A blow-up problem of a class of axisymmetric Navier-Stokes flows and Fujita equation

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Introduction

Axisymmetric NS eqs. (with infinite total energy)

vortex streched by a mean flow

Burgers vortex (1-celled, linear)

Sullivan vortex (2-celled, nonlinear)

Kida-Yanase (1999)

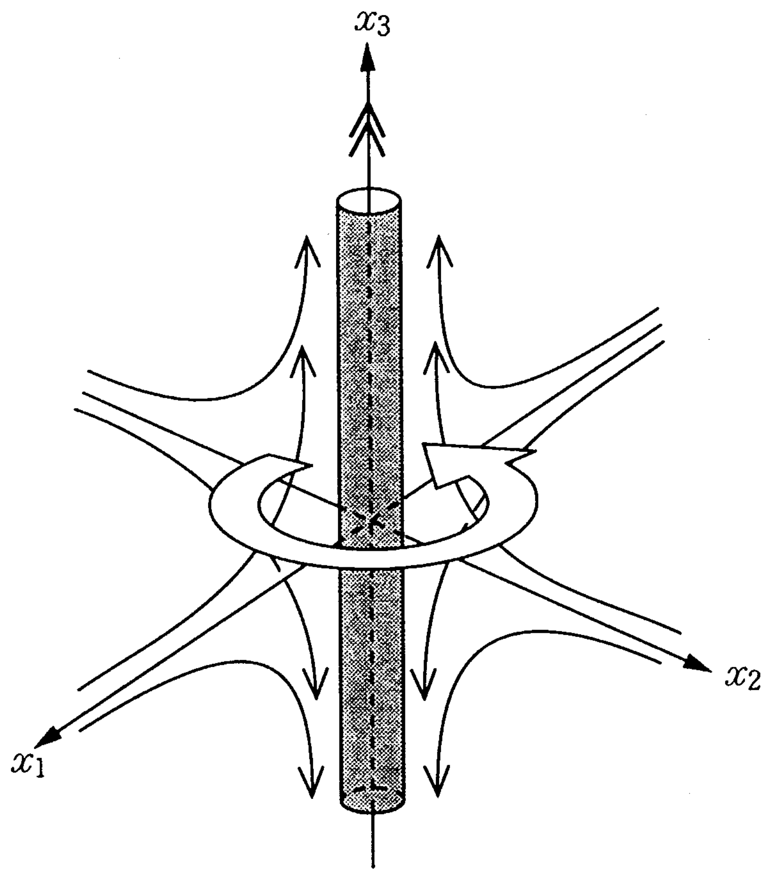

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図 9.3　バーガース渦管

円筒状部分は渦管の高渦度領域，矢印付き曲線は軸対称淀み点流の流線，2 重矢印は渦管の渦線，白抜き矢印は渦管の誘導する流れの向きを表す．

Donaldson and Sullivan (1960)

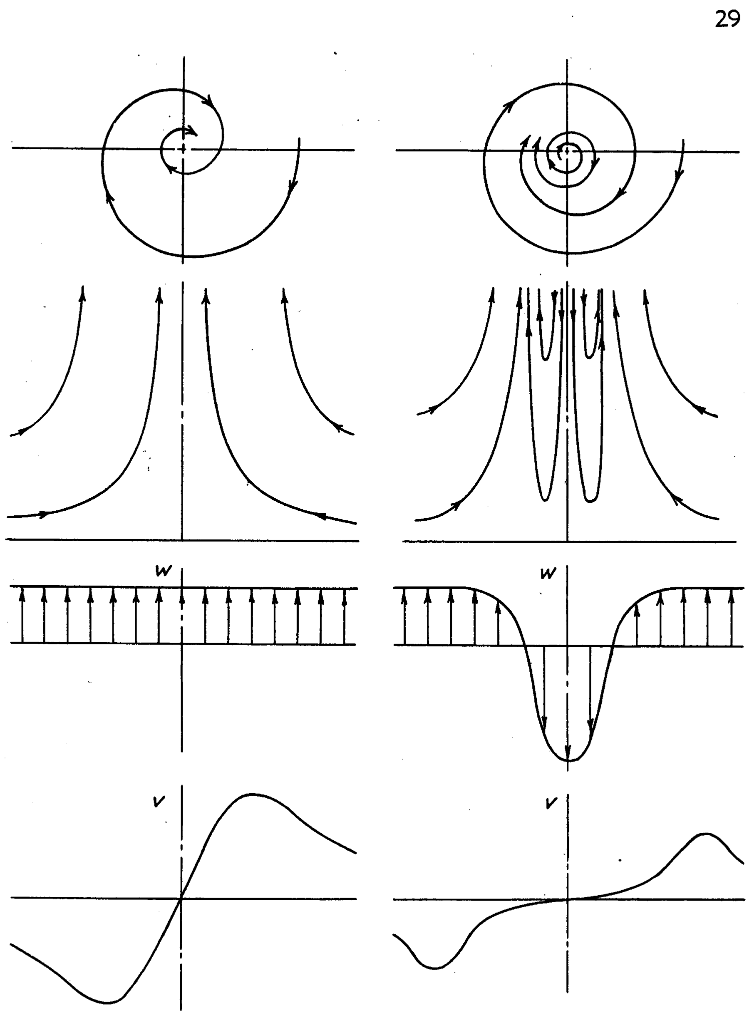

Fig. 4. Comparison of Burgers' and Sullivan's solutions.

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0. Basic equations

1. Survey on stationary solutions

2. Nonstationary blow-up solutions
not the \$ 10^6 prize problem

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AFOSR TN 60-1227 (1960)
Final report on AFOSR Contract No. AF49(638)-255

W.S. Lewellen, JFM 14(1962)420

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1961)

An approximate treatment of vortices of this type was originated by Einstein & Li (1951) and somewhat generalized by Deissler & Perlmutter (1958). In these analyses, the axial velocity is arbitrarily taken as a discontinuous function of the radius, and has a jump at the radius of the exhaust. Continuity is then used to determine a radial velocity which is independent of the axial coordinate. The tangential velocity is assumed to be a function of the radius only and can then be determined directly from the tangential momentum equation by simple quadrature.

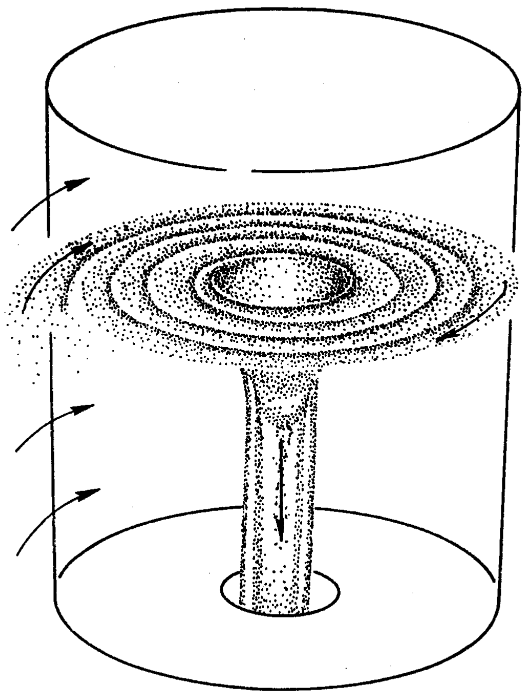


FIGURE 1. Sketch of vortex flow in which the fluid enters tangentially with high velocity, spirals radially inward and exits axially at some smaller radius.

Ranque-Hilsch tube

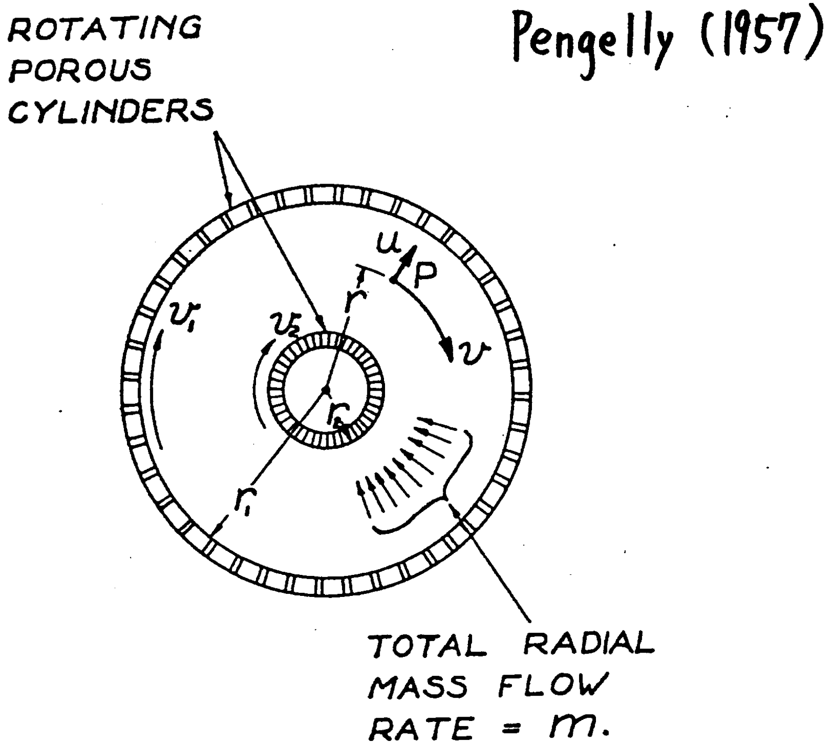


FIG. 1. Section of a two-dimensional vortex. Vortical flow exists in annulus between r_1 and r_2. Fluid enters through porous surface of outer cylinder and is absorbed at inner cylinder. Boundary conditions are established by angular velocities of the two cylinders.

0. Basic equations

Ansatz

axisymmetric Navier-Stokes eq. in cylindrical coordinates(r,ϕ,z)

$$u = (u_r, u_\phi, u_z) = (U(r,t), V(r,t), zW(r,t))$$

vorticity equation

$$\frac{\partial \omega_3}{\partial t} + U(r,t)\frac{\partial \omega_3}{\partial r} = W(r,t)\omega_3 + \nu \triangle \omega_3$$

Burgers vortex

$$(u_r, u_\phi, u_z) = \left(-\alpha r, \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{\alpha r^2}{2\nu}}\right), 2\alpha z\right), \text{ or } \omega_3 = \frac{\alpha\Gamma}{2\pi\nu}e^{-\frac{\alpha r^2}{2\nu}}$$

$$U(r,t) = -\alpha r,\ W(r,t) = 2\alpha$$

Sullivan vortex

$$(u_r, u_\phi, u_z) = \left(-\alpha r + \frac{6\nu}{r}\left(1 - e^{-\frac{\alpha r^2}{2\nu}}\right), \frac{\Gamma}{2\pi r}\frac{H(\frac{\alpha r^2}{2\nu})}{H(\infty)}, 2\alpha z\left(1 - 3e^{-\frac{\alpha r^2}{2\nu}}\right) \right)$$

$$H(r) \equiv \int_0^r \exp\left(-t + 3\int_0^t \frac{1-e^{-s}}{s} ds\right) dt$$

axisymmetric Navier-Stokes eqs.

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial r}-\frac{V^2}{r}=-\frac{\partial p}{\partial r}+\nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rU)\right)$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial r}+\frac{UV}{r}=\nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right)$$

$$z\left(\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial r}+W^2\right)=-\frac{\partial p}{\partial z}+\nu z\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)$$

continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + W = 0$$

$$\frac{\partial^2 p}{\partial r \partial z} = 0 \Rightarrow z\frac{\partial}{\partial r}\left[\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial r} + W^2 - \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right)\right] = 0$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial r} + W^2 - \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = -\bar{C}(z,t), \text{ in fact }, = -\bar{C}(t)$$

$$p(z,r,t) = \frac{\bar{C}(t)}{2}z^2 + \int^r \frac{V^2}{r}dr + \frac{1}{2}U^2$$

$$+\frac{\partial}{\partial t}\int^r U dr + \nu\frac{1}{r}\frac{\partial}{\partial r}(rU) + K$$

Relationship between U and W

$$W = -\frac{1}{r}\frac{\partial}{\partial r}(rU) \text{ or } U(r) = -\frac{1}{r}\int_0^r r'W(r')dr'$$

In $\mathbb{R}^2$, $\bar{C}(t) \equiv 0$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial r} + W^2 = \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right),$$

$$U(r) = -\frac{1}{r}\int_0^r r'W(r')dr'$$

Invisicd case $\nu = 0$, blow-up

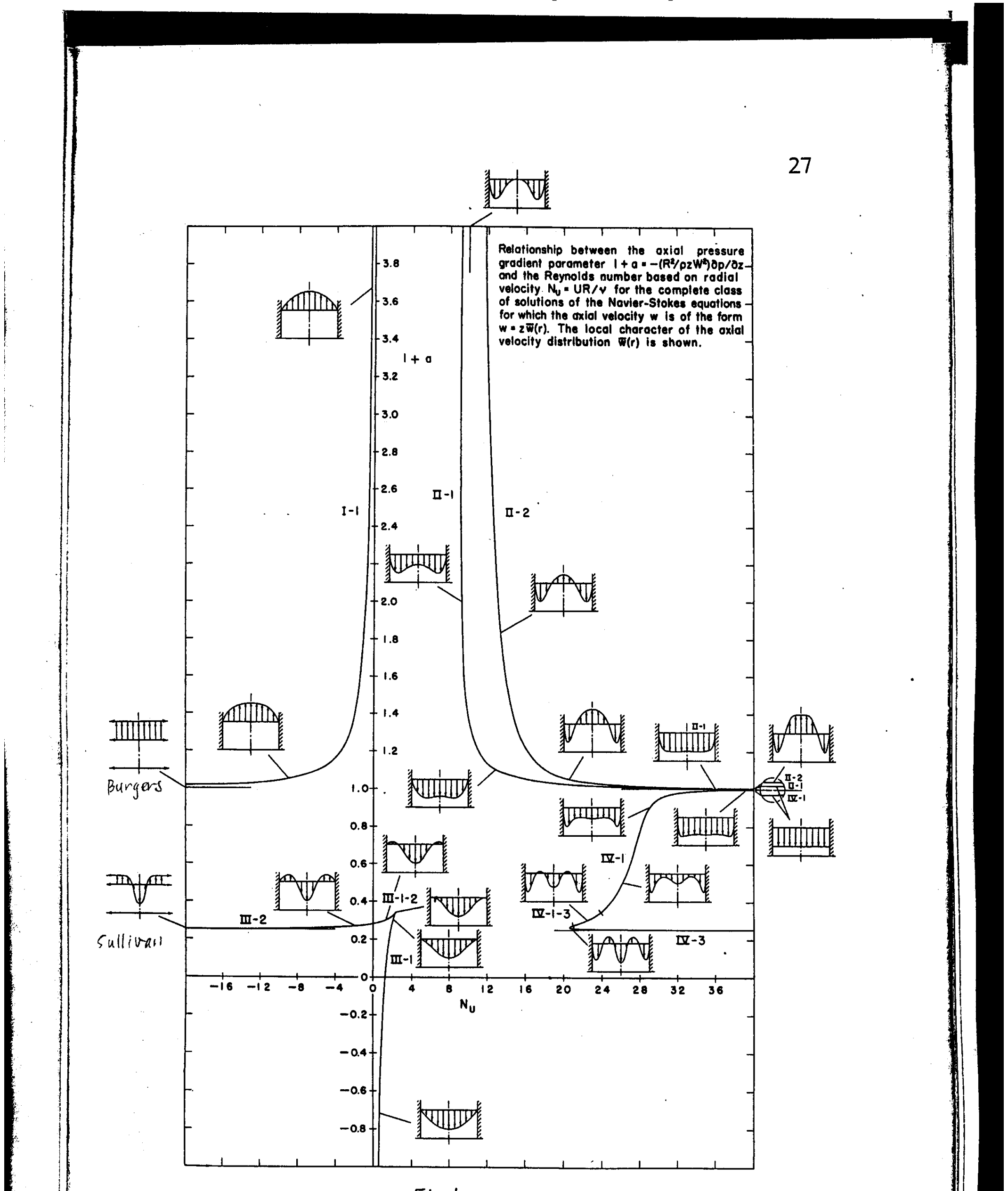
J.D. Gibbon, D.R. Moore, J.T. Stuart,

Nonlinearity 16 (2003), 1823.

GFD Physica D 132 (1999)497

OG, Phys. Fluids, 12(2000)3181

Donaldson and Sullivan (1960)



1. Stationary solutions (viscous)

Stationary Navier-Stokes eqs.

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + W = 0$$

$$\frac{U}{r}\frac{\partial}{\partial r}(rV) = \nu\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV)\right)$$

$$U\frac{\partial W}{\partial r} + W^2 - \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) = -\bar{C}$$

non-dimensionalization

$$x \equiv r^2/R^2, \; f(x) \equiv -2rU(r)/RS$$

$$g(x) \equiv RW(r)/S, \; h(x) \equiv rV(r)/\Gamma$$

$$f' = g$$

$$\frac{4\nu}{SR} x h'' + f h' = 0$$

$$f f'' - (f')^2 + \frac{4\nu}{SR}(x f'')' = \frac{R^2}{S^2}\bar{C}(t)$$

$(f(0) = h(0) = 0, \; f'(1) = 0, h(1) = 0)$

Transformation

$$X = cx, \;\; c^2 \equiv -\frac{R^4}{16\nu^2}\bar{C}(t)$$

$$F(X) = \frac{SR}{4\nu} f(x)$$

or

$$F(X) = -\frac{r}{2\nu}U(r), X = \sqrt{-\bar{C}(t)}\frac{r^2}{4\nu}$$

$$(XF'')' + FF'' - (F')^2 + 1 = 0$$

$$Xh''(X) + F(X)h'(X) = 0$$

$(F(0) = h(0) = 0,\ F'(c) = 0, h(c) = 0)$

$$A = F'(0),\ \ F''(0) = A^2 - 1$$

$F(X) = \pm X$ Burgers

$F(X) = X - 3(1 - e^{-X})$ Sullivan

2. Nonstationary solutions (viscous)

$$\omega \equiv -W$$

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial r} = \omega^2 + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right),$$

$$U(r) = \frac{1}{r}\int_0^r r'\omega(r')dr'$$

Comparison to reaction-diffusion eqs.

$$\frac{\partial u}{\partial t} = u^p + \triangle u, \;\; u(x,t=0) = u_0(x)(\geq 0)$$

Fujita equations (1966) $\mathbb{R}^N$

- If $1 < p < 1+2/N$, no global solutions except $u \equiv 0$ (blow-up)

- If $p > 1+2/N$, some global solutions for 'small' initial data, some blow-up for 'large' initial data

- $p = p_c = 1+2/N$ blowup

Meaning of the theorem

- If p is large, diffusion dominates over nonlinearity for small initial data

- decay rate vs. blowup rate

$$(\text{time})^{-N/2} \approx (\text{time})^{-1/(p-1)}$$

Are there any self-similar blow-up solutions ?

$$\omega = \frac{1}{T-t} f\left(\frac{r}{\sqrt{\nu(T-t)}}\right)$$

$$U = \sqrt{\frac{\nu}{T-t}} g\left(\frac{r}{\sqrt{\nu(T-t)}}\right)$$

$$g = \frac{1}{\xi}\int_0^{\xi} \eta f(\eta) d\eta, \; \xi = \frac{r}{\sqrt{\nu(T-t)}}$$

ODE

$$f + \frac{\xi}{2} f' + \frac{f'}{\xi}\int_0^{\xi} \eta f(\eta) d\eta - f^2 = \frac{1}{\xi}(\xi f')'$$

$$\frac{1}{2}(\xi^2 f)' + \frac{f'}{\xi}\int_0^\xi \eta f(\eta)d\eta - f^2 = \frac{1}{\xi}(\xi f')'$$

BC $f'(0) = 0, f(\infty) = 0$

By $\int_0^\infty d\xi$

$$2\int_0^\infty \xi f(\xi)^2 d\xi = 0 \Rightarrow f(\xi) \equiv 0$$

Trivial ones only

cf. constant sols. $f = 0,\ 1$

Consider L^p-norm

$$\omega(r,t) \geq 0$$

$$\|\omega\|_p^p = \int_0^\infty \omega(r)^p 2\pi r dr = \langle \omega(r)^p \rangle$$

$$\frac{d\|\omega\|_p^p}{dt} = (p+1)\|\omega\|_{p+1}^{p+1} - \nu(p-1)\left\langle \left(\frac{\partial \omega}{\partial r}\right)^2 \omega^{p-2} \right\rangle$$

If $U \equiv 0$

$$\frac{d\|\omega\|_p^p}{dt} = p\|\omega\|_{p+1}^{p+1} - \nu(p-1)\left\langle \left(\frac{\partial \omega}{\partial r}\right)^2 \omega^{p-2} \right\rangle$$

Numerics

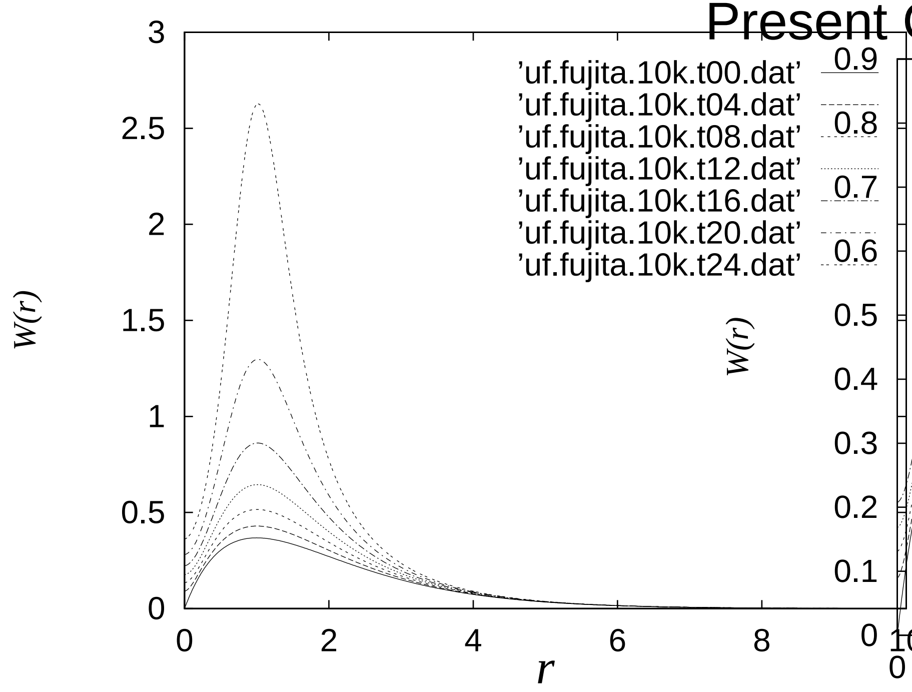
$$\nu = 1 \times 10^{-2}, \;\; dt = 2 \times 10^{-3}, dr = 1 \times 10^{-2}$$

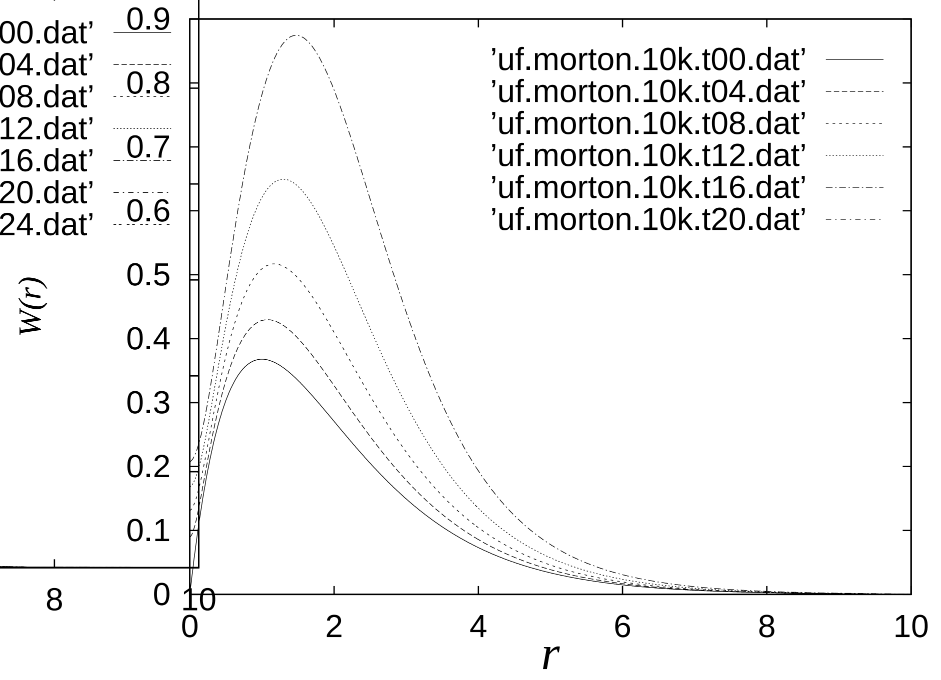
$$N = 10000, \; L = 100.$$

I.C.

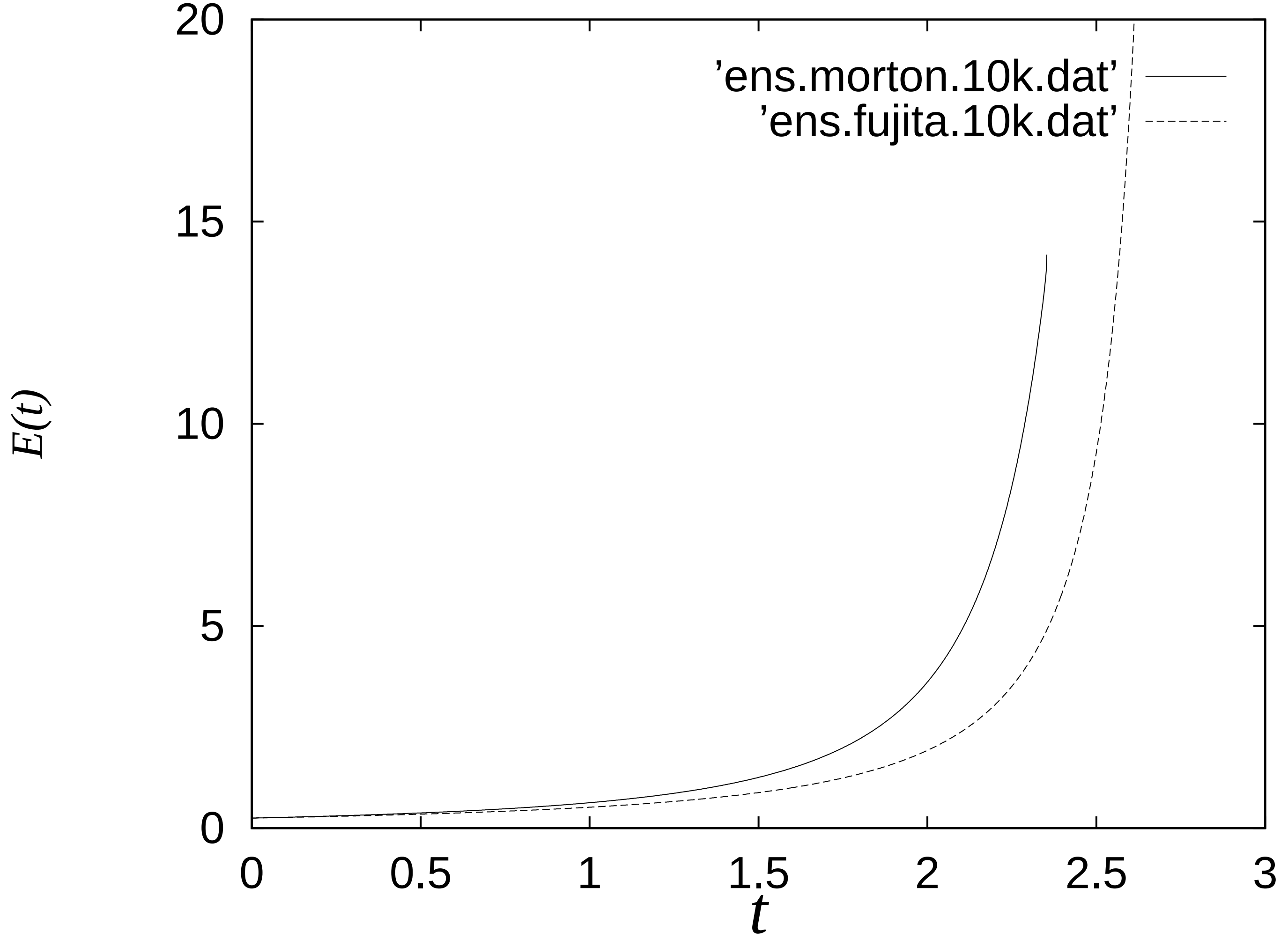
$$W(r) = re^{-r}$$

Present Class





Fujita eq.



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L^2-norm

Proof of blow-up(global nonexistence)

method of subsolution

Deng, Kwong and Levine(1992)

$$u_t = u_{xx} + \varepsilon u u_x + \frac{1}{2}(a\|u(\cdot,t)\|^{p-1} + b)u, \;\; a,\varepsilon \in \mathbb{R},\; b > 0, p > 1$$

in 2 steps

(i) comparison principle

$$\partial_t u = A(u) + F(u)$$

$$A(u) = \text{diffusion term}, F(u) = \text{nonlinear term}$$

$$\partial_t u - A(u) - F(u) \geq \partial_t v - A(v) - F(v) \Rightarrow u(\cdot,t) \geq v(\cdot,t),$$

(ii)construction of subsolution

$$\frac{\partial \underline{\omega}}{\partial t} \leq \underline{\omega}^2 - \underline{U}\frac{\partial \underline{\omega}}{\partial r} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\omega}}{\partial r}\right),$$

$$\frac{1}{\xi}(\xi \underline{f}')' - \underline{f} + \underline{f}^2 - \frac{\xi}{2}\underline{f}' - \frac{\underline{f}'}{\xi}\int_0^{\xi} \eta \underline{f}(\eta) d\eta \geq 0$$

$$\underline{f} = a\exp(-b\xi^2),\ a,b > 0$$

$$\left((a-(4b+1)) + b(a+4b)\xi^2\right)\underline{f} \geq 0$$

OK with $a > 4b+1$

$$\omega(r,t) \geq \underline{\omega}(r,t) = \frac{1}{T-t}\underline{f}(\xi)$$

$$
\begin{aligned}
\frac{\partial \underline{\omega}}{\partial t} &\leq 2\underline{\omega}^2 - \frac{1}{r}\frac{\partial}{\partial r}(r\underline{U\,\omega}) + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \underline{\omega}}{\partial r}\right), \\
\underline{U}(r) &= \frac{1}{r}\int_0^r r'\underline{\omega}(r')dr'
\end{aligned}
$$

Performing

$$
\int_0^t dt \int_0^\infty drr\phi(r,t)
$$

we find

$$
\int_0^\infty dt'\underline{\omega}\phi(r)rdr - \int_0^\infty \underline{\omega}_0(r)\phi(r,0)rdr
$$

$$
\leq \int_0^t dt' \int_0^\infty \left[\underline{\omega}\frac{\partial \phi}{\partial t'} + 2\underline{\omega}^2\phi + \underline{U}\,\underline{\omega}\frac{\partial \phi}{\partial r} + \nu\underline{\omega}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)\right] rdr
$$

$$D = \mathbb{R}^2 \times (0,T)$$

$$\int_0^\infty (\underline{\omega} - \overline{\omega})\phi(r) r dr \leq \int_0^\infty (\underline{\omega}_0(r) - \overline{\omega}_0(r))\phi(r,0) r dr$$

$$+ \int_0^t dt' \int_0^\infty \left[(\underline{\omega} - \overline{\omega}) \frac{\partial \phi}{\partial t'} + 2(\underline{\omega}^2 - \overline{\omega}^2)\phi + (\underline{U}\,\underline{\omega} - \overline{U}\overline{\omega}) \frac{\partial \phi}{\partial r} + \nu(\underline{\omega} - \overline{\omega}) \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) \right] r dr$$

$$\underline{\omega} \int_0^r r \underline{\omega}(r) dr - \overline{\omega} \int_0^r r \overline{\omega}(r) dr = (\underline{\omega} - \overline{\omega}) \int_0^r r \underline{\omega}(r) dr + \overline{\omega} \int_0^r r (\underline{\omega}(r) - \overline{\omega}(r)) dr$$

Thus

$$\int_0^\infty (\underline{\omega} - \overline{\omega})\phi(r) r dr \leq \int_0^\infty (\underline{\omega}_0(r) - \overline{\omega}_0(r))\phi(r,0) r dr$$

$$+\int_0^t dt' \int_0^\infty (\underline{\omega} - \overline{\omega}) \underbrace{\left[\frac{\partial \phi}{\partial t'} + A(r,t')\phi + B(r,t')\frac{\partial \phi}{\partial r} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)\right]}_{=0,\ \text{for some } \phi} r dr$$

$$+\int_0^t dt \int_0^\infty \overline{\omega}(r) \int_0^r r'(\underline{\omega}(r') - \overline{\omega}(r')) dr' \frac{\partial \phi}{\partial r} dr$$

where

$$A(r,t) = 2(\underline{\omega} + \overline{\omega}), B(r,t) = \frac{1}{r}\int_0^r r' \underline{\omega}(r') dr'$$

$$A_n,\ B_n \in C^\infty(D_T),\ A_n \to A,\ B_n \to B \text{ in } D_T \text{ as } n \to \infty$$

backward problem

$$\frac{\partial \phi_n}{\partial t} + A_n(r,t)\phi_n + B_n(r,t)\frac{\partial \phi_n}{\partial r} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_n}{\partial r}\right) = 0$$

$$\phi_n(\infty, t') = 0,\ 0 < t' < t$$

$$\phi_n(r,t) = \chi(r),\ r \geq 0$$

$$\chi(r) \in C_0^\infty(0,\infty),\ 0 \leq \chi \leq 1$$

$\phi = \lim_{n\to\infty} \phi_n$ with A_n, B_n replaced by A, B, $\phi \geq 0$

$$\int_0^\infty (\underline{\omega} - \overline{\omega})\chi(r) r dr \le M_1 \int_0^\infty (\underline{\omega}_0(r) - \overline{\omega}_0(r))^+ r dr$$

$$+ \int_0^t dt \int_0^\infty \overline{\omega}(r) \int_0^r r'(\underline{\omega}(r') - \overline{\omega}(r')) dr' \frac{\partial \phi}{\partial r} dr,$$

$$\int_0^r r'(\underline{\omega}(r') - \overline{\omega}(r')) dr' \le \int_0^r r'(\underline{\omega}(r') - \overline{\omega}(r'))^+ dr' \le \int_0^\infty r'(\underline{\omega}(r') - \overline{\omega}(r'))^+ dr'$$

$$|\text{the final term}| \le \int_0^t dt' \int_0^\infty \overline{\omega}(r) \int_0^\infty r'(\underline{\omega}(r') - \overline{\omega}(r'))^+ dr' \left| \frac{\partial \phi}{\partial r} \right| dr$$

$$\le M_2 M_3 \int_0^t dt' \int_0^\infty r'(\underline{\omega}(r') - \overline{\omega}(r'))^+ dr'$$

$M_1 = \sup_{D_T} |\phi|$, $M_2 = \sup \left| \frac{\partial \phi}{\partial r} \right|$, $M_3 = \int_0^\infty \overline{\omega}(r) dr$

$$\chi(r) = \begin{cases} 1, & \text{if } \underline{\omega}(r) - \overline{\omega}(r) > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_0^\infty (\underline{\omega}(r) - \overline{\omega}(r))^+ r dr \leq M_1 \int_0^\infty (\underline{\omega}_0(r) - \overline{\omega}_0(r))^+ r dr$$

$$+ M_2 M_3 \int_0^t dt' \int_0^\infty (\underline{\omega}(r') - \overline{\omega}(r'))^+ r' dr'$$

Gronwall

$$\int_0^\infty (\underline{\omega}(r) - \overline{\omega}(r))^+ r dr \leq M_1 e^{M_2 M_3 t} \int_0^\infty (\underline{\omega}_0(r) - \overline{\omega}_0(r))^+ r dr$$

QED

Summary

Burgers-Donaldson-Sullivan class

- Survey on stationary solutions

- Non-stationary solutions

  numerics suggests blow-up

  methods of reaction-diffusion eq. $\longrightarrow$ proof of blow-up for large IC

  Non-zero constant self-similar solution $\longrightarrow$ suggests asymptotic self-similar blow-up

THE REST IS SUPPLEMENTARY MATERIALS, WHICH ARE NOT USED IN THE TALK.

We must envisage a much broader class of flows in the entire x-space. Some sort of condition at infinity must, however, be imposed in order to insure reasonable behaviour of the flow. This is shown by the solutions

$$u_i = a_{i\nu} x_\nu, \;\; p = -(\dot{a}_{ik} + a_{i\alpha} a_{\alpha k}) x_i x_k$$

of the equations of flow where the a_{ik} are entirely arbitrary functions of t subject to the conditions

$$a_{ik} = a_{ki}, \; a_{\alpha\alpha} = 0.$$

Such a solution can become infinite at a finite moment of time. It can also start from the state of rest with $u = p = 0$ without staying at rest. Presumably no such things happen if a condition of the following sort is imposed:...

Hopf(1952)

Properties of solutions with finite energy are deeply rooted in our consciousness. What you will see may appear bizzard.

Infinite energy

$$u(x,y,z,t) = \left(\frac{y+z}{t-T}, \frac{z+x}{t-T}, \frac{x+y}{t-T}\right)$$

$$p(x,y,z,t) = -\frac{x^2+y^2+z^2}{(t-T)^2}$$

S.Childress and E. Spiegel

G.F.D. Duff “As Professor G. Duff points out, in case one is willing to accept infinite energies, one can easily do this just using (such an example).” Marsden, Ebin & Fisher (1972)

W.S. Lewellen “A solution for three dimensional vortex flows with strong circulation.”
J. Fluid Mech. 14 1962 420–432.

B.R. Morton “The Strength of Vortex and Swirling Core Flows” J. Fluid Mech. 38 1969 315-333

Fujita equation blow-up

Sketch of Proof

$$u(x,t) = e^{t\triangle} u(x,0) + \int_0^t e^{(t-s)\triangle} u(x,s)^p ds$$

$$e^{t\triangle} \phi(x) \equiv \int_{\mathbb{R}^n} G_t(x-y)\phi(y)dy$$

$$G_t(x) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

Crux inequality

$$t^{\frac{1}{p-1}} e^{t\triangle}\phi(x) \le C(p)$$

i.e.

$$t^{1/(p-1)} \underbrace{\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) \phi(y) dy}_{\to \|\phi\|_{L^1}} \leq C(p)(4\pi t)^{N/2}$$

$$\frac{1}{p-1} > \frac{N}{2} \Rightarrow \text{ contradiction}$$

cf. H.A. Levine, SIAM Review, 32(1990)262,

K. Deng & H.A. Levine, J. Math. Anal. Appl. 243(2000)85.

Proof

$$J(x,s) \equiv \int_{\mathbb{R}^n} G_{t-s}(x-y)u(y,s)dy$$

$$\frac{dJ(x,s)}{ds} = \int_{\mathbb{R}^n} G_{t-s}(x-y)(u(y,s))^p dy \geq \left(\int_{\mathbb{R}^n} G_{t-s}(x-y)u(y,s)dy \right)^p$$

$$\frac{dJ(x,s)}{ds} \geq J(x,s)^p$$

$$\frac{1}{p-1}\left(J(x,0)^{1-p} - J(x,t)^{1-p} \right) \geq t$$

$$t^{\frac{1}{p-1}} e^{t\triangle}\phi(x) \leq C(p)$$

Stability of Burgers vortex

D.G. Crowdy, A note on the linear stability of Burgers vortex. Stud. Appl. Math. 100 (1998), no. 2, 107–126.

W.O. Criminale, D.G. Lasseigne, T.L. Jackson, Vortex perturbation dynamics. Stud. Appl. Math. 98 (1997), no. 2, 99–120.

D.S.Nolan, B.F. Farrell, Generalized stability analyses of asymmetric disturbances in one- and two-celled vortices maintained by radial inflow. J. Atmospheric Sci. 56 (1999), no. 10, 1282–1307.

$$\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial r} = \omega^2 + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right),$$

$$U(r) = \frac{1}{r}\int_0^r r'\omega(r')dr'$$

Equivalently,

$$\frac{\partial \omega}{\partial t} = 2\omega^2 - \frac{1}{r}\frac{\partial}{\partial r}(rU\omega) + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right),$$

Critical case

$$t^{\frac{n}{2}} e^{t\triangle}\phi(x) \le C,\ \|\phi\|_{L^1} \le C',\ \|u(t)\|_{L^1} \le C'$$

Assume

$$\phi \ge kG_\alpha,\ k, \alpha > 0$$

$$u(t) \ge e^{t\triangle}\phi \ge e^{t\triangle}kG_\alpha$$

$$\|u(t)\|_{L^1} \ge \int_0^t \|e^{(t-s)\triangle}u(s)^p\|_{L^1} ds$$

$$\ge \int_0^t \|e^{(t-s)\triangle}(e^{s\triangle}kG_\alpha)^p\|_{L^1} ds$$

$$\ge \int_0^t \|(e^{s\triangle}kG_\alpha)^p\|_{L^1} ds$$

$$\left(e^{s\triangle}G_{\alpha}\right)^{p} = (G_{\alpha+s})^{p}$$

$$= [4\pi(s+\alpha)]^{-n(p-1)/2}p^{-n/2}G_{(s+\alpha)/p}$$

$$= [4\pi(s+\alpha)]^{-1}p^{-n/2}G_{(s+\alpha)/p}$$

$$\|u(t)\|_{L^1} \geq k^p p^{-N/2}(4\pi)^{-1}$$

$$\times \int_0^t (s+\alpha)^{-1}\|G_{(s+\alpha)/p}\|_{L^1}ds$$

$$= k^p p^{-N/2}(4\pi)^{-1}\log\left(\frac{t+\alpha}{\alpha}\right)$$

reaction-diffusion eqs with advection

$$\frac{\partial u}{\partial t} + (a \cdot \nabla) u^q = u^p + \triangle u,$$

$$p_c = \min\left(1 + \frac{2}{N}, 1 + \frac{2q}{N+1}\right)$$

Aguirre & Escobedo (1993)

$$\frac{\partial u}{\partial t} + (b \cdot \nabla) u = u^p + \triangle u,$$

$$u(x, t = 0) = u_0(x) (\geq 0)$$

$$(b \cdot \nabla) u = \nabla \cdot B, \ |B| \leq c u^p$$

$$1 < p < 1 + \frac{2}{N} \rightarrow \text{blowup}$$

Bandle-Levine (1994)