3 - q_{10}x_0^2x_2y_2 - q_{11}x_0^2x_1^3 - u_1x_0^3x_1^2 - u_2x_0^3y_1 - u_3x_0^3x_1x_2$$

$$+ t_3\lambda_5x_0^3x_1^2,$$

$$r_5^{(4)}: z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_2^2z - t_3x_0x_1^2z - t_4x_0x_1^3y_2 -$$

$$t_5x_0x_1^2x_2y_2 - t_6x_0x_1x_2y_2 - t_7x_0x_2^3y_2 - t_8x_0x_1y_2^2 - t_9x_0x_2y_2^2 - t_{10}x_0y_2z$$

$$- t_{11}x_0x_1^3y_1 - q_2x_0^2x_2z - q_4x_0^2x_1z - q_6x_0^2x_1^2y_2 - q_7x_0^2x_1x_2y_2 -$$

$$q_8x_0^2x_2^2y_2 - q_{10}x_0^2y_2^2 - q_{11}x_0^2x_1^2y_1 - u_1x_0^3x_1y_1 - u_2x_0^3z -$$

$$(u_3 + t_1\lambda_5)x_0^3x_1y_2 - t_2\lambda_5x_0^3x_2y_2.$$

These relations satisfy

$$s_1^{(4)}: x_1r_3^{(4)} - x_2r_2^{(4)} + y_1r_1^{(4)} - \lambda_5x_0^2r_1^{(4)} \equiv 0,$$

$$s_2^{(4)}: y_1r_3^{(4)} - y_2r_2^{(4)} + zr_1^{(4)} \equiv 0,$$

$$s_3^{(4)}: x_1r_5^{(4)} - y_1r_4^{(4)} - zr_2^{(4)} + Ar_1^{(4)} - t_1x_0x_1^2r_3^{(4)} - t_2x_0x_1x_2r_3^{(4)} -$$

$$t_3x_0x_1^2r_2^{(4)} - t_4x_0x_1^3r_1^{(4)} - t_5x_0x_1^2x_2r_1^{(4)} - t_6x_0x_1x_2^2r_1^{(4)} - t_7x_0x_2^3r_1^{(4)} -$$

$$t_8x_0x_1y_2r_1^{(4)} - t_9x_0x_2y_2r_1^{(4)} - t_{10}x_0zr_1^{(4)} - q_2x_0^2x_1r_3^{(4)} - q_4x_0^2x_1r_2^{(4)} -$$

$$q_6x_0^2x_1^2r_1^{(4)} - q_7x_0^2x_1x_2r_1^{(4)} - q_8x_0^2x_2^2r_1^{(4)} - q_{10}x_0^2y_2r_1^{(4)} - u_2x_0^3r_2^{(4)} -$$

$$u_3x_0^3x_1r_1^{(4)} \equiv 0,$$

$$s_4^{(4)}: x_2r_5^{(4)} - y_2r_4^{(4)} - zr_3^{(4)} + Br_1^{(4)} + t_1x_0x_1y_2r_1^{(4)} - t_1x_0x_1x_2r_3^{(4)} +$$

$$t_2x_0x_2y_2r_1^{(4)} - t_2x_0x_2^2r_3^{(4)} - t_3x_0x_1^2r_3^{(4)} + t_{11}x_0x_1^3r_1^{(4)} - q_2x_0^2x_2r_3^{(4)} +$$

$$q_2x_0^2y_2r_1^{(4)} - q_4x_0^2x_1r_3^{(4)} + q_{11}x_0^2x_1^2r_1^{(4)} + u_1x_0^3x_1r_1^{(4)} - u_2x_0^3r_3^{(4)} -$$

$$u_3x_0^3x_2r_1^{(4)} \equiv 0,$$

$$s_5^{(4)}: y_1r_5^{(4)} - zr_4^{(4)} - Ar_3^{(4)} - Br_2^{(4)} + t_1x_0x_1zr_1^{(4)} + t_2x_0x_2zr_1^{(4)} +$$

$$t_4x_0x_1^3r_3^{(4)} + t_5x_0x_1^2x_2r_3^{(4)} + t_6x_0x_1x_2^2r_3^{(4)} + t_7x_0x_2^3r_3^{(4)} + t_8x_0x_1y_2r_3^{(4)} +$$

$$t_9x_0x_2y_2r_3^{(4)} + t_{10}x_0zr_3^{(4)} + t_{11}x_0x_1^3r_2^{(4)} - q_2x_0^2r_5^{(4)} + q_2x_0^2zr_1^{(4)} +$$

$$q_6x_0^2x_1^2r_3^{(4)} + q_7x_0^2x_1x_2r_3^{(4)} + q_8x_0^2x_2^2r_3^{(4)} + q_{10}x_0^2y_2r_3^{(4)} + q_{11}x_0^2x_1^2r_2^{(4)} +$$

$$u_1x_0^3x_1r_2^{(4)} + u_3x_0^3x_1r_3^{(4)} + t_1\lambda_5x_0^3x_1r_3^{(4)} + y_2\lambda_5x_0^3x_2r_3^{(4)} \equiv 0.$$

Step ℓ : $R^{(\ell)} \to R^{(\ell+1)}$

As we extend from ℓC to $(\ell+1)C$ and work modulo higher powers of x_0 there are fewer monomials of lower degrees. The calculations get somewhat easier and so the final ring required shall be written down here. We conclude that,

$$R(7C,\tilde{D}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z] /(x_0^7, \tilde{I}),$$

where $\tilde{I}$ is generated by

$$\tilde{r}_1: x_1y_2 - x_2 y_1,$$

$$\tilde{r}_2 : x_1z - y_1^2 - \lambda_5 x_0^2 y_1,$$

$$\tilde{r}_3 : x_2z - y_1y_2 - \lambda_5 x_0^2 y_2,$$

$$\begin{aligned}\tilde{r}_4 : \quad & y_1z - x_2A - x_1B - t_1x_0x_1^2y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1^2y_1 - \\ & t_4x_0x_1^3x_2 - t_5x_0x_1^2x_2^2 - t_6x_0x_1x_2^3 - t_7x_0x_2^4 - t_8x_0x_1x_2y_2 \\ & - t_9x_0x_2^2y_2 - t_{10}x_0x_2z - t_{11}x_0x_1^4 - \lambda_5x_0^2z - q_2x_0^2x_1y_2 - \\ & q_4x_0^2x_1y_1 - q_6x_0^2x_1^2x_2 - q_7x_0^2x_1x_2^2 - q_8x_0^2x_2^3 - q_{10}x_0^2x_2y_2 - \\ & q_{11}x_0^2x_1^3 - u_1x_0^3x_1^2 - u_2x_0^3y_1 - u_3x_0^3x_1x_2 + t_3\lambda_5x_0^3x_1^2 - \\ & vx_0^4x_1 - v_1x_0^4x_2 + u_2\lambda_5x_0^5,\end{aligned}$$

$$\tilde{r}_5 : \; z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_2^2z - t_3x_0x_1^2z -$$

$$t_4x_0x_1^3y_2 - t_5x_0x_1^2x_2y_2 - t_6x_0x_1x_2y_2 - t_7x_0x_2^3y_2 -$$

$$t_8x_0x_1y_2^2 - t_9x_0x_2y_2^2 - t_{10}x_0y_2z - t_{11}x_0x_1^3y_1 - q_2x_0^2x_2z$$

$$- q_4x_0^2x_1z - q_6x_0^2x_1^2y_2 - q_7x_0^2x_1x_2y_2 - q_8x_0^2x_2^2y_2 -$$

$$q_{10}x_0^2y_2^2 - q_{11}x_0^2x_1^2y_1 - u_1x_0^3x_1y_1 - u_2x_0^3z -$$

$$(u_3 - t_1\lambda_5)x_0^3x_1y_2 - t_2\lambda_5x_0^3x_2y_2 - (v_1 - \lambda_5q_1)x_0^4y_2 -$$

$$(v - \lambda_5q_2)x_0^4y_1$$

These relations satisfy the syzygies below

$$\tilde{s}_1 : x_1\tilde{r}_3 - x_2\tilde{r}_2 + y_1\tilde{r}_1 + \lambda_5x_0^2\tilde{r}_1 \equiv 0,$$

$$\tilde{s}_2 : y_1\tilde{r}_3 - y_2\tilde{r}_2 + z\tilde{r}_1 \equiv 0,$$

$$\tilde{s}_3 : x_1\tilde{r}_5 - y_1\tilde{r}_4 - z\tilde{r}_2 + A\tilde{r}_1 + t_1x_0x_1^2\tilde{r}_3 + t_2x_0x_1x_2\tilde{r}_3 + t_3x_0x_1^2\tilde{r}_2 +$$

$$t_4x_0x_1^3\tilde{r}_1 + t_5x_0x_1^2x_2\tilde{r}_1 + t_6x_0x_1x_2^2\tilde{r}_1 + t_7x_0x_2^3\tilde{r}_1 + t_8x_0x_1y_2\tilde{r}_1 +$$

$$t_9x_0x_2y_2\tilde{r}_1 + t_{10}x_0z\tilde{r}_1 + q_2x_0^2x_1\tilde{r}_3 + q_4x_0^2x_1\tilde{r}_2 + q_6x_0^2x_1^2\tilde{r}_1 +$$

$$q_7x_0^2x_1x_2\tilde{r}_1 + q_8x_0^2x_2^2\tilde{r}_1 + q_{10}x_0^2y_2\tilde{r}_1 + u_2x_0^3\tilde{r}_2 + u_3x_0^3x_1\tilde{r}_1 \equiv 0$$

$$\tilde{s}_4: x_2\tilde{r}_5 - y_2\tilde{r}_4 - z\tilde{r}_3 + B\tilde{r}_1 - t_1x_0x_1y_2\tilde{r}_1 + t_1x_0x_1x_2\tilde{r}_3 - t_2x_0x_2y_2\tilde{r}_1$$

$$+ t_2x_0x_2^2\tilde{r}_3 + t_3x_0x_1^2\tilde{r}_3 - t_{11}x_0x_1^3\tilde{r}_1 + q_2x_0^2x_2\tilde{r}_3 - q_2x_0^2y_2\tilde{r}_1 +$$

$$q_4x_0^2x_1\tilde{r}_3 - q_{11}x_0^2x_1^2\tilde{r}_1 - u_1x_0^3x_1\tilde{r}_1 + u_2x_0^3\tilde{r}_3 + u_3x_0^3x_2\tilde{r}_1 \equiv 0,$$

$$\tilde{s}_5: y_1\tilde{r}_5 - z\tilde{r}_4 - A\tilde{r}_3 - B\tilde{r}_2 - t_1x_0x_1z\tilde{r}_1 - t_2x_0x_2z\tilde{r}_1 - t_4x_0x_1^3\tilde{r}_3 -$$

$$t_5x_0x_1^2x_2\tilde{r}_3 - t_6x_0x_1x_2^2\tilde{r}_3 - t_7x_0x_2^3\tilde{r}_3 - t_8x_0x_1y_2\tilde{r}_3 - t_9x_0x_2y_2\tilde{r}_3 -$$

$$t_{10}x_0z\tilde{r}_3 - t_{11}x_0x_1^3\tilde{r}_2 + q_2x_0^2\tilde{r}_5 - q_2x_0^2z\tilde{r}_1 - q_6x_0^2x_1^2\tilde{r}_3 - q_7x_0^2x_1x_2\tilde{r}_3 -$$

$$q_8x_0^2x_2^2\tilde{r}_3 - q_{10}x_0^2y_2\tilde{r}_3 - q_{11}x_0^2x_1^2\tilde{r}_2 - u_1x_0^3x_1\tilde{r}_2 - u_3x_0^3x_1\tilde{r}_3 -$$

$$t_1\lambda_5x_0^3x_1\tilde{r}_3 - y_2\lambda_5x_0^3x_2\tilde{r}_3 \equiv 0.$$

At this stage the algorithm terminates and we can write:

$$R(X, K_X) \cong \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z] / \tilde{I}.$$

As with the curve we can fit the structure of this ring into a Pfaffian one. At the moment the relations are not quite written in such a way that a skew symmetric matrix with appropriate Pfaffians can be found. By adding multiples of $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_3$ to $\tilde{r}_4$ and $\tilde{r}_5$ this may be remedied and the matrix M, in the statement of the theorem will deliver a generating set for $\tilde{I}$.

□

14).Surfaces of type III.

Theorem (14.1)

Let X be a surface with $p_g = 3$ and $K_X^2 = 4$ such that the canonical system has two base points. Then the presentation of the ring R(C, D) given in Theorem (9.4) is formative and informative in degree -1. In fact

$$R(X, K_X) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/\tilde{I}.$$

The degrees of the generators are,

$$\deg (x_i, y_j, z_k) = (1, 2, 3) \text{ respectively.}$$

$\tilde{I}$ is generated by $\tilde{r}_1,\ldots,\tilde{r}_9$ as follows:

$$\text{Let } A = \begin{bmatrix} x_1 & y_1 & x_2^2-\alpha_{21}x_0x_2-\alpha_{23}x_0^2 & z_1 \\ x_2 & x_1^2-\alpha_{11}x_0x_1-\alpha_{13}x_0^2 & y_2 & z_2 \end{bmatrix}.$$

Then $\tilde{r}_1,\ldots,\tilde{r}_6$ are given by,

$$\text{rank}A \leq 1$$

$\tilde{r}_7, \tilde{r}_8, \tilde{r}_9$ are given by,

$$\begin{bmatrix} \tilde{r}_7 & \tilde{r}_8 \\ \tilde{r}_8 & \tilde{r}_9 \end{bmatrix} = APA^t,$$

where $P = (P_{ij})$ is a symmetric matrix with entries

$$P_{11} = -h - \delta_1 x_0 x_2 y_1 - \delta_2 \alpha_{21} x_0^2 x_1^2 - \delta_3 x_0 x_2 y_2 - \delta_{11} x_0 x_1 y_2 +$$

$$\delta_{11} \alpha_{11} x_0^2 y_2 - \delta_{15} x_0^2 y_1 - \delta_{18} x_0^2 y_2 - \delta_{13} x_0 z_1 - \delta_{16} x_0^3 x_1 +$$

$$\delta_{24} x_0^3 x_1 + \delta_{25} x_0^3 x_2 - \delta_{22} x_0^2 x_2^2 + \delta_{28} x_0^4,$$

$$P_{12} = \tfrac{1}{2}(-\delta_5 x_0 y_1 - \delta_6 x_0 y_2 + \delta_{27} x_0^3),$$

$$P_{13} = \tfrac{1}{2}(- \delta_2 x_0 y_1 - \delta_4 x_0 y_2 + \delta_{26} x_0^3),$$

$$P_{14} = \tfrac{1}{2}(-\delta_{13} \alpha_{13} x_0^2 + \delta_{23} x_0^2),$$

$$P_{22} = -\lambda y_1 - \delta_{10} x_0 x_2 - \delta_{21} x_0^2,$$

$$P_{23} = -\tfrac{1}{2}\delta_{17} x_0^2,$$

$$P_{24} = -\tfrac{1}{2}\delta_{12} x_0,$$

$$P_{33} = -\mu y_2 - \delta_4 \alpha_{21} x_0^2 - \delta_{19} x_0^2,$$

$$P_{34} = -\tfrac{1}{2}\delta_9 x_0,$$

$$P_{44} = 1.$$

Remark (14.2)

As was remarked in the proof of Theorem (9.4) relations generated by rank $M \leq 1$ and MPM^T satisfy syzygies given by $(M^*M)PM^T \equiv M^*(MPM^T)$. This shows that this presentation is formative. For reference the relations $\tilde{r}_1,\ldots,\tilde{r}_6$ in the statement of the theorem are reproduced below:

$\tilde{r}_1$: $x_1^3 - x_2y_1 - \alpha_{11}x_0x_1^2 - \alpha_{13}x_0^2x_1$,

$\tilde{r}_2$: $x_2^3 - x_1y_2 - \alpha_{21}x_0x_2^2 - \alpha_{23}x_0^2x_2$,

$\tilde{r}_3$: $y_1y_2 - x_1^2x_2^2 + \alpha_{11}x_0x_1x_2^2 + \alpha_{21}x_0x_1^2x_2 + \alpha_{13}x_0^2x_2^2 +$

$\alpha_{23}x_0^2x_1^2 - \alpha_{11}\alpha_{21}x_0^2x_1x_2 - \alpha_{21}\alpha_{13}x_0^3x_2 - \alpha_{23}\alpha_{11}x_0^3y_1 -$

$\alpha_{23}\alpha_{13}x_0^4$,

$\tilde{r}_4$: $x_1z_2 - x_2z_1$,

$\tilde{r}_5$: $x_1^2z_1 - y_1z_2 - \alpha_{11}x_0x_1z_1 - \alpha_{13}x_0^2z_1$,

$\tilde{r}_6$: $x_2^2z_2 - y_2z_1 - \alpha_{21}x_0x_2z_2 - \alpha_{23}x_0^2z_2$.

Proof of (14.1)

Let $C \in |K_X|$ be given by the vanishing of $x_0 \in H^0(X, K_X)$ and let $D = K_{X|C}$. In Theorem (9.4) we showed that

$$R(C,D) = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I,$$

where degree $x_i, y_i, z_i = 1, 2, 3$, and I is generated by $r_1,\ldots,r_9$ as follows. The relations below are $\pm$ those given in (9.4).

Degree 3 $r_1 : x_1^3 - x_2y_1$,

$r_2 : x_2^3 - x_1y_2$,

Degree 4 $r_3 : y_1y_2 - x_1^2x_2^2$,

$r_4 : x_1z_2 - x_2z_1$,

Degree 5 $\quad r_5 : x_1^2 z_1 - y_1 z_2,$

$r_6 : x_2^2 z_2 - y_2 z_1,$

Degree 6 $\quad r_7 : z_1^2 - \lambda y_1^3 - \mu x_2^4 y_2 - x_1^2 h,$

$r_8 : -z_1 z_2 + \lambda x_1^2 y_1^2 + \mu x_2^2 y_2^2 + x_1 x_2 h,$

$r_9 : z_2^2 - \lambda x_1^4 y_1 - \mu y_2^3 - x_2^2 h.$

The first 6 relations were written determinantally as follows,

$$\text{rank } M = \text{rank} \begin{bmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1$$

These give rise to determinantal syzygies as follows,

$\Sigma_1 : x_1 r_3 + y_1 r_2 + x_2^2 r_1 \equiv 0,$

$\Sigma_2 : -x_2 r_3 - x_1^2 r_2 - y_2 r_1 \equiv 0,$

$\Sigma_3 : x_1 r_6 - x_2^2 r_4 - z_1 r_2 \equiv 0,$

$\Sigma_4 : x_1 r_5 + y_1 r_4 - z_1 r_1 \equiv 0,$

$\Sigma_5 : -x_2 r_5 - x_1^2 r_4 + z_2 r_1 \equiv 0,$

$\Sigma_6 : -x_2 r_6 + y_2 r_4 + z_2 r_2 \equiv 0,$

$\Sigma_7 : -y_1 r_6 - x_2^2 r_5 - z_1 r_3 \equiv 0,$

$$\Sigma_8 : x_1^2 r_6 + y_2 r_5 + z_2 r_3 \equiv 0,$$

and it was found that the syzygies involving r_7, r_8 and r_9 are the following,

$$\Sigma_9 : x_2 r_7 + x_1 r_8 + z_1 r_4 - \lambda y_1^2 r_1 + \mu x_2^2 y_2 r_2 \equiv 0,$$

$$\Sigma_{10} : -x_2 r_8 - x_1 r_9 + z_2 r_4 - \lambda x_1^2 y_1 r_1 + \mu y_2^2 r_2 \equiv 0,$$

$$\Sigma_{11} : x_1^2 r_7 + y_1 r_8 - z_1 r_5 - \mu x_2^2 y_2 r_3 + x_1 h r_1 \equiv 0,$$

$$\Sigma_{12} : -x_1^2 r_8 - y_1 r_9 - z_2 r_5 - \mu y_2^2 r_3 + x_2 h r_1 \equiv 0,$$

$$\Sigma_{13} : y_2 r_7 + x_2^2 r_8 + z_1 r_6 + \lambda y_1^2 r_3 - x_1 h r_2 \equiv 0,$$

$$\Sigma_{14} : -y_2 r_8 - x_2^2 r_9 + z_2 r_6 + \lambda x_1^2 y_1 r_3 - x_2 h r_2 \equiv 0,$$

$$\Sigma_{15} : z_2 r_7 + z_1 r_8 - \lambda y_1^2 r_5 + \mu x_2^2 y_2 r_6 + x_1 h r_4 \equiv 0,$$

$$\Sigma_{16} : -z_2 r_8 - z_1 r_9 - \lambda x_1^2 y_1 r_5 + \mu y_2^2 r_6 + x_2 h r_4 \equiv 0.$$

Remember that we wrote down these relations in a format dependent only on the matrix M and a symmetric matrix. We shall use this format to shorten the extension-deformation calculations.

We now use the extension-deformation theory to extend this ring to $R(2C, D^{(2)})$ then to $R(3C,D^{(3)})$, $R(4C,D^{(4)})$ etc. If at each stage we can show that the relations lift to ones in the same format for some l_{ij}',P_{ij}' where l_{ij}' reduces mod x_0 to l_{ij}, P_{ij}' reduces mod x_0 to P_{ij} then the lifted relations will automatically satisfy syzygies Σ_1',...,Σ_{16}' such that Σ_i' reduces mod x_0 to Σ_i as required. This will shorten some of the checking procedures along

the way.

Step 0 : Simplifications

First we do some basis changes to simplify the calculations. Suppose

$$R(X,K_X) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/\tilde{I},$$

where $\tilde{I}$ is generated by,

$$\tilde{r}_1 : x_1^3 - x_2y_1 - a_1x_0,$$

$$\tilde{r}_2 : x_2^3 - x_1y_2 - a_2x_0,$$

$$\tilde{r}_3 : y_1y_2 - x_1^2x_2^2 - b_1x_0,$$

$$\tilde{r}_4 : x_1z_2 - x_2z_1 - b_2x_0,$$

$$\tilde{r}_5 : x_1^2z_1 - y_1z_2 - c_1x_0,$$

$$\tilde{r}_6 : x_2^2z_2 - y_2z_1 - c_2x_0,$$

$$\tilde{r}_7 : z_1^2 - \lambda y_1^3 - \mu x_2^4y_2 - x_1^2h - d_1x_0,$$

$$\tilde{r}_8 : -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h - d_2x_0,$$

$$\tilde{r}_9 : z_2^2 - \lambda x_1^4y_1 - \mu y_2^3 - x_2^2h - d_3x_0.$$

Here the a_i, b_i, c_i and d_i are polynomials in x_0, x_1, y_1, y_2, z_1 and z_2 so that the $\tilde{r}_i$ satisfy syzygies $\tilde{\Sigma}_i$ such that $\tilde{\Sigma}_i$ reduces mod x_0 to Σ_i. By changing basis we can assume that

$$a_1 = \alpha_{11}x_1^2 + \alpha_{12}y_2 + \alpha_{13}x_0x_1 + \alpha_{14}x_0^2,$$

$$a_2 = \alpha_{21}x_2^2 + \alpha_{22}y_1 + \alpha_{23}x_0x_2 + \alpha_{24}x_0^2,$$

$$b_2 = \beta_{21}z_1 + \beta_{22}z_2 + \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0^3.$$

The c_i are degree 4 and so (with a chosen basis for $H^0(X,4K_X)$) we can write

$$c_1 = \gamma_{1,1}x_1^2y_1 + \gamma_{1,2}x_1x_2y_1 + \gamma_{1,3}x_2^2y_1 + \gamma_{1,4}x_1^2y_2 + \gamma_{1,5}x_1x_2y_2 + \gamma_{1,6}x_2^2y_2$$

$$+ \gamma_{1,7}y_1^2 + \gamma_{1,8}y_1y_2 + \gamma_{1,9}y_2^2 + \gamma_{1,10}x_1z_1 + \gamma_{1,11}x_2z_1 + \gamma_{1,12}x_2z_2 +$$

$$\gamma_{1,13}x_0x_1^2x_2 + \gamma_{1,14}x_0x_1x_2^2 + \gamma_{1,15}x_0x_1y_1 + \gamma_{1,16}x_0x_1y_2 + \gamma_{1,17}x_0x_2y_1 +$$

$$\gamma_{1,18}x_0x_2y_2 + \gamma_{1,19}x_0z_1 + \gamma_{1,20}x_0z_2 + \gamma_{1,21}x_0^2x_1^2 + \gamma_{1,22}x_0^2x_1x_2 + \gamma_{1,23}x_0^2x_2^2$$

$$+ \gamma_{1,24}x_0^2y_1 + \gamma_{1,25}x_0^2y_2 + \gamma_{1,26}x_0^3x_1 + \gamma_{1,27}x_0^3x_2 + \gamma_{1,28}x_0^4,$$

and similarly for c_2 with $\gamma_{1,i}$ replaced by $\gamma_{2,i}$. Since $h^0(X,5K_X) = 44$ we could also write down expressions for the d_i depending on 44 parameters before we begin the calculations (but we won't).

Step 1 : $R \to R^{(2)}$

We begin the calculation of $R(2C,D^{(2)})$. Consider

$$\Sigma_1^{(2)}: x_1r_3^{(2)} + y_1r_2^{(2)} + x_2^2r_1^{(2)} \in x_0I^{(2)}.$$

Then

$$x_1b'_1 + y_1a'_2 + x_2^2a'_1 \in I.$$

That is,

$$x_1b'_1 + y_1a'_2 + x_2^2a'_1 = p_1r_3 + p_2r_4 + p_3x_1r_1 + p_4x_2r_1 + p_5x_1r_2 + p_6x_2r_2,$$

where $p_i \in \mathbb{C}$.

Comparing coefficients we see at once that:

$$\alpha_{12}=\alpha_{22}=p_2=p_3=p_5=p_6=0 \text{ and } \alpha_{11}=-p_1,\ \alpha_{21}=-p_4,$$

$$b'_1= -\alpha_{11}x_1x_2^2 - \alpha_{21}x_2x_1^2.$$

These coefficients also satisfy

$$\Sigma_2^{(2)}: x_2r_3^{(2)} + x_1^2r_2^{(2)} + y_2r_1^{(2)} \in x_0I^{(2)}.$$

In the same way Σ_3 must lift to a syzygy in $R(2C,D^{(2)})$:

$$\Sigma_3 : x_1c'_2 - x_2^2b'_2 - z_1a'_2 \in I.$$

Substituting what we already know from Step 0 about a'_2 and b'_2 we get

$$x_1c'_2 - \beta_{21}x_2^2z_1 - \beta_{22}x_2^2z_2 - \alpha_{21}x_2^2z_1 \in I.$$

This leads us to the equations between the coefficients

$$\beta_{22}=0,\ c'_2= (\beta_{21} + \alpha_{21})x_2z_2 .$$

However, looking at

$$\Sigma_4 : -x_1c'_1 - y_1b'_2 + z_1a'_1 \in I,$$

we find that

$$\beta_{21}= 0 ,\ c'_1 = \alpha_{11}x_1z_1.$$

So far we have deduced that,

$$a'_1 = \alpha_{11}x_1^2,$$

$$a'_2 = \alpha_{21}x_2^2,$$

$$b'_1 = -\alpha_{11}x_1x_2^2 - \alpha_{21}x_1^2x_2,$$

$$b'_2 = 0,$$

$$c'_1 = \alpha_{11}x_1z_1,$$

$$c'_2 = \alpha_{21}x_2z_2.$$

Incidentally, the relations $r_1^{(2)},\ldots,r_6^{(2)}$ are still determinantal in form. In fact they are given by,

$$\operatorname{rank}\begin{bmatrix} x_1 & y_1 & x_2^2 - \alpha_{21}x_0x_2 & z_1 \\ x_2 & x_1^2 - \alpha_{11}x_0x_1 & y_2 & z_2 \end{bmatrix} \leq 1.$$

From this we can see that all the determinantal syzygies are lifted by these choices of polynomial.

The calculations get much longer when we consider the nondeterminantal syzygies, $\Sigma_9,\ldots,\Sigma_{16}$. In order to lift Σ_9 we need,

$$x_2d'_1 + x_1d'_2 + z_1b'_2 - \lambda y_1^2a'_1 + \mu x_2^2y_2a'_2 \in I.$$

We can write

$$d'_1 = \delta_1x_1y_2^2 + \bar{d}'_1$$

$$d'_2 = \lambda\alpha_{11}x_1y_1^2 - (\mu\alpha_{21} + \delta_1)x_2y_2^2 + \bar{d}'_2$$

where

$$x_2\bar{d}'_1 + x_1\bar{d}'_2 \in I$$

Since d_1 and d_2 are degree 5 , this equation is degree 6. Taking a basis of $H^0(5K_{X|C})$ we can write

$$d'_i = \delta_{i,1}x_1^2x_2y_1 + \delta_{i,2}x_1x_2^2y_1 + \delta_{i,3}x_1^2x_2y_2 + \delta_{i,4}x_1x_2^2y_2 + \delta_{i,5}x_1y_1^2 +$$

$$\delta_{i,6}x_1y_1y_2 + \delta_{i,7}x_1y_2^2 + \delta_{i,8}x_1x_2z_1 + \delta_{i,9}x_2^2z_1 + \delta_{i,10}x_2y_1^2 + \delta_{i,11}x_2y_1y_2 +$$

$$\delta_{i,12}x_2y_2^2 + \delta_{i,13}y_1z_1 + \delta_{i,14}y_1z_2 + \delta_{i,15}y_2z_1 + \delta_{i,16}y_2z_2.$$

Similarly taking a basis for the degree 6 part of I we find that we have to solve,

$$x_2\bar{d}'_1 + x_1\bar{d}'_2 = t_1y_1r_3 + t_2y_2r_3 + t_3x_1^2r_4 + t_4x_1x_2r_4 + t_5x_2^2r_4 + t_6y_1r_4 +$$

$$t_7y_2r_4 + t_8x_1^3r_1 + t_9x_1x_2^2r_1 + t_{10}x_1^2x_2r_1 + t_{11}x_2^3r_1 + t_{12}x_1y_1r_1$$
$$+ t_{13}x_1y_2r_1 + t_{14}x_2y_1r_1 + t_{15}x_2y_2r_1 + t_{16}z_1r_1 + t_{17}z_2r_1 +$$

$$t_{18}x_1x_2^2r_2 + t_{19}x_1^2x_2r_2 + t_{20}x_2^3r_2 + t_{21}x_1y_1r_2 + t_{22}x_1y_2r_2 +$$
$$t_{23}x_2y_1r_2 + t_{24}x_2y_2r_2 + t_{25}z_1r_2 + t_{26}z_2r_2$$

It is found that

$t_i = 0$ for i = 1, 2, 4, 5, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 23, 26.

After renumbering we see that

$$d'_1 = \delta_1x_1^2x_2y_1 + \delta_2x_1x_2^2y_1 + \delta_3x_1^2x_2y_2 + \delta_4x_1x_2^2y_2 + \delta_5x_1y_1^2 +$$

$$\delta_6x_1y_1y_2 + \delta_7x_1y_2^2 + \delta_8x_1x_2z_1 + \delta_9x_2^2z_1 + \delta_{10}x_2y_1^2 +$$
$$\delta_{11}x_2y_1y_2 + \delta_{12}y_1z_1 + \delta_{13}y_1z_2 + \delta_{14}y_2z_1.$$

$$d'_2 = -\delta_1 x_1 x_2^2 y_1 - \delta_2 x_1 y_1 y_2 - \delta_3 x_1 x_2^2 y_2 - \delta_4 x_1 y_2^2 - \delta_5 x_2 y_1^2 -$$

$$\delta_6 x_2 y_1 y_2 - (\delta_7 + \mu\alpha_{21}) x_2 y_2^2 - \delta_8 x_2^2 z_1 - \delta_9 y_2 z_1 - \delta_{10} x_1^2 x_2 y_1$$

$$- \delta_{11} x_1^2 x_2 y_2 - \delta_{12} y_1 z_2 - \delta_{13} x_1 x_2 z_1 - \delta_{14} y_2 z_2 + \lambda\alpha_{11} x_1 y_1^2.$$

Now some restrictions are placed on d'_3 using our calculated values for a'_1, a'_2, b'_2, d'_2 and the syzygy Σ_{10}.

$$\Sigma_{10} : -x_2 d'_2 - x_1 d'_3 - \lambda\alpha_{11} x_1^4 y_1 + \mu\alpha_{21} x_2^2 y_2^2 \in I$$

That is ,

$$\delta_1 x_1 x_2^3 y_1 + \delta_2 x_1 x_2 y_1 y_2 + \delta_3 x_1 x_2^3 y_2 + \delta_4 x_1 x_2 y_2^2 + \delta_5 x_2^2 y_1^2 + \delta_6 x_2^2 y_1 y_2 + (\delta_7$$

$$+ \mu\alpha_{21}) x_2^2 y_2^2 + \delta_8 x_2^3 z_1 + \delta_9 x_2 y_2 z_1 + \delta_{10} x_1^2 x_2^2 y_1 + \delta_{11} x_1^2 x_2^2 y_2 + \delta_{12} x_2 y_1 z_2 +$$

$$\delta_{13} x_1 x_2^2 z_1 + \delta_{14} x_2 y_2 z_2 - \lambda\alpha_{11} x_1 x_2 y_1^2 - x_1 \bar{d}_3 - \lambda\alpha_{11} x_1^4 y_1 + \mu\alpha_{21} x_2^2 y_2^2 \in I$$

Writing out the righthand side of this equation in full and comparing coefficients gives,

$$d'_2 = -\delta_1 x_1 x_2^2 y_1 - \delta_2 x_1 y_1 y_2 - \delta_3 x_1 x_2^2 y_2 - \delta_4 x_1 y_2^2 - \delta_5 x_2 y_1^2 -$$

$$\delta_6 x_2 y_1 y_2 - \delta_9 y_2 z_1 - \delta_{10} x_1^2 x_2 y_1 - \delta_{11} x_1^2 x_2 y_2 - \delta_{12} y_1 z_2 -$$

$$\delta_{13} x_1 x_2 z_1 + \lambda\alpha_{11} x_1 y_1^2 + \mu\alpha_{21} x_2 y_2^2,$$

$$d'_3 = -2\lambda\alpha_{11}x_2y_1^2 + \delta_1x_1y_1y_2 + \delta_2x_2y_1y_2 + \delta_3x_1y_2^2 + \delta_4x_2y_2^2 +$$

$$\delta_5x_1^2x_2y_1 + \delta_6x_1^2x_2y_2 + \delta_9y_2z_2 + \delta_{10}x_1x_2^2y_1 + \delta_{11}x_1x_2^2y_2 +$$

$$\delta_{12}x_1x_2z_1 + \delta_{13}x_2^2z_1.$$

Consequently,

$$\delta_8=\delta_{14}=0,\ \delta_7=-2\mu\alpha_{21}$$

We have shown that the following expressions for $r_1^{(2)},\ldots,r_9^{(2)}$ are necessary to extend the syzygies $\Sigma_1,\ldots,\Sigma_{10}$. They have been written in the formative presentation of the statement of the theorem. This will prove that they also extend the syzygies $\Sigma_{11},\ldots,\Sigma_{16}$.

$r_1^{(2)},\ldots,r_6^{(2)}$ are given by

$$\text{rank } M^{(2)} = \text{rank}\begin{bmatrix} x_1 & -y_1 & x_2^2 - \alpha_{21}x_0x_2 & -z_1 \\ -x_2 & x_1^2 - \alpha_{11}x_0x_1 & -y_2 & z_2 \end{bmatrix} \leq 1,$$

$r_7^{(2)}$, $r_8^{(2)}$, $r_9^{(2)}$ are given by

$$M^{(2)}P^{(2)}M^{(2)T} = \begin{bmatrix} r_7^{(2)} & r_8^{(2)} \\ r_8^{(2)} & r_9^{(2)} \end{bmatrix},$$

where $P^{(2)}$ is the symmetric matrix with coefficients,

$$P_{11} = -h - \delta_1x_0x_2y_1 - \delta_3x_0x_2y_2 - \delta_{13}x_0z_1 - \delta_{11}x_0x_1y_2,$$

$$P_{12} = \tfrac{1}{2}(\delta_5x_0y_1 + \delta_6x_0y_2),$$

$$P_{13} = \tfrac{1}{2}(-\delta_2x_0y_1 - \delta_4x_0y_2),$$

$P_{14} = 0,$

$P_{22} = -\lambda y_1 - \delta_{10}x_0x_2,$

$P_{23} = 0,$

$P_{24} = -\frac{1}{2}\delta_{12}x_0,$

$P_{33} = -\mu y_2,$

$P_{34} = \frac{1}{2}\delta_9 x_0,$

$P_{44} = 1.$

These relations satisfy the syzygies $\Sigma_1^{(2)},\ldots,\Sigma_{16}^{(2)}$ following from the format (see Remark (9.5)). Below are listed $\Sigma_1^{(2)}$, $\Sigma_2^{(2)}$, $\Sigma_3^{(2)}$, $\Sigma_5^{(2)}$, $\Sigma_9^{(2)}$ and $\Sigma_{10}^{(2)}$ which we use in step 2.

$$\Sigma_1^{(2)} : x_1r_3^{(2)} + y_1r_2^{(2)} + x_2^2r_1^{(2)} - \alpha_{21}x_0x_2r_1^{(2)} \equiv 0,$$

$$\Sigma_2^{(2)} : -x_2r_3^{(2)} - x_1^2r_2^{(2)} - y_2r_1^{(2)} + \alpha_{11}x_0x_1r_1^{(2)} \equiv 0,$$

$$\Sigma_3^{(2)} : x_1r_6^{(2)} - x_2^2r_4^{(2)} - z_1r_2^{(2)} + \alpha_{21}x_0x_2r_4^{(2)} \equiv 0,$$

$$\Sigma_5^{(2)} : -x_2r_5^{(2)} - x_1^2r_4^{(2)} + z_2r_1^{(2)} + \alpha_{11}x_0x_1r_4^{(2)} \equiv 0,$$

$$\Sigma_9^{(2)} : x_2r_7^{(2)} + x_1r_8^{(2)} + z_1r_4^{(2)} - \lambda y_1^2r_1^{(2)} + \mu x_2^2y_2r_2^{(2)} + \delta_2x_0x_1y_1r_2^{(2)} +$$

$$\delta_4x_0x_1y_2r_2^{(2)} + \delta_9x_0z_1r_2^{(2)} - \delta_{10}x_0x_2y_1r_1^{(2)} - \delta_{11}x_0x_2y_2r_1^{(2)} -$$

$$\delta_{12}x_0y_1r_4^{(2)} - \delta_{13}x_0x_2r_5^{(2)} + \mu\alpha_{21}x_0x_2y_2r_2^{(2)} \equiv 0,$$

$$\Sigma_{10}^{(2)} : -x_2r_8^{(2)} - x_1r_9^{(2)} + z_2r_4^{(2)} - \lambda x_1^2y_1r_1^{(2)} + \mu y_2^2r_2^{(2)} + \delta_1x_0x_1y_1r_2^{(2)} +$$

$$\delta_3x_0x_1y_2r_2^{(2)} - \delta_5x_0x_2y_1r_1^{(2)} - \delta_6x_0x_2y_2r_1^{(2)} - \delta_9x_0y_2r_4^{(2)} - \delta_{12}x_0x_2r_5^{(2)}$$

$$- \lambda\alpha_{11}x_0x_1y_1r_1^{(2)} \equiv 0.$$

Hence we have extended R(C,D) to

$$R(2C,D^{(2)})=\mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/(x_0^2, I^{(2)})$$

where $I^{(2)}$ is generated by $r_1^{(2)},\ldots,r_9^{(2)}$.

Step 2 : $R^{(2)} \to R^{(3)}$

We extend $R(2C,D^{(2)})$ to $R(3C,D^{(3)})$ in the same way. First we extend the determinantal syzygies.

Begin by extending Σ_1 to a syzygy $\Sigma_1^{(3)}$ in $R(3C,D^{(3)})$

$$\Sigma_1^{(3)}: x_1r_3^{(3)} + y_1r_2^{(3)} + x_2^2r_1^{(3)} - \alpha_{21}x_0x_2r_1^{(3)} \in x_0I^{(3)}$$

That is,

$$x_1b''_1 + y_1\alpha_{23}x_2 + x_2^2\alpha_{13}x_1 - \alpha_{11}\alpha_{21}x_1^2x_2 \in I$$

is of degree 3. We deduce that

$$b''_1 = -\alpha_{13}x_2^2 + \alpha_{11}\alpha_{21}x_1x_2 - \alpha_{23}x_1^2.$$

This expression for b''_1 also lifts Σ_2 to $\Sigma_2^{(3)}$. To lift Σ_3 we need

$$x_1c''_2 - x_2^2b''_2 - z_1\alpha_{23}x_2 \in I.$$

Bearing in mind our simplifications at step 0 we set about finding solutions for this expression. To begin with let

$$c''_2 = \bar{c}''_2 + x_2^2 f \, , \; b''_2 = \bar{b}''_2 + x_1 f$$

where f is degree 1. Since the possibilities that c''_2 contains a term in x_2^3 and that b_2'' contains a term in x_1^2 have been excluded by our previous simplifications, f=0.

Thus we need

$$x_1\bar{c}''_2 - x_2^2\bar{b}''_2 - z_1\alpha_{23}x_2 \in I.$$

Comparing with a general degree 4 element of I gives,

$$c''_2 = \alpha_{23}z_2 + \gamma_2 x_1^2 x_2$$

$$b''_2 = \gamma_2 y_1.$$

To lift to $\Sigma_5^{(3)}$ we need

$$x_2 c''_1 + x_1^2 b''_2 - z_2\alpha_{13}x_1 \in I,$$

that is,

$$x_2 c''_1 + \gamma_2 x_1^2 y_1 - z_2\alpha_{13}x_1 \in I.$$

It is easy to deduce that,

$$\gamma_2 = 0$$

$$c''_1 = \alpha_{13}z_1,$$

A simple check shows that these expressions for $a''_1, a''_2, b''_1, b''_2, c''_1$ and c''_2 also lift the remaining determinantal syzygies.

Now we see if they will lift the nondeterminantal syzygies and calculate expressions for d''_1,d''_2 and d''_3. First consider

$$\Sigma_9^{(3)}: x_2d''_1 + x_1d''_2 + z_1b''_2 - \lambda y_1^2a''_1 + \mu x_2^2y_2a''_2 + \delta_2\alpha_{21}x_1x_2^2y_1$$

$$+ \delta_4\alpha_{21}x_1x_2^2y_2 + \delta_9\alpha_{21}x_2^2z_1 - \delta_{10}\alpha_{11}x_1^2x_2y_1 -$$

$$\delta_{11}\alpha_{11}x_1^2x_2y_2 - \delta_{13}\alpha_{11}x_1x_2z_1 + \mu\alpha_{21}^2x_2^3y_2 \in I.$$

Comparing this with a general degree 5 element of I gives

$$x_2d''_1 + x_1d''_2 - \lambda\alpha_{13}x_1y_1^2 + \mu\alpha_{23}x_2^3y_2 + \delta_2\alpha_{21}x_1x_2^2y_1 +$$

$$\delta_4\alpha_{21}x_1x_2^2y_2 + \delta_9\alpha_{21}x_2^2z_1 - \delta_{10}\alpha_{11}x_1^2x_2y_1 -$$

$$\delta_{11}\alpha_{11}x_1^2x_2y_2 - \delta_{13}\alpha_{11}x_1x_2z_1 + \mu\alpha_{21}^2x_2^3y_2$$

$=$

$$t_1r_6 + t_2r_5 + t_3x_1r_4 + t_4x_2r_4 + t_5x_1^2r_1 + t_6x_1x_2r_1 +$$

$$t_7x_2^2r_1 + t_8y_1r_1 + t_9y_2r_1 + t_{10}x_1^2r_2 + t_{11}x_1x_2r_2 +$$

$$t_{12}x_2^2r_2 + t_{13}y_1r_2 + t_{14}y_2r_2.$$

Equating coefficients we get

$$t_1=t_2=t_3=t_5=t_6=t_7=t_{10}=t_{11}=t_{12}=0,$$

$$d''_1 = \delta_{15}x_1^2y_1 + \delta_{16}x_1x_2y_1 + \delta_{17}x_2^2y_1 + \delta_{18}x_1^2y_2 + \delta_{19}x_1x_2y_2 +$$

$$\delta_{20}x_2^2y_2 + \delta_{21}y_1^2 + \delta_{22}y_1y_2 + \delta_{23}x_1z_1 + \delta_{24}x_2z_1$$

$$d''_2 = (\delta_{10}\alpha_{11}-\delta_{15})x_1x_2y_1 - (\delta_2\alpha_{21} + \delta_{16})x_2^2y_1 - \delta_{17}y_1y_2 +$$

$$(\delta_{11}\alpha_{11}- \delta_{18})x_1x_2y_2 - (\delta_{19} + \delta_4\alpha_{21})x_2^2y_2 -$$

$$(\mu\alpha_{21}^2 + \mu\alpha_{23} + \delta_{20})y_2^2 - \delta_{21}x_1^2y_1 - \delta_{22}x_1^2y_2 +$$

$$(\delta_{13}\alpha_{11} - \delta_{23})x_2z_1 - (\delta_9\alpha_{21} + \delta_{24})x_2z_2 + \lambda\alpha_{13}y_1^2$$

Now by considering

$$\Sigma_{10}^{(3)}: x_2d''_2 + x_1d''_3 - z_2b''_2 + \lambda x_1^2y_1a''_1 - \mu y_2^2a''_2 - \delta_1\alpha_{21}x_1x_2^2y_1$$

$$- \delta_3\alpha_{21}x_1x_2^2y_2 + \delta_5\alpha_{11}x_1^2x_2y_1 + \delta_6\alpha_{11}x_1^2x_2y_2 +$$

$$\delta_{12}\alpha_{11}x_1x_2z_1 + \lambda\alpha_{11}^2x_1^3y_1 \in I$$

and, in the same way as before, comparing coefficients we find that

$$\delta_{20}=-2\mu\alpha_{23} - \mu\alpha_{21}^2,\ \delta_{24}=-\delta_9\alpha_{21},$$

and are left with

$$d''_1 = \delta_{15}x_1^2y_1 + \delta_{16}x_1x_2y_1 + \delta_{17}x_2^2y_1 + \delta_{18}x_1^2y_2 + \delta_{19}x_1x_2y_2 -$$

$$(2\mu\alpha_{23} + \mu\alpha_{21}^2)x_2^2y_2 + \delta_{21}y_1^2 + \delta_{22}y_1y_2 + \delta_{23}x_1z_1 -$$

$$\delta_9\alpha_{21}x_2z_1,$$

$$d''_2 = (\delta_{10}\alpha_{11}-\delta_{15})x_1x_2y_1 - (\delta_2\alpha_{21}+\delta_{16})x_2^2y_1 - \delta_{17}y_1y_2 +$$

$$(\delta_{11}\alpha_{11}-\delta_{18})x_1x_2y_2 - (\delta_{19}+\delta_4\alpha_{21})x_2^2y_2 + \mu\alpha_{23}y_2^2 -$$

$$\delta_{21}x_1^2y_1 - \delta_{22}x_1^2y_2 + (\delta_{13}\alpha_{11} - \delta_{23})x_2z_1 + \lambda\alpha_{13}y_1^2,$$

$$d''_3 = (\delta_1\alpha_{21} - \delta_{10}\alpha_{11} + \delta_{15})x_2^2y_1 + (\delta_2\alpha_{21}+\delta_{16})y_1y_2 +$$

$$\delta_{17}x_1^2y_2 + (\delta_3\alpha_{21} - \delta_{11}\alpha_{11} + \delta_{18})x_2^2y_2 + (\delta_4\alpha_{21}+\delta_{19})y_2^2 +$$

$$(-\delta_5\alpha_{11}+\delta_{21})x_1x_2y_1 + (-\delta_6\alpha_{11}+\delta_{22})x_1x_2y_2 +$$

$$(\delta_{23} - \delta_{13}\alpha_{11})x_2z_2 - \delta_{12}\alpha_{11}x_2z_1 - \lambda(2\alpha_{13}+\alpha_1{}_1^2)x_1^2y_1.$$

So we have proved that $r_1^{(3)},\ldots,r_9^{(3)}$ must be of the following form in order for the syzygies $\Sigma_1,\ldots,\Sigma_{10}$ to lift:

rank $M^{(3)}$ =

$$\text{rank}\begin{bmatrix} x_1 & -y_1 & x_2^2-\alpha_{21}x_0x_2-\alpha_{23}x_0^2 & -z_1 \\ -x_2 & x_1^2-\alpha_{11}x_0x_1-\alpha_{13}x_0^2 & -y_2 & z_2 \end{bmatrix}\leq 1,$$

gives

$$r_1^{(3)} : x_1^3 - x_2y_1 - \alpha_{11}x_0x_1^2 - \alpha_{13}x_0^2x_1,$$

$$r_2^{(3)} : x_2^3 - x_1y_2 - \alpha_{21}x_0x_2^2 - \alpha_{23}x_0^2x_2,$$

$$r_3^{(3)} : y_1y_2 - x_1^2x_2^2 + \alpha_{11}x_0x_1x_2^2 + \alpha_{21}x_0x_1^2x_2 + \alpha_{13}x_0^2x_2^2 + \alpha_{23}x_0^2x_1^2 -$$

$$\alpha_{11}\alpha_{21}x_0^2x_1x_2,$$

$$r_4^{(3)} : x_1z_2 - x_2z_1,$$

$$r_5^{(3)} : x_1^2z_1 - y_1z_2 - \alpha_{11}x_0x_1z_1 - \alpha_{13}x_0^2z_1,$$

$$r_6^{(3)} : x_2^2z_2 - y_2z_1 - \alpha_{21}x_0x_2z_2 - \alpha_{23}x_0^2z_2,$$

and we may write, $r_7^{(3)}$, $r_8^{(3)}$, $r_9^{(3)}$, as

$$M^{(3)}P^{(3)}M^{(3)T} = \begin{bmatrix} r_7^{(3)} & r_8^{(3)} \\ r_8^{(3)} & r_9^{(3)} \end{bmatrix}.$$

Here $P^{(3)}$ is the symmetric matrix given by reducing the coefficients in the matrix P mod x_0^3 (P as in the statement of the theorem).

Since these relations have been presented in the form given in the statement of the theorem they satisfy syzygies $\Sigma_1^{(3)},\ldots,\Sigma_{16}^{(3)}$ such that $\Sigma_i^{(3)}$ reduces mod x_0 to Σ_i.

So we have demonstrated that

$$R(3C,D^{(3)}) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/(x_0^3, I^{(3)}),$$

where $I^{(3)}$ is generated by $r_1^{(3)},\ldots,r_9^{(3)}$.

Step ℓ : $R^{(\ell)} \to R^{(\ell+1)}$

We continue algorithmically extending $R(3C,D^{(3)})$ to $R(4C,D^{(4)})$,$R(4C,D^{(4)})$ to $R(5C,D^{(5)})$ etc until we have reached $R(7C,D^{(7)})$.

At this point the algorithm ends and we will, in fact have the ideal $\tilde{I}$ generated as in the statement of the theorem. The remainder of the calculations are similar to the above and the result has been stated in Theorem (14.1).

□

15).Surfaces of type III_a and III_b.

Theorem (15.1)

Let X be a surface with $p_g=3$, $K_S^2=4$ and such that the canonical linear system contains a fixed (-2)-cycle. Then Lemma (9.1) shows that X is a surface of type III_a or III_b. In each case the presentation of the ring R(C, D) given in Theorem (9.4_a) is formative and informative in degree -1. In fact for X a surface of type III_a we have

$$R(X, K_X) \quad \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/\tilde{I}.$$

The degrees of the generators are,

$$\deg (x_i, y_j, z_k) = (1, 2, 3) \text{ respectively.}$$

$\tilde{I}$ is generated by $\tilde{r}_1,\ldots,\tilde{r}_9$ as follows:

Let $A = \begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 - \alpha_{11}x_0x_1 - \alpha_{13}x_0^2 & y_2 - \alpha_{23}x_0^2 & z_2 \end{bmatrix}$.

Then $\tilde{r}_1,\ldots,\tilde{r}_6$ are given by,

$$\text{rank}A \leq 1$$

$\tilde{r}_7$, $\tilde{r}_8$, $\tilde{r}_9$ are given by

$$\begin{bmatrix} \tilde{r}_7 & \tilde{r}_8 \\ \tilde{r}_8 & \tilde{r}_9 \end{bmatrix} = APA^t,$$

where P is a symmetric matrix, (cf. 14.1), and the P_{ij} are,

$$P_{11} = h + \delta_6 x_0 x_2^2(x_1 - \alpha_{11}x_0) + \delta_1 x_0 x_2^3 + \delta_{11} x_0 z_2 +$$

$$\delta_7' x_0^2 x_2(x_1 - \alpha_{11}x_0) + \delta_1' x_0^2 x_2^2 + \delta_4' x_0^2 y_1 + \delta_6'' x_0^3(x_1 - \alpha_{11}x_0)$$

$$+ \delta_1'' x_0^3 x_2 + \delta_1''' x_0^4,$$

$$P_{12} = \tfrac{1}{2}(\delta_5 x_0 x_2^2 + \delta_6' x_0^2 x_2 + \delta_5'' x_0^3),$$

$$P_{13} = \tfrac{1}{2}(\delta_4 x_0 x_2^2 - f x_0^2(x_1 - \alpha_{11}x_0) + \delta_5' x_0^2 x_2 + \delta_4'' x_0^3),$$

$$P_{14} = \tfrac{1}{2}(\delta_{16} x_0(x_1 - \alpha_{11}x_0) + \delta_9 x_0 x_2 + \delta_{10}' x_0^2),$$

$$P_{22} = f x_0^2,$$

$$P_{23} = \tfrac{1}{2}(\delta_9' x_0^2),$$

$$P_{24} = \tfrac{1}{2}(\delta_{15} x_0),$$

$$P_{33} = \lambda y_1 + \mu y_2 + \delta_8' x_0^2,$$

$$P_{34} = \tfrac{1}{2}(\delta_{14} x_0),$$

$$P_{44} = -1.$$

In the above, h, λ and μ are as in Theorem (9.4_a) and δ_i, δ_i',..., $f \in \mathbb{C}$.

The case where X is a surface of type III_b is analogous.

Remark (15.2)

We only present the calculation for surfaces of type III_a here as this calculation illustrates some points of interest. In the previous cases the presentations we have considered have been formative and in fact *all* extensions of R(C, D) in degree -1 have had a presentation with the same format. That is all the extensions $R(nC, D^{(n)})$ are given by an ideal $I^{(n)}$ which is generated by x_0^n and relations given by the same presentation as those in R(C, D). In Theorem (15.1) however, as in the example of Pinkham, this is not true. The ring $R(X, K_X)$ has the same presentation as R(C, D) but not all the intermediate rings $R(nC, D^{(n)})$ have. There is an obstruction at step 7 forcing exactly the conditions upon the ideal which are required to give $R(X, K_X)$ the formative presentation. The format given in Remark (9.5) does survive at every step however.

Proof of (15.1)

Let X be a surface of type III_a, E be the fixed (-2)-cycle and $K_X = \Gamma + E$. In Theorem (9.4_a) we calculated the ring R(C,D) for $D=K_{X|C}$ and found that

$$R(C,D) = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/I,$$

where degree $(x_i, y_j, z_k) = (1, 2, 3)$ and I is generated by

$$\text{rank } M = \text{rank}\begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1$$

giving

degree 3 $\quad r_1: x_1^3 - x_2y_2,$

$\quad r_2: x_1y_2 - x_2y_1,$

degree 4 $r_3: y_2^2 - x_1^2 y_1,$

$r_4: x_1 z_2 - x_2 z_1,$

degree 5 $r_5: y_2 z_2 - x_1^2 z_1,$

$r_6: y_1 z_2 - y_2 z_1,$

and

degree 6 $r_7: -z_1^2 + \lambda y_1^3 + \mu y_1^2 y_2 + x_1^2 h,$

$r_8: -z_1 z_2 + \lambda y_1^2 y_2 + \mu y_1 y_2^2 + x_1 x_2 h,$

$r_9: -z_2^2 + \lambda y_1 y_2^2 + \mu y_2^3 + x_2^2 h,$

where, $\lambda, \mu \in \mathbb{C}$

and

$$h = \lambda_1 x_1^4 + \lambda_2 x_1^3 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1 x_2^3 + \lambda_5 y_1^2 + \lambda_6 y_1 y_2$$

$$+ \lambda_7 y_2^2 + \lambda_8 x_1^2 y_2, \lambda_i \in \mathbb{C}.$$

These relations satisfy the syzygies $\Sigma_1,\ldots,\Sigma_{16}$ following

$\Sigma_1: x_1 r_3 - y_2 r_2 + y_1 r_1 \equiv 0,$

$\Sigma_2: x_2 r_3 - x_1^2 r_2 + y_2 r_1 \equiv 0,$

$\Sigma_3: x_1 r_6 - y_1 r_4 + z_1 r_2 \equiv 0,$

$\Sigma_4: x_1 r_5 - y_2 r_4 + z_1 r_1 \equiv 0,$

$\Sigma_5: x_2 r_5 - x_1^2 r_4 + z_2 r_1 \equiv 0,$

$\Sigma_6: x_2 r_6 - y_2 r_4 + z_2 r_2 \equiv 0,$

$\Sigma_7: y_2 r_6 - y_1 r_5 + z_1 r_3 \equiv 0,$

$\Sigma_8: x_1^2 r_6 - y_2 r_5 + z_2 r_3 \equiv 0,$

$$\Sigma_9\colon -x_2r_7 + x_1r_8 \equiv (\lambda y_1 + \mu y_2)y_1r_2 - z_1r_4,$$
$$\Sigma_{10}\colon -x_2r_8 + x_1r_9 \equiv (\lambda y_1 + \mu y_2)y_2r_2 - z_2r_4,$$

$$\Sigma_{11}\colon -x_1^2r_7 + y_2r_8 \equiv -x_1hr_1 + (\lambda y_1 + \mu y_2)y_1r_3 - z_1r_5,$$

$$\Sigma_{12}\colon -x_1^2r_8 + y_2r_9 \equiv -x_2hr_1 + (\lambda y_1 + \mu y_2)y_2r_3 - z_2r_5,$$

$$\Sigma_{13}\colon -y_2r_7 + y_1r_8 \equiv -x_1hr_2 - z_1r_6,$$
$$\Sigma_{14}\colon -y_2r_8 + y_1r_9 \equiv -x_2hr_2 - z_2r_6,$$
$$\Sigma_{15}\colon -z_2r_7 + z_1r_8 \equiv -x_1hr_4 - (\lambda y_1 + \mu y_2)y_1r_6,$$
$$\Sigma_{16}\colon -z_2r_8 + z_1r_9 \equiv -x_2hr_4 - (\lambda y_1 + \mu y_2)y_2r_6.$$

Step 0 : Simplifications

We begin our extension at this point after first making some simplifying basis changes. We wish to produce a ring

$$R(X,K_X) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/\tilde{I}$$

with $\tilde{I}$ generated by (with a few sign changes in comparison with the curve ring):

$$\tilde{r}_1\colon x_1^3 - x_2y_2 - x_0a_1\,,$$

$$\tilde{r}_2\colon x_1y_2 - x_2y_1 - x_0a_2\,,$$

$$\tilde{r}_3\colon x_1z_2 - x_2z_1 - x_0b_1\,,$$

$$\tilde{r}_4\colon x_1^2y_1 - y_2^2 - x_0b_2\,,$$

$$\tilde{r}_5\colon x_1^2z_1 - y_2z_2 - x_0c_1\,,$$

$$\tilde{r}_6\colon y_1z_2 - y_2z_1 - x_0c_2\,,$$

$$\tilde{r}_7\colon z_1^2 - \lambda y_1^3 - \mu y_1^2y_2 - x_1^2h - x_0d_1\,,$$

$$\tilde{r}_8: z_1z_2 - \lambda y_1^2y_2 - \mu y_1y_2^2 - x_1x_2h - x_0d_2 ,$$

$$\tilde{r}_9: z_2^2 - \lambda y_1y_2^2 - \mu y_2^3 - x_2^2h - x_0d_3 ,$$

where deg $(a_i, b_i, c_i, d_i) = (2, 3, 4, 5)$ such that the $\tilde{r}_i$ satisfy syzygies $\tilde{\Sigma}_j$ and $\tilde{\Sigma}_j$ reduces mod x_0 to Σ_j.

By changing coordinates we can assume that

$$a_1 = \alpha_{11}x_1^2 + \alpha_{12}y_1 + \alpha_{13}x_0x_1 + \alpha_{14}x_0^2,$$

$$a_2 = \alpha_{21}x_1^2 + \alpha_{22}y_1 + \alpha_{23}x_0x_1 + \alpha_{24}x_0^2,$$

$$b_2 = \beta_{21}z_1 + \beta_{22}z_2 + \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0^3.$$

The calculations are in practice somewhat more manageable if we begin by calculating the determinantal relations so that they lift the syzygies $\Sigma_1,\ldots,\Sigma_8$ to any order. In other words we begin by using the algorithm of Section 6 on the ring

$$\bar{R} = \mathbb{C}[x_1,x_2,y_1,y_2,z_1,z_2]/\bar{I},$$

where $\bar{I} = (r_1,\ldots,r_6)$. Having lifted $r_1,\ldots,r_6$ in this ring the calculation to lift them from R is simplified.

Step 1 : $\bar{R} \to \bar{R}^{(2)}$

Taking Σ_1, we produce expressions for a_1, a_2 and b_1 which will lift this single syzygy.

$$\bar{\Sigma}_1 : x_1b_1 - y_2a_2 + y_1a_1 \in \bar{I}$$

that is mod x_0,

$$\beta_{11}x_1^3x_2 + \beta_{12}x_1^2x_2^2 + \beta_{13}x_1x_2^3 + \beta_{14}x_1^2y_1 + \beta_{15}x_1x_2y_1 +$$

$$\beta_{16}x_1x_2y_2 + \beta_{17}x_1z_1 + \beta_{18}x_1z_2 - \alpha_{21}x_1^2y_2 - \alpha_{22}y_1y_2 +$$

$$\alpha_{11}x_1^2y_1 + \alpha_{12}y_1^2$$

=

$$t_1y_2^2 - t_1x_1^2y_1 + t_2x_1z_2 - t_2x_2z_1 + t_3x_1^2y_2 - t_3x_1x_2y_1 +$$

$$t_4x_1x_2y_2 - t_4x_2^2y_1 + t_5x_1^4 - t_5x_1x_2y_2 + t_6x_1^3x_2 - t_6x_2^2y_2$$

where $t_i \in \mathbb{C}$. Equating coefficients we get:

$$a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1 + \alpha_{14}x_0^2,$$

$$a_2 = \alpha_{21}x_1^2 + \alpha_{23}x_0x_1 + \alpha_{24}x_0^2,$$

$$b_1 = -\alpha_{11}x_1y_1 + \alpha_{21}x_2y_1 + \beta_{11}'x_0x_1^2 + \beta_{12}'x_0x_1x_2 +$$

$$\beta_{13}'x_0x_2^2 + \beta_{14}'x_0y_1 + \beta_{15}'x_0y_2 + \beta_{11}''x_0^2x_1 +$$

$$\beta_{12}''x_0^2x_2 + \beta_1'''x_0^3.$$

Substituting this back into

$$\bar{\Sigma}_1 : x_1b_1 - y_2a_2 + y_1a_1 \in \bar{I}$$

and continuing the calculation eventually leaves

$$a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1,$$

$$a_2 = \alpha_{21}x_1^2 + \alpha_{23}x_0x_1,$$

$$b_1 = -\alpha_{11}x_1y_1 + \alpha_{21}x_2y_1 + \alpha_{21}^2x_0x_1^2 - \alpha_{13}x_0y_1 +$$

$$\alpha_{23}x_0y_2 + \alpha_{21}\alpha_{23}x_0^2x_1.$$

We must check that these formulas satisfy

$$\bar{\Sigma}_2 : x_2b_1 - x_1^2a_2 + y_2a_1 \in \bar{I}$$

It is easy to check that this is the case only if $\alpha_{21} = 0$.

Now using these formulas and the remaining syzygies $\bar{\Sigma}_3,\ldots,\bar{\Sigma}_8$ we calculate b_2, c_1 and c_2.

$$\bar{\Sigma}_3 : x_1c_2 - y_1b_2 + z_1a_2 \in \bar{I}$$

gives

$$\gamma_{2,1}x_1^3x_2^2 + \gamma_{2,2}x_1^2x_2^3 + \gamma_{2,3}x_1x_2^4 + \gamma_{2,4}x_1^3y_1 + \gamma_{2,5}x_1^2x_2y_1 + \gamma_{2,6}x_1x_2^2y_1 +$$

$$\gamma_{2,7}x_1x_2^2y_2 + \gamma_{2,8}x_1y_1^2 + \gamma_{2,9}x_1y_1y_2 + \gamma_{2,10}x_1^2z_1 + \gamma_{2,11}x_1x_2z_1 +$$

$$\gamma_{2,12}x_1x_2z_2 - \beta_{21}y_1z_1 - \beta_{22}y_1z_2 + \alpha_{11}x_1^2z_1 \in I.$$

Setting the above expression equal to a general degree 5 element of I and equating coefficients gives

$$c_2 = x_0c_2',$$

$$b_2 = \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0^3.$$

Continuing in this way with $\bar{\Sigma}_3$ and $\bar{\Sigma}_4$ gives us

$$a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1,$$

$$a_2 = \alpha_{23}x_0x_1,$$

$$b_1 = -\alpha_{11}x_1y_1 - \alpha_{13}x_0y_1 + \alpha_{23}x_0y_2,$$

$$b_2 = 0\,,$$

$$c_1 = -\alpha_{11}x_1z_1 - \alpha_{13}x_0z_1\,,$$

$$c_2 = -\alpha_{23}x_0z_1.$$

The relations corresponding to these formulas are given by the following matrix and rank condition which means we do not need to check the remaining determinantal syzygies as they follow automatically from this form of the equations.

$$\text{rank}\begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 - \alpha_{11}x_0x_1 - \alpha_{13}x_0^2 & y_2 - \alpha_{23}x_0^2 & z_2 \end{bmatrix} \leq 1,$$

gives the 6 relations we have thus far calculated though we do not of course know yet whether the full generality of these equations will lift the remaining (non-determinantal) syzygies for any values of d_1,d_2 and d_3.

In fact the remainder of this calculation goes through as in the previous cases except for the existence of an obstruction in the final stage. To make this clear we produce a ring in Proposition (15.2) which extends R modulo x_0^6 but cannot be extended further. This ring exhibits all the obstructions occuring in the calculation.

□

Proposition (15.2)

Let R = R(C, D) be as in Theorem (9.4_a). In the notation of the extension algorithm, we denote by $R^{(n)}$ any ring which is formed from R at step (n-1). The rings given below (dependent on α, β, γ, δ, ε) are extensions of R

$$R^{(6)} = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/(I^{(6)}, x_0^6)$$

where $I^{(6)}$ is generated by:

$r_1^{(6)},\ldots,r_6^{(6)}$ given by

$$\operatorname{rank}\begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 - \alpha x_0 x_1 - \beta x_0^2 & y_2 - \gamma x_0^2 & z_2 \end{bmatrix} \leq 1,$$

and $r_7^{(6)}$, $r_8^{(6)}$, $r_9^{(6)}$ given by

$$r_7^{(6)} : z_1^2 - (\lambda y_1^2 + \mu y_1 y_2)y_1 - x_1^2 h - \delta x_0^4 y_1 - \varepsilon x_0^4 y_2,$$

$$r_8^{(6)} : z_1 z_2 - (\lambda y_1^2 + \mu y_1 y_2)(y_2 - \gamma x_0^2) - x_1 x_2 h - \delta x_0^4 y_2 - \varepsilon x_0^4 (x_1^2 - \alpha x_0 x_1),$$

$$r_9^{(6)} : z_2^2 - (\lambda y_1 + \mu y_2)(y_2 - \gamma x_0^2)^2 - x_2^2 h - \delta x_0^4 (x_1^2 - \alpha x_0 x_1) - \varepsilon x_0^4 (x_1 - \alpha x_0) x_2.$$

The above rings $R^{(6)}$ will extend to some $R^{(7)}$ if and only if

$$\delta\gamma = -\varepsilon\beta.$$

Proof

$R^{(6)}$ extends R because it is given in the formative presentation of Remark (9.5).

To extend $R^{(6)}$ to a ring $R^{(7)}$ the following syzygies must lift to syzygies in $R^{(7)}$:

$$\Sigma_9 : -x_2 r_7^{(6)} + x_1 r_8^{(6)} - z_1 r_4^{(6)} + (\lambda y_1^2 + \mu y_1 y_2) r_2^{(6)} + \delta x_0^4 r_2^{(6)} + \varepsilon x_0^4 r_1^{(6)},$$

$$\Sigma_{10} : -x_2 r_8^{(6)} + x_1 r_9^{(6)} - z_2 r_4^{(6)} + (\lambda y_1 + \mu y_2)(y_2 - \gamma x_0^2) r_2^{(6)} + \delta x_0^4 r_1^{(6)}.$$

Let $$r_7' = r_7^{(6)} + x_0^6 d_1,$$

$$r_8' = r_8^{(6)} + x_0^6 d_2,$$

$$r_9' = r_9^{(6)} + x_0^6 d_3.$$

It is easy to see that to lift Σ_9 we must have

$$\Sigma_9 : -x_0^6 x_2 d_1 + x_0^6 x_1 d_2 - \delta\gamma x_0^6 x_1 - \varepsilon\beta x_0^6 x_1 \in x_0^6 I.$$

This forces

$$d_1 = 0$$

and

$$d_2 = \delta\gamma + \varepsilon\beta.$$

To lift Σ_{10} we need

$$\Sigma_{10} : -x_0^6 x_2 d_2 + x_0^6 x_1 d_3 - \delta\beta x_0^6 x_1 \in x_0^6 I.$$

In turn this forces

$$d_2 = 0$$

and

$$d_3 = \delta\beta.$$

These conditions imply that the ring $R^{(6)}$ can only be extended to some $R^{(7)}$ if

$$\delta\gamma = -\varepsilon\beta.$$

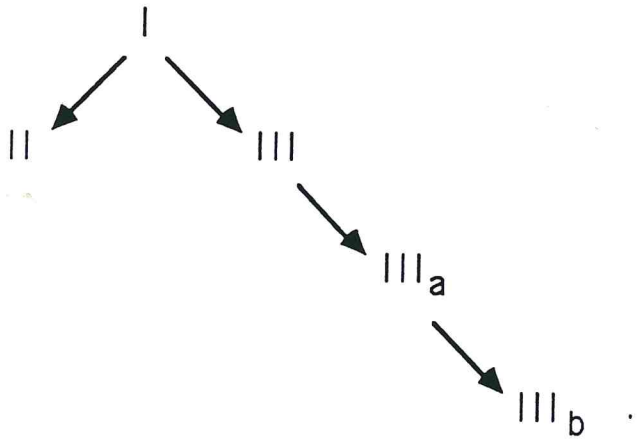
□

CHAPTER 5:

Geometrical Consequences of Calculations.

16. Deformations of surfaces and their moduli space.

In this section we draw some geometrical conclusions about surfaces with $p_g = 3$, $K^2 = 4$ and their moduli space. We determine in some cases whether surfaces of one type deform to surfaces of another. We can produce explicit families of surfaces $\{X_t\}_{t\in\mathbb{C}}$ parametrised by $t \in \mathbb{C}$ such that $\forall t\neq 0$, X_t is of type A but for $t=0$, X_t is of type B whenever A and B are connected by an arrow in the following diagram:



These are produced using the explicit form of the canonical rings given in Chapter 4.

Theorem (16.1)

The above hierarchy holds between surfaces with $p_g = 3$, $K^2 = 4$.

Proof.

I $\longrightarrow$ II.

We exhibit an explicit family of surfaces $\{X_t\}_{t\in T}$, with X_t of type I when $t\neq 0$, and of type II when $t=0$.

Let

$$X_t = \text{Proj } \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z]/I_t$$

where I_t is generated by $r_1^t,\ldots,r_5^t$ as follows,

r_1^t: $x_1y_2 - x_2y_1 + t(z - x_0f_1)$,

r_2^t: $x_1z - y_1(y_1 + \lambda_5x_0^2) + t(A - x_0f_2)$,

r_3^t: $x_2z - y_2(y_1 + \lambda_5x_0^2) - t(B + x_0f_3)$,

r_4^t: $(y_1 + \lambda_5x_0^2)(z - x_0f_1) - x_2(A - x_0f_2) - x_1(B + x_0f_3)$,

r_5^t : $(z - x_0f_1)z - y_2(A - x_0f_2) - y_1(B + x_0f_3)$.

Here A, B, f_1, f_2, f_3, λ_5 are as in Theorem (13.1). This is in fact the the ideal given by the 4×4 Pfaffians of the matrix (see section 4):

$$M_t = \begin{bmatrix} 0 & t & x_1 & x_2 & y_1 + \lambda_5x_0^2 \\ -t & 0 & y_1 & y_2 & z \\ -x_1 & -y_1 & 0 & -z + x_0f_1 & -A + x_0f_2 \\ -x_2 & -y_2 & z - x_0f_1 & 0 & B + x_0f_3 \\ -y_1-\lambda_5x_0^2 & -z & A - x_0f_2 & -B - x_0f_3 & 0 \end{bmatrix}.$$

When $t\neq 0$, we can eliminate z from these relations using r_1^t, this gives

$$z = x_0f_1 - (x_1y_2 - x_2y_1)/t.$$

Substituting into all the other relations we get that

$$r_4^t = (x_1 r_3^t - x_2 r_2^t)/t,$$

$$r_5^t = (y_1 r_3^t - y_2 r_2^t)/t$$

and r_2^t and r_3^t are general degree 4 relations in x_0,x_1,x_2,y_1,y_2. Thus for $t\neq 0$ we have a surface of type I but for $t=0$ we have the standard ring for a surface of type II.

In this case the Pfaffian form of the relations for a surface of type II lends itself to a quick and easy way of writing down a deformation, however for surfaces of type III writing down an explicit flat deformation is rather a large calculation.

I $\longrightarrow$ III

In Proposition (10.2) we have exhibited a structured deformation for the canonical curves in this case. However for the surfaces such a deformation has not yet been found. We therefore refer to [Ho1] Theorem (2.3) to show that a deformation does exist. We hope to find an explicit example at a later date.

III $\longrightarrow$ III$_a$.

Let

$$X_t = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/I_t,$$

where I_t is generated by $r_1^t,\ldots,r_9^t$ as follows:

$r_1^t,\ldots,r_6^t$ are given by

$$\text{rank } M_t = \text{rank} \begin{bmatrix} x_1 & y_1 & y_2 & z_1 \\ x_2 & g_1 & y_1 + \alpha x_0^2 + t y_2 & z_2 \end{bmatrix} \leq 1.$$

Here $g_1 = x_1^2 + \alpha_{11} x_0 x_1 + \alpha_{13} x_0^2$, $\alpha_{11}, \alpha_{13} \in \mathbb{C}$.

r_7^t, r_8^t, r_9^t are given by

$$M_t P (M_t)^T = \begin{bmatrix} r_7^t & r_8^t \\ r_8^t & r_9^t \end{bmatrix},$$

for some symmetric matrix P.

Notice that X_0 is precisely a surface of type III_a. For $t \neq 0$ it is easy to see that a series of row and column operations with changes of basis will put the matrix M_t into the form which gives a surface of type III (see (10.1)).

$III_a \rightarrow III_b$

A deformation can be found analogous to the one for curves in the canonical linear system (see (10.1)). That is, considering the form of the rings given in Theorem (15.1), the deformation is given by the parameter λ.

□

Number of moduli.

There are several possible ways of calculating the dimension of the moduli spaces. In [Ho1] Horikawa computed the dimension of $H^1(X,\Theta_X)$. It is also possible to count the number of parameters needed for the construction of the curves in $\mathbb{P}^2$ forming the branch locus of the double cover in cases III, III_a and III_b(see section 17). Since we have calculated the

form of the canonical ring for all these surfaces, we can count the number of parameters on which this construction depends. We aim to show the table below. In general the calculations do not give a rigorous proof of the dimensions of the moduli spaces, but a lower bound. [Ho1] gives 32 for the dimension of the moduli space for surfaces of type I which then serves as an upper bound for the other types.

Type of surface	Number of moduli
I	32
II	31
III	31
III_a	30
III_b	29

Type I

$$X = \text{Proj } \mathbb{C}[x_0,x_1,x_2,y_1,y_2]/<Q_1,Q_2>$$

where Q_1 and Q_2 are generic degree 4 polynomials. We must count the number of parameters on which Q_1 and Q_2 depend and take into account that we are only interested in the subvariety of the weighted projective space $\mathbb{P}(1^3, 2^2)$ which they define. We do not wish to count surfaces which only differ from each other by an automorphism of $\mathbb{P}(1^3,2^2)$.

The total number of degree 4 monomials in $x_0, x_1, x_2, y_1, y_2 = 30$

Number of parameters for $Q_1 = 30$

We can divide Q_1 by any nonzero coefficient and reduce it modulo Q_2, therefore we need only count 28 parameters. In the same way we count 28 for Q_2.

Let A = group of automorphisms of $\mathbb{P}(1^3,2^2)$. Then we can check that A acts faithfully on $\{(Q_1, Q_2)\}$ and rank A = 24.

Therefore the number of moduli = 28 + 28 - 24 = 32.

Type II

X = Proj $\mathbb{C}[x_0,x_1,x_2,y_1,y_2,z]/I$, where I is generated by the 4×4 Pfaffians of the following matrix:

$$\begin{bmatrix} 0 & 0 & l_1 & l_2 & q_1 \\ 0 & 0 & q_2 & q_3 & c_1 \\ -l_1 & -q_2 & 0 & c_2 & f_1 \\ -l_2 & -q_3 & -c_2 & 0 & f_2 \\ -q_1 & -c_1 & -f_1 & -f_2 & 0 \end{bmatrix}$$

where the l_i are generic degree 1, q_i are generic degree 2, c_i are generic degree 3 and the f_i are generic degree 4. By exactly analogy to the previous calculation we compute the number of variables on which the matrix depends and take into account that we are only interested in the subvariety of $\mathbb{P}(1^3, 2^2, 3)$ which they define. We then consider the number of automorphisms of the ambient weighted projective space and whether it acts faithfully on the set of ideals.

Number of parameters which entries in matrix depend upon = 130

Taking each relation in turn we are allowed to divide through by a nonzero coefficient and reduce modulo the linearly independent relations of the same degree. A short calculation shows that this allows us to take off 58.

Rank of group of automorphisms of $\mathbb{P}[1^3, 2^2, 3] = 41$

Assuming that the group acts faithfully on our set of ideals, the number of moduli = 130 - 58 - 41 = 31.

Type III

By Remark (9.5) we have that

$$X = \text{Proj}\mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/I,$$

where I is generated by

$$\text{rank}\begin{bmatrix} l_1 & q_1 & q_2 & c_1 \\ l_2 & q_3 & q_4 & c_2 \end{bmatrix} \leq 1,$$

and

$$r_7: l_1A_1 + q_1A_2 + q_2A_3 + c_1A_4$$

$$r_8: l_2A_1 + q_3A_2 + q_4A_3 + c_2A_4$$

which is **identically** equal (in $\mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]$) to

$$r_8': l_1B_1 + q_1B_2 + q_2B_3 + c_1B_4$$

$$r_9: l_2B_1 + q_3B_2 + q_4B_3 + c_2B_4$$

where degree l_i = 1, degree q_i = 2, degree c_i, A_4, B_4 = 3, degree A_2,B_2,A_3,B_3 = 4 and degree A_1,B_1 = 5. All the polynomials can be general under the restriction imposed by $r_8 \equiv r_8'$.

Now we count the number of parameters required to construct the matrix and $A_1,...,A_4$ and then the freedom of choice in forming $B_1,...,B_4$.

Number of parameters required for l_1, q_1, q_2, c_1 = 37,

Number of parameters required for l_2, q_3, q_4, c_2 = 37,

Number of parameters required for $A_1,...,A_4$ = 156,

The expression $l_1B_1 + q_1B_2 + q_2B_3 + c_1B_4$ has 193 separate monomials. Thus the freedom of choice for $B_1,...,B_4$ = 193 - number of degree 6 monomials

= 193 - 115

$= 78.$

Therefore we have, so far, $156 + 78 + 37 + 37 = 308$ parameters.

As before, taking each relation in turn we can divide through by a nonzero coefficient and then reduce modulo the linearly independent relations of the same degree. This means that we can take off

$$2 + 2 + 8 + 8 + 22 + 22 + 51 + 51 + 51 = 217.$$

Rank of group of automorphisms of $\mathbb{P}(1^3,2^2,3^2) = 60$.

Assuming that the group acts faithfully, the total number of parameters on which our construction depends is $308 - 217 - 60 = 31$ as required.

Types III_a and III_b

These follow in the same way as case III.

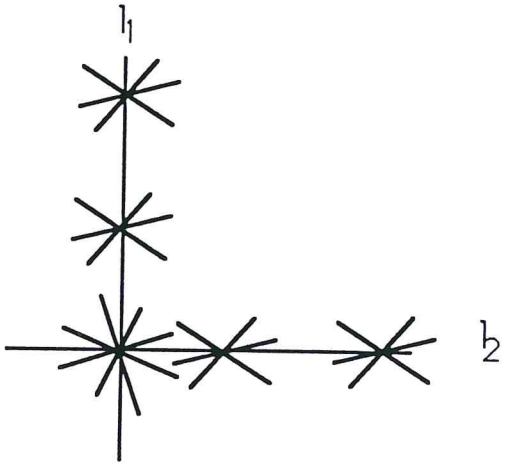
17. The 1-canonical map.

In [Ho1] Horikawa shows that surfaces of type III and III_a are birationally equivalent to double covers of $\mathbb{P}^2$, and he describes the branch locus in each case. We show the same result by an explicit calculation of the branch locus and do the case III_b in addition (see Introduction).

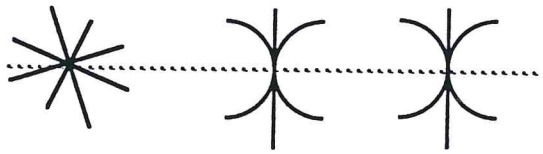
Theorem (17.1).

Let X be a surface of type III, III_a or III_b. Then X is birationally equivalent to a double cover of $\mathbb{P}^2$ with branch curve as follows:

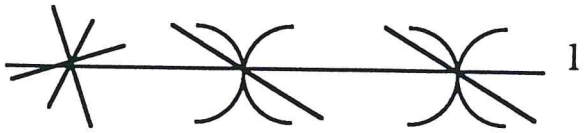
Type III. A degree 12 curve, made up of a degree 10 curve f_{10} and two lines $l_1 : (x_1 = 0)$, $l_2 : (x_2 = 0)$. f_{10} has two triple points (possibly infinitely near) on each l_i and a quadruple point at $l_1 \cap l_2$.



Type III_a. A degree 10 curve, having two triple tacnodes (possibly infinitely near) and a quadruple point all lying on a line.



Type III_b. A degree 10 curve, made up of a degree 9 curve f_9 and a line $l : (x_2 = 0)$. f_9 has 3 triple points lying on l as in the diagram below.



Proof

In each case the 1-canonical map gives a double cover of $\mathbb{P}^2$ and we can recover the branch locus from the canonical ring by completing the square. We rewrite the relation $\tilde{r}_7$ as

$$\tilde{r}_7 : z'^2 - h_6$$

and eliminate the variables y_1, y_2, z_2. By making these eliminations we find the generators and relations of the integral closure of $\mathbb{C}[x_0,x_1,x_2]$ in $R(X, K_X)$.

We consider each case in turn.

Type III.

Refering to Theorem (14.1) we see that we can write

$$\tilde{r}_7 : (z_1 - \tfrac{1}{2}\delta_{12}x_0x_1y_1 - \tfrac{1}{2}\delta_9x_0g_2 - \tfrac{1}{2}\lambda_{15}x_1^3 - \tfrac{1}{2}\lambda_{16}x_1^2x_2 - \tfrac{1}{2}\lambda_{18}x_1x_2^2 -$$

$$\tfrac{1}{2}\delta_{13}x_0x_1^2 + \tfrac{1}{2}\delta_{13}\alpha_{11}x_0^2x_1 - \tfrac{1}{2}\delta_{23}x_0^2x_1)^2$$

$$= \tfrac{1}{4}\delta_{12}^2x_0^2y_1^2 + \tfrac{1}{4}\delta_9^2x_0^2g_2^2 + \tfrac{1}{4}\lambda_{15}^2x_1^6 + \tfrac{1}{4}\lambda_{16}^2x_1^4x_2^2 + \tfrac{1}{4}\lambda_{18}^2x_1^2x_2^4 + \tfrac{1}{4}\delta_{13}^2x_0^2x_1^4 +$$

$$\tfrac{1}{4}\delta_{13}^2\alpha_{11}^2x_0^4x_1^2 + \tfrac{1}{4}\delta_{23}^2x_0^4x_1^2 + x_1A_1' + y_1A_2' + g_2A_3',$$

where

$$g_1 = x_1^2 - \alpha_{11}x_0x_1 - \alpha_{13}x_0^2,$$

$$g_2 = x_2^2 - \alpha_{21}x_0x_2 - \alpha_{23}x_0^2$$

and

$$A_1' = x_1(x_1P_{11} + y_1P_{12} + g_2P_{13} + z_1P_{14}),$$
$$A_2' = y_1(x_1P_{12} + y_1P_{22} + g_2P_{23} + z_1P_{24}),$$
$$A_3' = g_2(x_1P_{13} + y_1P_{23} + g_2P_{33} + z_1P_{34})$$

after the removal of all monomials involving z_1 and z_2. Now we use r_1 and r_2 to write $y_1 = x_1g_1/x_2$ and $y_2 = x_2g_2/x_1$ respectively. Substituting this into the above and clearing the denominator gives

$$r_7' : x_1x_2(x_2z_1 - g)^2 = f_{10},$$

for some degree 10 polynomial $f_{10}(x_0,x_1,x_2)$ and some g derived from the expression for $\tilde{r}_7$ above.

By multiplying both sides by x_1x_2 we produce the required equation,

$$r_7'' : (z')^2 = f_{12}$$

where $f_{12} = x_1x_2f_{10}$.

Consider the intersection of the curve f_{10} with the two lines l_1: ($x_1 = 0$) and l_2: ($x_2 = 0$). We find that

$$l_1 \cap f_{10} = \{ [x_0,x_1,x_2] \in \mathbb{P}^2 \mid x_1 = 0 , -\mu x_2^4 g_2^3 = 0 \},$$

and

$$l_2 \cap f_{10} = \{ [x_0,x_1,x_2] \in \mathbb{P}^2 \mid x_2 = 0 , -\lambda x_1^4 g_1^3 = 0 \}.$$

Thus f_{10} intersects l_1 and l_2 as stated in the theorem.

Type III_a.

Working as in the previous case we find that we can write

$$r_9' : x_2^4(z_2 - g)^2 = f_{10},$$

which we can write down as

$$r_9'' : (z')^2 = f_{10}$$

where

$$f_{10} = (x_1g_1 - \alpha_{23}x_0^2x_2)^2(\lambda x_1^2g_1 - \lambda\alpha_{23}x_0^2x_1x_2 + \mu x_1g_1x_2 + \delta_8'x_0^2x_2^2) +$$

$$x_2(\text{other terms})$$

Considering the line $l : (x_2 = 0)$ we find that it meets $f_{10} = 0$ where $-\lambda x_1^4g_1^3 = 0$. This shows that there is a quadruple point at [1:0:0] and triple points at [1:P:0] and [1:Q:0] where P and Q are the roots of $x_1^2 - \alpha_{11}x_1 - \alpha_{13} = 0$. The tangent lines at [1:P:0] are given by

$$P\lambda(x_1(P - Q)P - \alpha_{23}x_2)^3 = 0.$$

Similarly at [1:Q:0] the tangent lines are given by

$$Q\lambda(x_1(Q - P)Q - \alpha_{23}x_2)^3 = 0.$$

This proves the theorem in this case.

Type III_b

Working as in case III_a we now have $\lambda = 0$. Thus r_9 becomes

$$r_9'' : (z')^2 = x_2 f_9$$

where

$$f_9 = (x_1 g_1 - \alpha_{23} x_0^2 x_2)^2(\mu x_1 g_1 + \delta_8' x_0^2 x_2) + x_2(\text{other terms}).$$

Considering the line $(x_2 = 0)$ we find that it meets $f_9 = 0$ where $\mu x_1^3 g_1^3 = 0$. This shows that there are triple points at [1:0:0], [1:P:0] and [1:Q:0] where P and Q are the roots of $x_1^2 - \alpha_{11}x_1 - \alpha_{13} = 0$. The tangent lines at [1:P:0] are given by

$$(Px_1(P - Q) - \alpha_{23}x_2)^2(\mu P x_1(P - Q) + \delta_8' x_2) = 0,$$

and at [1:Q:0] by

$$(Qx_1(Q - P) - \alpha_{23}x_2)^2(\mu Q x_1(Q - P) + \delta_8' x_2) = 0.$$

This proves the theorem in this case.

□

Another consequence of our calculations is that it is possible to find specific curves on the surfaces. Surfaces of type II, for example, can be shown to contain an elliptic curve which can be contracted to an elliptic Gorenstein singularity of type k=1, [R1]. We show this in the next theorem.

Theorem (17.2)

Let X be a surface of type II. Then there is an elliptic curve $E \subset X$ and a numerical quintic $\bar{X}$ with a single elliptic Gorenstein singularity P of type $k = 1$ and a contraction map $\pi : X \dashrightarrow \bar{X}$ such that $\pi(E) = P$.

Proof

Refer to Theorem (13.1) for our description of the canonical ring of X. Consider the subvariety of X given by

$$x_1 = x_2 = y_1 + \lambda_5 x_0^2 = 0.$$

Then $r_1,\ldots,r_4$ are all trivially satisfied by these conditions and r_5 gives the following equation,

$$r_5 : \lambda_5 x_0^2(B + v x_0^4 + q_2 x_0^2 y_2 - \lambda_5 q_4 x_0^4) +$$

$$y_2(-A - t_{10} x_0 z - v_1 x_0^4 - q_{10} x_0^2 y_2) - z(-z + u_2 x_0^3).$$

This is the equation of a degree 6 curve, E in $\mathbb{P}(1,2,3)$. It is easy to check that it is nonsingular and hence elliptic. The base point of the canonical system is given by the equation $z^2 = \lambda y_2^3$. Of course, this point lies on E and $|K_X|$ goes through E only at this single point and is trivial off it. This is enough to prove the theorem (see [R1]).

□

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