Analytical solution for nonlinear Schrödinger vortex reconnection

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Analysis of the nonlinear Schrödinger vortex reconnection is given in terms of the coordinate-time power series. The lowest order terms in these series correspond to the solution to the linear Schrödinger equation and provide several interesting properties of the reconnection process, in particular a non-singular character of the reconnection, the anti-parallel configuration of the vortex filaments and the square-root law of approach to the reconnection point. As the whole, the infinite power series represent a fully nonlinear analytic solution in a finite volume including the reconnection point which is valid for finite time provided that the initial condition is an analytic function. The series solutions are free from the periodicity artifacts and discretization error of the direct computational approaches and they are easy to analyze using a computer algebra program.

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1. INTRODUCTION

Vortex solutions of the nonlinear Schrödinger (NLS) equation are of interest in nonlinear optics [1–3] and in the theory of Bose-Einstein condensates [4] (BEC). The NLS equation is also often used to describe turbulence in superfluid helium. [5] NLS is a nice model in this case because the vortex quantization appears naturally in this model and because its large-scale limit is the compressible Euler equation describing classical inviscid fluids. [6,7] At short scales, the NLS equation allows for a "quantum uncertainty principle" which allows the vortex reconnection process without introduction of viscosity or any other dissipation. Numerically, NLS vortex reconnection was studied by Koplik and Levine[8] and, more recently, by Leadbeater et al.[9] and, for a non-local version of NLS equation, by Berloff et al.[4] In applications to superfluid turbulence, the NLS equation was directly computed by Nore et al.[10] Such cryogenic turbulence consists of repeatedly reconnecting vortex tangles, with each reconnection event resulting in the generation of Kelvin waves on the vortex cores [11] and a sound emission. [9] These two small-scale processes are very hard to correctly compute in direct simulations of 3D NLS turbulence due to the numerical resolution problems. A popular way to avoid this problem is to compute vortex tangles by a Biot-Savart method which is derived from the Euler equation and to use a simple reconnection rule each time the vortex filaments approach closer than a critical ("quantum") distance. This approach was pioneered by Schwarz [6] and it has been further developed by Samuels et al.[12] In this case, it is important to prescribe realistic vortex reconnection rules. Therefore, the elementary vortex reconnection events have to be carefully studied and parameterized. Numerically, such a study was performed by Leadbeater et al., [9] the present paper is devoted to analytical study of these NLS vortex reconnection events.

The analytical approach of this paper is based on expanding the solution in powers of small distance, from the reconnection point, and small time, measured from the reconnection moment. The idea is to exploit the fact that the nonlinearity is small near the reconnecting point when the vortex filaments have approached to a distances less than that of the vortex cores. This smallness of the nonlinearity just stems from the definition of vortices in NLS (curves where $\Psi = 0$) and the continuity of Ψ. Their core size is of the order of the distance over which $\Psi \to 1$ (where $\Psi = 1$ represents the background condensate). Therefore, at distances much smaller than the core size, close to the reconnection point of two vortices, Ψ is small provided it is continuous. Thus, to the first approximation the solution near the reconnection point can be described by a linear solution which, already at this level, contains some very important information about the reconnection process: (1) that the reconnection proceeds smoothly without any singularity formation, (2) that in the immediate vicinity of the reconnection the vortices are strictly anti-parallel and (3) just before the reconnection event the distance between the vortices decreases as $|t|^{-1/2}$, where t is the time measured from the reconnection moment. Note that result (1) could surprise those who draw their intuition from the vortex collapse in Euler equations (which is believed to be singular). On the other hand, results (2) and (3) are remarkably similar to the numerical and theoretical results for the Euler vortex collapse. [13–15]

In section II of this paper we examine the local analysis of the reconnection process by deriving a linear solution and in section III consider its properties. The linear solution describes many, but not all the important properties of vortex reconnection. In particular, it cannot describe solutions outside the vortex cores and, therefore, it cannot describe the far-field sound radiation produced by the reconnection. On the other hand, one can substitute the linear solution back into the NLS equation and find the first nonlinear correction to this solution. Recursively repeating this procedure, one can recover the fully nonlinear solution in terms of infinite coordinate and time series. This derivation is discussed in detail in section IV. The series produced are a general solution to a Cauchy initial value problem. Thus, by Cauchy-Kowalevski theorem, [16] these series define an analytic function (with a finite convergence radius) provided the initial conditions are analytic. The generation of such a suitable initial condition is addressed in section V. Our series representation of the solution to the NLS equation is exact, and therefore will include such properties as sound emission. However, due to the finite radius of convergence of the analytic solution, one is unable to observe a far-field sound emission directly. In this paper, we use *Mathematica* to compute some examples of the fully nonlinear solutions for the vortex reconnection. The results of which are presented in section VI.

Let us summarise the advantages and disadvantages that our analytical solution has with respect to those being computed via direct numerical simulations (DNS). Firstly, our analytical solutions are obtained as a general formula, simultaneously applicable for a broad class of initial vortex positions and orientations. Secondly, our analytical solutions are not affected by any periodicity artifacts which are typical for DNS using spectral methods and by a discretization error. On the other hand, our analytical solutions are only available for a finite distance from the vortex lines (of the order of the vortex core size) because their defining power series have a finite radius of convergence.

2. LOCAL ANALYSIS OF THE RECONNECTION

Let us start with the defocusing NLS equation written in the non-dimensional form,

$$i\Psi_t + \Delta\Psi + (1 - |\Psi|^2)\Psi = 0. \tag{1}$$

Suppose that in vicinity of the point $\mathbf{r} = (x, y, z) = (0, 0, 0)$ at $t = t_0$ we have $\Psi = \Psi_0$ such that $Re\Psi_0 = z$, and $Im\Psi_0 = az + bx^2 - cy^2$, where a, b and c are some positive constants. For such initial conditions the geometrical location of the vortex filaments, $\Psi = 0$, is given by two intersecting straight lines, $z = 0$ and $y = \pm\sqrt{b/c}\,x$.

In the small vicinity of the point $\mathbf{r} = 0$, deep inside the vortex core (where $\Psi_0 \approx 0$), we can ignore the nonlinear term found in equation (1). Further, by a simple transformation $\Psi = \Phi e^{it}$ we can eliminate the third term Ψ and obtain $i\Phi_t + \Delta\Phi = 0$. (This just corresponds to multiplying our solution by a phase, it does not alter its properties, but does make the following analysis simpler). It is easy to see that the intial condition has not changed under this transformation, $\Psi_0 = \Phi_0$. Advancing our system to small time $t - t_0$, we find $Re\,\Phi = Re\,\Psi_0 - (t-t_0)\,\Delta Im\,\Psi_0$ and $Im\,\Phi = Im\,\Psi_0 + (t-t_0)\,\Delta Re\,\Psi_0$, or

$$\begin{aligned} Re\Phi &= z - 2(b-c)\,(t-t_0), \\ Im\Phi &= az + bx^2 - cy^2. \end{aligned} \qquad (2)$$

For both $t - t_0 < 0$ and $t - t_0 > 0$ the set of vortex lines, $\Phi = 0$, is given by two hyperbolas. A bifurcation happens at $t = t_0$ where these hyperbolas degenerate into the two intersecting lines (see Fig. 1). This bifurcation corresponds to the reconnection of the vortex filaments. Thus, we have constructed a local (in space and time) NLS solution corresponding to the vortex reconnection. Obviously, this solution corresponds to a smooth function Φ at the reconnection point. It should be stressed that this is not an assumption, but just the way in which we have chosen to construct our solution. However, we do believe that this observed smoothness may be expected to be common to NLS vortex reconnection processes. If this is the case then all such reconnecting vortices can locally be described by the presented solution because intersection of a hyperbola and a moving plane provide a generic local bifurcation describing reconnection in the case of smooth fields.

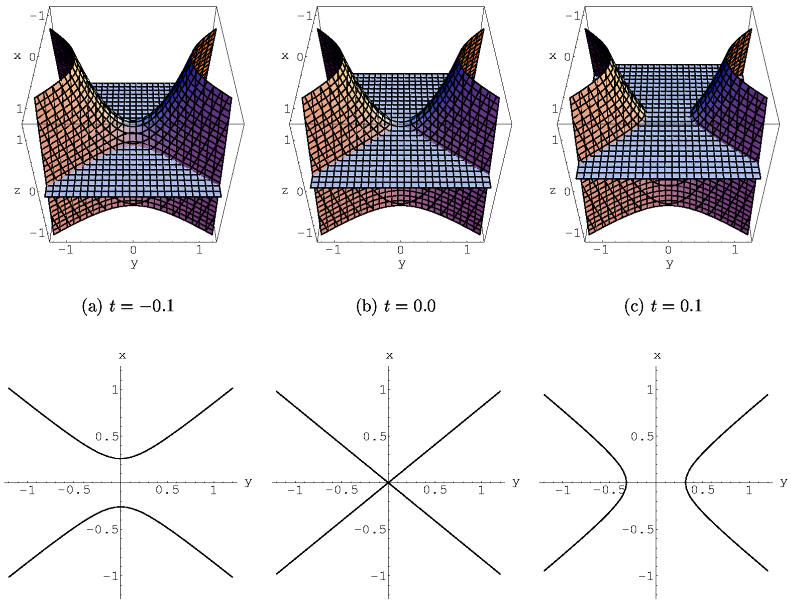
3. PROPERTIES OF THE VORTEX RECONNECTION

The local linear solution (2) reveals several important properties of the reconnection of NLS vortices.

1. Whatever initial orientation of the vortex filaments was, the reconnecting parts of these filaments align so that they approach each other in an anti-parallel configuration. Amazingly similar effect has been observed in numerical Biot-Savart simulations of thin vortex filaments in inviscid incompressible fluid.[13,15] Later we will return to discussion the possible origins of this similarity.

2. It follows from Eq. (2) that the reconnecting parts of the vortex filaments approach each other as $\sqrt{t - t_0}$. Exactly the same scaling behavior, for approaching thin filaments in incompressible fluids, has been given by the theory of Siggia and Pumir [13–15] and has been observed numerically in Biot-Savart computations. [13,15]

3. The nonlinearity plays a minor role in the late stages of vortex reconnection in NLS. This is a simple manifestation of the fact that in the close



(a) $t = -0.1$ (b) $t = 0.0$ (c) $t = 0.1$

(d) $t = -0.1$ (e) $t = 0.0$ (f) $t = 0.1$

Fig. 1. Linear solution Eq. (2) of the nonlinear Schrödinger equation for $a = 1$, $b = 3$, $c = 2$ and $t_0 = 0$. Sub-figures (a), (b) and (c) show the intersection of the real (plane) and imaginary (hyperbolic paraboloid) parts of Eq. (2) at successive times $t = -0.1$, $t = 0.0$ and $t = 0.1$ respectively. Sub-figures (d), (e) and (f) show the corresponding lines of intersection where $\Psi = 0$; reconnection occurs at $t = t_0 = 0$.

spatio-temporal vicinity of the reconnection point, $\Psi \approx 0$, so that the dynamics are almost linear. This property can be also reformulated as follows.
4. No singularity is observed in the process of reconnection according to the solution (2): both the real and imaginary part of Ψ behave continuously in space and time. This property of vortex reconnection in NLS is in drastic contrast to the singularity formation in the vortex collapse events described by the Euler equations. We will return to the discussion of this below.
5. Distinct from incompressible fluids, no viscous dissipation is needed for the NLS vortices to reconnect. Here, the dispersion does the same job of breaking the topological constraints (related to the Kelvin circulation theorem) as viscosity does in a normal fluid.

4. NONLINEAR SOLUTION

We will now move on to consider the full NLS equation. We will use a recursion relationship to compute the solution assuming that $x, y, z \sim \epsilon$ and $t \sim \epsilon^3$ (for simplicity, we take $t_0 = 0$). The solution we obtain will therefore be of the form $\Psi = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \cdots$, where $\Psi^{(n)} \sim \epsilon^n$. The above ϵ scaling of x, y, z and t has been chosen to generate a recursion relationship when substituted in the NLS equation (1). Of course we could have chosen a different ϵ dependence, however, as the final series representation of our solution contains an infinite number of terms, this would just correspond to the same solution but with a suitable re-ordering.

Consider the NLS equation (1). Firstly, we note that $\partial_t \sim \epsilon^{-3}$ and $\triangle \sim \epsilon^{-2}$ and therefore $i\Psi_t^{(m)} \sim \epsilon^{m-3}$, $\triangle\Psi^{(n)} \sim \epsilon^{n-2}$, $\left[|\Psi|^2\Psi\right]^{(p)} \sim \epsilon^p$ and $\Psi^{(q)} \sim \epsilon^q$, where

$$[|\Psi|^2\Psi]^{(p)} = \sum_{i,j=1}^{p} \Psi^{*(i)}\Psi^{(j)}\Psi^{(p-i-j)}. \tag{3}$$

Matching the terms, by setting $m = n+1$ and $p = q = n-2$, and integrating we find

$$\Psi^{(n+1)} = \Psi_0^{(n+1)} + i\int_0^t \left[\triangle\Psi^{(n)} + \Psi^{(n-2)} - [|\Psi|^2\Psi]^{(n-2)}\right] dt, \tag{4}$$

where $\Psi_0^{(n)}$ are arbitrary n^{th} order functions of coordinate which appear as constants of integration with respect to time. The full nonlinear solution of the Cauchy initial value problem can now be obtained by matching $\Psi_0^{(n)}$ to the n^{th} order components of the initial condition at $t = 0$ obtained via the Taylor expansion in coordinate. Let us assume that the initial condition is an analytic function so that it can be represented by power series in coordinates with a non-zero volume of convergence. Then, by Cauchy-Kowalevski theorem, the function Ψ will remain analytic for non-zero time. In the other words, the solution can also be represented as a power series with a non-zero domain of convergence in space and time. Remarkably, the recursion relation Eq. (4) is precisely the means by which one can write down the terms of the power-series representation of the fully nonlinear solution to the NLS equation, with an arbitrary analytical initial condition Ψ_0.

5. INITIAL CONDITION

Our next step is to construct a suitable initial condition for study of reconnecting vortices which will have to be formulated in term of the power series. We start by formulating the famous line vortex solution to the steady state NLS equation [5] in terms of a power series. Substituting $\Psi = Ae^{i\theta}$ into Eq. (1), we find $\triangle A - A|\nabla\theta|^2 + A - A^2 = 0$, where we have used the fact that $\nabla\theta \cdot \nabla A = 0$ and $\triangle\theta = 0$. We can simplify this equation, since $A = A(r)$ and therefore, $\triangle A = \frac{1}{r}\partial_r(rA_r)$. However, we also note that $|\nabla^2| = \frac{1}{r^2}$ since $\theta_x = -y/r^2$ and $\theta_y = -x/r^2$. Therefore, we have $\frac{1}{r}\partial_r(rA_r) - \frac{A}{r^2} - A^3 + A = 0$. We will solve this equation using another recursive method. We'd like to get a solution of the form $A = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots = \sum_n A^{(n)}$. (However, we can set a_0 to zero on physical grounds, since we require $\Psi = 0$ at $r = 0$). As before $\frac{1}{r}\partial_r(rA_r^{(m)}) = m^2 a_m r^{m-2} \sim \epsilon^{m-2}$, $\frac{A^{(n)}}{r^2} = a_n r^{n-2} \sim \epsilon^{n-2}$, $[A^3]^{(p)} \sim \epsilon^p$ and $A^{(q)} = a_q r^q \sim \epsilon^q$, where,

$$[A^3]^{(p)} = \sum_{i,j=1}^{p} A^{(i)} A^{(j)} A^{(p-i-j)}. \tag{5}$$

Again, by matching the powers of r we can get a recursion relationship for a_n. Setting $m = n$ and $p = q = n - 2$ we obtain

$$a_n = \frac{f_{n-2} - a_{n-2}}{(n^2 - 1)}, \tag{6}$$

where $f_p = [A^3]^{(p)}/r^p$.

We should note that $a_{2n} = 0$ for all n. Therefore, taking a power of r out of our expansion for $A(r)$ we find,

$$\Psi = A(r)r^{i\theta} = rg(r)e^{i\theta}, \tag{7}$$

where $g(r) = g(r^2) = a_1 + a_3 r^2 + a_5 r^4 + \cdots = \sum_{n=1}^{\infty} a_{2n-1} r^{2n-2}$. Further, $re^{i\theta}$ is complex so we can write $re^{i\theta} = x + iy$ and hence our prototype solution, for a vortex pointing along the z-axis, is $\Psi = (x + iy)g(x^2 + y^2)$.

We can manipulate this prototype solution to get an initial condition for our vortex reconnection problem. Our initial condition Ψ_0 will be made up of two vortices, Ψ_1 and Ψ_2, a distance $2d$ and angle 2α apart. Namely, we will take the initial condition to be the product of Ψ_1 and Ψ_2 [Koplik, Ref. 8], that is $\Psi_0 = \Psi_1\Psi_2$.

Firstly, We would like the vortices in the (x, y) plane. We can do this by transforming our coordinates $x \to y$, $y \to z$ and $z \to x$. This will give us a vortex pointing along the x-axis $\Psi = (y + iz)g(y^2 + z^2)$. The vortex can now be rotated by angle α to the x-axis in the (x, y) plane via $x \to x\cos\alpha - y\sin\alpha$ and $y \to y\cos\alpha + x\sin\alpha$. Finally, we shift the whole vortex in the z direction by a distance d using $z \to z - d$ we finally obtain $\Psi_1 = [y\cos\alpha + x\sin\alpha + i(z - d)]g((y\cos\alpha + x\sin\alpha)^2 + (z - d)^2)$. In a similar manner, Ψ_2 is a vortex at angle $-\alpha$ and shifted by $-d$ in the z direction, $\Psi_2 = [y\cos\alpha - x\sin\alpha + i(z + d)]g((y\cos\alpha - x\sin\alpha)^2 + (z + d)^2)$.

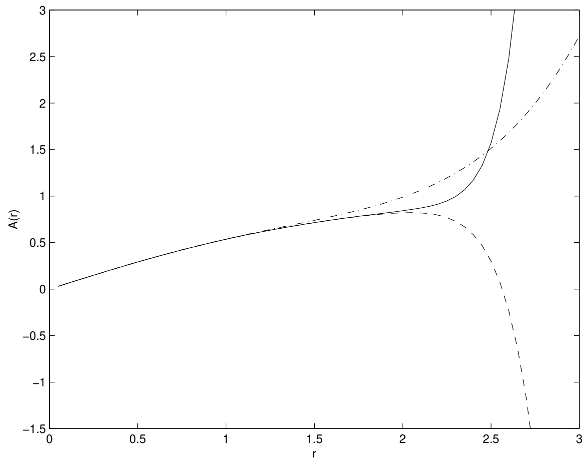


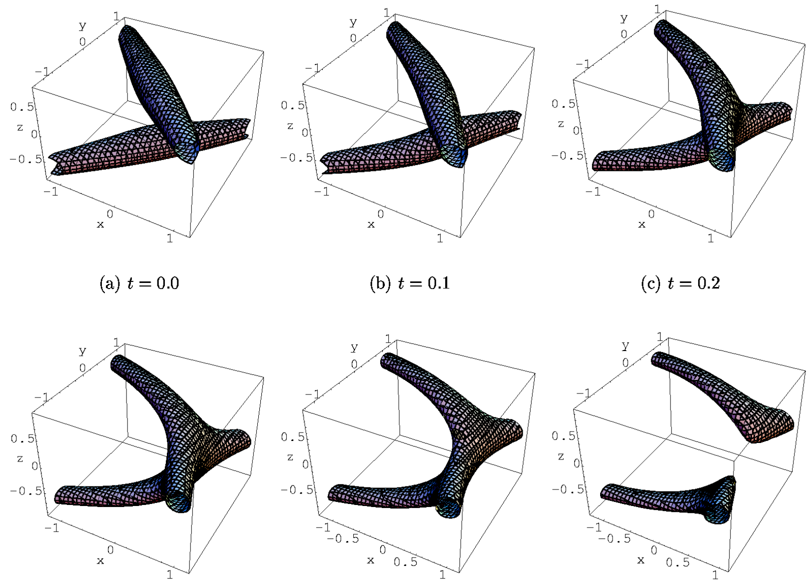
Fig. 2. The initial condition for the prototype vortex solution is constructed via an appropriate expansion for $A = A(r)$, Eq. (6). Here we can see the expansion for $A = A(r)$ truncated at three different orders of n; $n = 5$ (dash-dot line), $n = 15$ (dashed line) and $n = 21$ (solid line) with $a_1 = 0.6$. At higher orders one would see the existence of a finite radius of convergence at $r \approx 2.5$.

6. RESULTS

It would be difficult and time consuming to expand the analytical solution, derived in the previous section, by hand. Thankfully, we can use a computer to perform the necessary algebra. What follows is an example solution of the reconnection of two initially separated vortices.

Firstly, we need to consider the validity and accuracy of our initial condition. Fig. 2, shows the prototype solution A=A(r) for a single vortex, at various different orders. Increasing the order will obviously improve accuracy. However, one should note that at higher order there is evidence of a finite radius of convergence r_c. This will restrict the spatial region of validity for our full t-dependent solution. Our prototype solution also has a dependence on a_1. In the following simulation we have chosen $a_1 = 0.6$ numerically, so that $\Psi \to 1$ as r increases (although it will obviously diverge near r_c), therefore matching the definition of a NLS vortex, as outlined in section I. Of course the validity of the full t-dependent solution will be restricted, in the spatial sense, by the initial condition's region of convergence. The region of convergence will evolve, remaining finite during a finite interval of time (by the Cauchy-Kowalevski theorem), but then may shrink to zero afterwards.

We will now discuss an example solution. One should note, however,



(a) $t = 0.0$ (b) $t = 0.1$ (c) $t = 0.2$

(d) $t = 0.3$ (e) $t = 0.4$ (f) $t = 0.5$

Fig. 3. Sub-figures (a) to (f) show the evolution of two initially separated vortices in time t. This realization is for $|\Psi| = 0.1$, $a_1 = 0.6$, $d = 0.6$ and $\alpha = \Pi/4$. The reconnection and separation events are clearly evident.

that even using a computer it can be time consuming constructing a general high order solution due to the huge number of terms the recursive formulae generate. By specifying some of the initial parameters before hand, for example the separation distance d, the solution can be generated much more quickly. As we only wish to demonstrate this method, we will not consider a high order solution in this paper. In our example simulation below, we used Mathematica to perform the necessary algebra in generating a nonlinear solution up to $O(\epsilon^6)$. (One should note that although the prototype solution (7) for a single vortex has $a_{2n} = 0$ for all n, our initial condition is made up of two such vortices. i.e. two series multiplied together. Therefore, there will be non-zero cross terms of order $O(\epsilon^6)$ in our initial condition).

Our choice of parameters will be $d = 0.6$ and $\alpha = \pi/4$. This corresponds to two vortices, initially separated by a distance 1.2, at right angles to each other. Fig. 3 shows the evolution of the iso-surface $|\Psi| = 0.1$ in time, demonstrating reconnection and then separation. Examining this solution in detail we can clearly see evidence of some of the properties mentioned earlier. Firstly, we see there is a smooth reconnection of the two vortices without any singularity formation. Secondly, in the vicinity of the reconnection the vortices are indeed anti-parallel.

7. CONCLUSION

In this paper we presented a local analysis of the NLS reconnection processes. We showed that many interesting properties of the reconnection can be seen already at the level of the linear solution. We derived a recursion formula Eq. (4) that gives the fully nonlinear solution of the initial value problem in a finite volume around the reconnection point for a finite period of time. In fact, formula (4) can describe a much wider class of problems than just the vortex reconnection. Of interest, for example, solutions describing creation or annihilation of NLS vortex rings. This process is easy to describe by considering vortex rings that lie on the intersection of a paraboloid and a plane crossing its minimum point at the creation/annihilation moment of time. Also, the method of expansion around the reconnection point can be used for other evolution equations, e.g., the Ginzburg-Landau equation. These applications will be considered in future.

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