A Weak Turbulence Theory for Incompressible Magnetohydrodynamics

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Abstract. We derive a weak turbulence formalism for incompressible MHD. Three–wave interactions lead to a system of kinetic equations for the spectral densities of energy and helicity. The kinetic equations conserve energy in all wavevector planes normal to the applied magnetic field $B_0\,\hat{\mathbf{e}}_\parallel$. Numerically and analytically, we find energy spectra $E^+ \sim k_\perp^{n_s}$, $E^- \sim k_\perp^{n_{-s}}$ such that $n_s + n_{-s} = -4$, where $E^\pm$ are the spectra of the Elsässer variables $\mathbf{z}^\pm = \mathbf{v} \pm \mathbf{b}$ in the two-dimensional case ($k_\parallel = 0$). The constants of the spectra are computed exactly and found to depend on the amount of correlation between the velocity and the magnetic field. Comparison with several numerical simulations and models is also made.

1 Introduction and General Discussion

Magnetohydrodynamic (MHD) turbulence plays an important role in many astrophysical situations [34], ranging from the solar wind [23], to the Sun [35], the interstellar medium [14] and beyond [50], as well as in laboratory devices such as tokamaks (see *e.g.* [46] [43] [12] [44]). A very instrumental step in recognizing some of the features that distinguished MHD turbulence from hydrodynamic turbulence was taken independently in the early sixties by Iroshnikov and Kraichnan in 1965 (IK ; [19] [17]). They argued that the destruction of phase coherence by Alfvén waves traveling in opposite directions along local large eddy magnetic fields introduces a new time scale and a slowing down of energy transfer to small scales. They pictured the scattering process as being principally due to three wave interactions. Assuming 3D isotropy, dimensional analysis then leads to the prediction of a $k^{-3/2}$ Kolmogorov finite energy flux spectrum.

However, it is clear, and it has been a concern to Kraichnan and others throughout the years, that the assumption of local three dimensional isotropy is troublesome. Indeed numerical simulations and experimental measurements both indicate that the presence of strong magnetic fields make MHD turbulence strongly anisotropic. Anisotropy is manifested in a two dimensionalization of the turbulence spectrum in a plane transverse to the locally dominant magnetic field and in inhibiting spectral energy transfer along the direction parallel to the field [27] [29] [30]. Replacing the 3D isotropy assumption by a 2D one, and retaining of the rest of the IK picture, leads to the dimensional analysis prediction of a $k_\perp^{-2}$ spectrum ($\mathbf{B_0} = B_0\,\hat{\mathbf{e}}_\parallel$, the applied magnetic field, $k_\parallel = \mathbf{k}\cdot\hat{\mathbf{e}}_\parallel$, $\mathbf{k}_\perp = \mathbf{k} - k_\parallel\,\hat{\mathbf{e}}_\parallel$, $k_\perp = |\mathbf{k}_\perp|$) although recent numerical simulations by Ng and Bhattacharjee [31] [32] to confirm this spectrum were inconclusive. Indeed, it is also difficult to determine the exact spectral slope from experimental data.

A major controversy in the debate of the universal features of MHD turbulence was introduced by Sridhar and Goldreich (SG [40]) in 1994. They challenged that part of IK thinking which viewed Alfvén wave scattering as a three wave interaction process, an assumption implicit in the IK derivation of the $k^{-3/2}$ spectrum. SG argue that, in the inertial range where amplitudes are small, significant energy exchange between Alfvén waves can only occur for resonant three wave interactions. Moreover, their argument continues, because one of the fluctuations in such a resonant triad has zero Alfvén frequency, the three wave coupling is empty. They conclude therefore that the long time dynamics of weak MHD fields are determined by four wave resonant interactions.

This conclusion is false. In this paper, we will show that resonant three wave interactions are non empty and lead to a relaxation to universal behavior and significant spectral energy redistribution. Moreover, weak turbulence theory provides a set of closed kinetic equations for the long time evolution of the eight power spectra, corresponding to total energy $e^s(\mathbf{k})$, parallel energy $\Phi^s(\mathbf{k})$, magnetic and pseudo magnetic helicities $R^s(\mathbf{k})$, $I^s(\mathbf{k})$ constructed from the Elsässer fields $\mathbf{z^s} = \mathbf{v} + s\mathbf{b}$, $s = \pm 1$, where $\mathbf{v}$ and $\mathbf{b}$ are the fluctuating velocity and Alfvén velocity respectively. The latter is defined such that $\mathbf{b} = \mathbf{B}/\sqrt{\mu_0\rho_0}$, where ρ_0 is the uniform density and μ_0 the magnetic permeability. We will also show that a unique feature of Alfvén wave weak turbulence is the existence of additional conservation laws. One of the most important is the conservation of energy on all wavevector planes perpendicular to the applied field $\mathbf{B_0}$. There is no energy transfer between planes. This extra symmetry means that relaxation to universal behavior only takes place as function of $k_\perp$ so that, in the inertial range (or window of transparency), $e^s(\mathbf{k}) = f(k_\|)k_\perp^{-p}$ where $f(k_\|)$ is non universal.

Because weak turbulence theory for Alfvén waves is not straightforward and because of the controversy raised by SG, it is important to discuss carefully and understand clearly some of the key ideas before outlining the main results. We therefore begin by giving an overview of the theory for the statistical initial value problem for weakly nonlinear MHD fields.

1.1 Alfvén weak turbulence : the kinematics, the asymptotic closure and some results

The starting point is a kinematic description of the fields. We assume that the Elsässer fields $\mathbf{z^s}(\mathbf{x}, t)$ are random, homogeneous, zero mean fields in the three spatial coordinates $\mathbf{x}$. This means that the n-point correlation functions between combinations of these variables estimated at $\mathbf{x_1}, ..., \mathbf{x_n}$ depend only on the relative geometry of the spatial configuration. We also assume that for large separation distances $|\mathbf{x_i} - \mathbf{x_j}|$ along **any** of the three-spatial directions, fluctuations are statistically independent. We will also discuss the case of strongly two dimensional fields for which there is significant correlation along the direction of the applied magnetic field. We choose to use cumulants rather than moments, to which the cumulants are related by a one-to-one map. The choice is made for two reasons. The first is that they are exactly those combinations of moments which are asymptotically zero for all large separations. Therefore they have well defined and, at least initially before long distance correlations can be built up by nonlinear couplings, smooth Fourier transforms. We will be particularly interested in the spectral densities

$$q_{jj'}^{ss'}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \langle z_j^s(\mathbf{x}) z_{j'}^{s'}(\mathbf{x}+\mathbf{r}) \rangle \, e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{1}$$

of the two point correlations. (Remember, $z_j^s(\mathbf{x})$ has zero mean so that the second order cumulants and moments are the same.) The second reason for the choice of cumulants as dependent variables is that, for joint Gaussian fields, all cumulants above second order are identically zero. Moreover, because of linear wave propagation, initial cumulants of order three and higher decay to zero in a time scale $(b_0 k_\|)^{-1}$ where $\mathbf{b_0} = \mathbf{B_0}/\sqrt{\mu_0\rho_0}$ is the Alfvén velocity ($b_0 = |\mathbf{b_0}|$) and $k_\|^{-1}$ a dominant parallel length scale in the initial field. This is a simple consequence of the Riemann-Lebesgue lemma ; all Fourier space cumulants become multiplied by fast nonvanishing oscillations because of linear wave properties and these oscillations give rise to cancelations upon integration. Therefore, the statistics approaches a state of joint Gaussianity. The amount by which it differs, and the reason for a nontrivial relaxation of the dynamics, is determined by the long time cumulative response generated by nonlinear couplings of the waves. The special manner in which third and higher cumulants are regenerated by nonlinear processes leads to a natural asymptotic closure of the statistical initial value problem.

Basically, because of the quadratic interactions, third order cumulants (equal to third order moments) are regenerated by fourth order cumulants and binary products of second order ones. But the only long time contributions arise from a subset of the second order products which lie on certain resonant manifold defined by zero divisors. It is exactly these terms which appear in the kinetic equations which describe the evolution of the power spectra of second order moments over time scales $(\epsilon^2 \, b_0 k_\parallel)^{-1}$. Here, ϵ is a measure of the strength of the nonlinear coupling. Likewise, higher order cumulants are nonlinearly regenerated principally by products of lower order cumulants. Some of these small divisor terms contribute to frequency renormalization and others contribute to further (*e.g.* four wave resonant interactions) corrections of the kinetic equations over longer times.

What are the resonant manifolds for three wave interactions and, in particular, what are they for Alfvén waves? They are defined by the divisors of a system of weakly coupled wavetrains $a_j e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_s(k_j)t)}$, with $\omega_s(k_j)$ the linear wave frequency, s its level of degeneracy, which undergo quadratic coupling. One finds that triads $\mathbf{k}$, $\boldsymbol{\kappa}$, $\mathbf{L}$ which lie on the resonant manifold defined for some choice of s, s', s'', by

$$\begin{aligned} \mathbf{k} &= \boldsymbol{\kappa} + \mathbf{L} \,, \\ \omega_s(\mathbf{k}) &= \omega_{s'}(\boldsymbol{\kappa}) + \omega_{s''}(\mathbf{L}) \,, \end{aligned} \tag{2}$$

interact strongly (cumulatively) over long times $(\epsilon^2 \, \omega_0)^{-1}$, ω_0 a typical frequency. For Alfvén waves, $\omega_s(\mathbf{k}) = s\mathbf{b_0} \cdot \mathbf{k} = s b_0 k_\parallel$ when $s = \pm 1$ (Alfvén waves of a given wavevector can travel in one of two directions) and b_0, the Alfvén velocity, is the strength of the applied field. Given the dispersion relation, $\omega = s\mathbf{b_0} \cdot \mathbf{k}$, one might ask why there is any weak turbulence for Alfvén waves at all because for, $s = s' = s''$, (2) is satisfied for all triads. Furthermore, in that case, the fast oscillations multiplying the spectral cumulants of order $N+1$ in the evolution equation for the spectral cumulant of order N disappear so that there is no cancelation (phase mixing) and therefore no natural asymptotic closure. However, the MHD wave equations have the property that the coupling coefficient for this interaction is identically zero and therefore the only interactions of importance occur between oppositely traveling waves where $s' = -s$, $s'' = s$. In this case, (2) becomes

$$2s\mathbf{b_0} \cdot \boldsymbol{\kappa} = 2sb_0\kappa_{\|} = 0\,. \tag{3}$$

The third wave in the triad interaction is a fluctuation with zero Alfvén frequency. SG incorrectly conclude that the effective amplitude of this zero mode is zero and that therefore the resonant three wave interactions are null.

Although some of the kinetic equations will involve principal value integral (PVI) with denominator $s\omega(\mathbf{k}) + s\omega(\boldsymbol{\kappa}) - s\omega(\mathbf{k}-\boldsymbol{\kappa}) = 2sb_0\kappa_{\|}$, whose meaning we discuss later, the majority of the terms contain the Dirac delta functions of this quantity. The equation for the total energy density contains only the latter implying that energy exchange takes place by resonant interactions. Both the resonant delta functions and PVI arise from taking long time limits $t \to \infty$, $\epsilon^2 t$ finite, of integrals of the form

$$\begin{aligned} &\int F(k_{\|}, \epsilon^2 t)\ \left(e^{(2isb_0k_{\|}t)} - 1\right)\ (2isb_0k_{\|})^{-1}\, dk_{\|} \\ &\sim \int F(k_{\|}, \epsilon^2 t)\ \left(\pi sgn(t)\,\delta(2sb_0k_{\|}) + i\mathcal{P}(\tfrac{1}{2sb_0k_{\|}})\right)\ dk_{\|}\ . \end{aligned} \tag{4}$$

Therefore, implicit in the derivation of the kinetic equations is the assumption that $F(k_{\|}, \epsilon^2 t)$ is relatively smooth near $k_{\|} = 0$ so that $F(k_{\|}, \epsilon^2 t)$ remains nearly constant for $k_{\|} \sim \epsilon^2$. In particular, the kinetic equation for the total energy density

$$e^s(\mathbf{k}_\perp, k_{\|}) = \Sigma_{j=1}^3 q_{jj}^{ss}(\mathbf{k}_\perp, k_{\|}) \tag{5}$$

is the integral over $\boldsymbol{\kappa}_\perp$ of a product of a combination of $q_{jj'}^{ss}(\mathbf{k}_\perp - \boldsymbol{\kappa}_\perp, k_{\|})$ with $Q^{-s}(\boldsymbol{\kappa}_\perp, 0) = \Sigma_{p,m} k_p k_m\, q_{pm}^{-s-s}(\boldsymbol{\kappa}_\perp, 0)$. Three observations (O1,2,3) and two questions (Q1,2) arise from this result.

O1– Unlike the cases for most systems of dispersive waves, the resonant manifolds for Alfvén waves **foliate** wavevector space. For typical dispersion relations, a wavevector $\boldsymbol{\kappa}$, lying on the resonant manifold of the wavevector $\mathbf{k}$, will itself have a different resonant manifold, and members of that resonant manifold will again have different resonant manifolds. Indeed the union of all such manifolds will fill $\mathbf{k}$ space so that energy exchange occur throughout all of $\mathbf{k}$ space.

O2– In contrast, for Alfvén waves, the kinetic equations for the total energy density contains $k_{\|}$ as a parameter which identifies which wavevector plane perpendicular to $\mathbf{B_0}$ we are on. Thus the resonant manifolds for all wavevectors of a given $\bar{k}_\|$ is the plane $k_\| = \bar{k}_\|$. The resonant manifolds foliate **k**-space.

O3– Further, conservation of total energy holds for each $k_\|$ plane. There is energy exchange between energy densities having the same $k_\|$ value but not between those having different $k_\|$ values. Therefore, relaxation towards a universal spectrum with constant transverse flux occurs in wavevector planes perpendicular to the applied magnetic field. The dependence of the energy density on $k_\|$ is nonuniversal and is inherited from the initial distribution along $k_\|$.

Q1– If the kinetic equation describes the evolution of power spectra for values of $k_\|$ outside of a band of order ϵ^ξ, $\xi < 2$, then how does one define the evolution of the quantities contained in $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ so as to close the system in $k_\|$?

Q2– Exactly what is $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$? Could it be effectively zero as SG surmise ? Could it be possibly singular with singular support located near $k_\| = 0$ in which case the limit (4) is suspect ?

To answer the crucially important question 2, we begin by considering the simpler example of a one dimensional, stationary random signal $u(t)$ of zero mean. Its power spectrum is $f(\omega)$ the limit of the sequence $f_L(\omega) = \frac{1}{2\pi}\int_{-L}^{L}\langle u(t)u(t+\tau)\rangle e^{-i\omega\tau}\,d\tau$ which exists because the integrand decays to zero as $\tau \to \pm\infty$. Ergodicity and the stationarity of $u(t)$ allows us to estimate the average $R(\tau) = \langle u(t)u(t+\tau)\rangle$ by the biased estimator

$$R_L(\tau) = \frac{1}{2L}\int_{-L+|\tau|/2}^{L-|\tau|/2} u(t-\tau/2)u(t+\tau/2)\,dt$$

with mean $E\{R_L(\tau)\} = (1-\tau/2L)R(\tau)$. Taking L sufficiently large and assuming a sufficiently rapid decay so that we can take $R_L(\tau) = 0$ for $|\tau| > 2L$ means that $R_L(\tau)$ is simply the convolution of the signal with itself. Furthemore the Fourier transform $S_L(\omega)$ can then be evaluated as

$$S_L(\omega) = \int_{-2L}^{2L} R_T(\tau)e^{-i\omega\tau}\,d\tau = \frac{1}{2L}\,|\int_{-L}^{L} u(t)e^{-i\omega t}\,dt|^2\,.$$

For sufficiently large L, the expected value of $S_L(\omega)$ is $S(\omega)$, the Fourier transform of $R(\tau)$ although the variance of this estimate is large. Nevertheless $S_L(\omega)$, and in particular $S_L(0)$, is generally non zero and measures the power in the low frequency modes. To make the connection with Fourier space, we can think of replacing the signal $u(t)$ by the periodic extension of the truncated signal $\tilde{u}_L(t) = u(t)$, $|t| < L$; $\tilde{u}_L(t+2L)$ for $|t| > L$. The zero mode of the

Fourier transform $a_L(0) = \frac{1}{2L}\int_{-L}^{L} \tilde{u}(t)\,dt$ is a nonzero random variable and, while its expected value (for large L) is zero, the expected value of its square is certainly not zero. Indeed the expected value of $S_L(0) = 2L\,a_L^2(0)$ has a finite nonzero value which, as $L \to \infty$, is independent of L as $a_L(0)$ has mean zero and a standard deviation proportional to $(2L)^{-1/2}$. Likewise for Alfvén waves, the power associated with the zero mode $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ is nonzero and furthermore, for the class of three dimensional fields in which correlations decay in all directions, $Q^{-s}(\boldsymbol{\kappa}_\perp, k_\|)$ is smooth near $k_\| = 0$. Therefore, for these fields, we may consider $Q^{-s}(\boldsymbol{\kappa}_\perp, 0)$ as a limit of $Q^{-s}(\boldsymbol{\kappa}_\perp, k_\|)$ as $\frac{k_\|}{k_{\perp 0}} \to 0$ and $\frac{1}{\epsilon^2}\frac{k_\|}{k_{\perp 0}} \to \infty$. Here $k_{\perp 0}$ is some wavenumber near the energy containing part of the inertial range. Therefore, in this case, we solve first the nonlinear kinetic equation for $lim_{k_\| \to 0}\ e_k^s(k_\perp, k_\|)$, namely for very oblique Alfvén waves, and having found the asymptotic time behavior of $e^s(k_\perp, 0)$, then return to solve the equation for $e^s(k_\perp, k_\|)$ for finite $k_\|$.

Assuming isotropy in the transverse $k_\perp$ plane, we find universal spectra $c_n^s k_\perp^{n_s}$ for $E^s(k_\perp)$ ($\int E^s(k_\perp, 0)\,dk_\perp = \int e^s(k_\perp)\,d\mathbf{k}_\perp$), corresponding to finite fluxes of energy from low to high transverse wavenumbers. Then $e^s(k_\perp, k_\|) = 2\pi f^s(k_\|) c_n^s k_\perp^{n_s - 1}$ where $f^s(k_\|)$ is not universal. These solutions each correspond to energy conservation. We find that convergence of all integrals is guaranteed for $-3 < n_s, n_{-s} < -1$ and that

$$n_{-s} + n_s = -4 \tag{6}$$

which means, that for no directional preference, $n_+ = n_- = -2$. These solutions have finite energy, *i.e.* $\int E\,dk_\perp$ converges. If we interpret them as being set up by a constant flux of energy from a source at low $k_\perp$ to a sink at high $k_\perp$, then, since they have finite capacity and can only absorb a finite amount of energy, they must be set up in finite time. When we searched numerically for the evolution of initial states to the final state, we found a remarkable result which we yet do not fully understand. Each $E^s(k_\perp)$ behaves as a propagating front in the form $E^s(k_\perp) = (t_0 - t)^{1/2} E_0(k_\perp (t_0 - t)^{3/2})$ and $E_0(l) \sim l^{-7/3}$ as $l \to +\infty$. This means that for $t < t_0$, the $E^s(k_\perp)$ spectrum had a tail for $k_\perp < (t_0 - t)^{-3/2}$ with stationary form $k_\perp^{-7/3}$ joined to $k_\perp = 0$ through a front $E_0(k_\perp (t_0 - t)^{3/2})$. The $7/3$ spectrum is steeper than the $+2$ spectrum. Amazingly, as t approached very closely to t_0, disturbances in the high $k_\perp$ part of the $k_\perp^{-7/3}$ solution propagated back along the spectrum, rapidly turning it into the finite energy flux spectrum $k_\perp^{-2}$. We neither understand the origin nor the nature of this transition solution, nor do we understand the conservation law involved with the second equilibrium solution of the kinetic equations. Dimensional analysis suggests that it is associated with the conservation of the spectral density $k_\perp^{-2/3} E^s(k)$ and that would seem to connote a nonlocal quantity in physical space. However, it is more likely determined as a condition on the front solution of the integro-differential equation. The almost constant $k_\perp^{-7/3}$ spectrum would appear to allow a nonlocal interaction whereby the energy at the developing front can be instantaneously supplied by the reservoir at small k. Once the connection to infinity is made, however the circuit between source and sink is closed and the finite flux energy spectrum takes over.

To this point we have explained how MHD turbulent fields for which correlations decay in all directions relax to quasiuniversal spectra via the scattering of high frequency Alfvén waves with very oblique, low frequency ones. But there is another class of fields that it is also important to consider. There are homogeneous, zero mean random fields which have the anisotropic property that correlations in the direction of applied magnetic field do not decay with increasing separation $\mathbf{B_0} \cdot (\mathbf{x_1} - \mathbf{x_2})$. For this case, we may think of decomposing the Elsässer fields as

$$z_j^s(\mathbf{x}_\perp, x_\parallel) = \bar{z}_j^s(\mathbf{x}_\perp) + \hat{z}_j^s(\mathbf{x}_\perp, x_\parallel) \tag{7}$$

where the $\hat{z}_j^s(\mathbf{x}_\perp, x_\parallel)$ have the same properties of the fields considered heretofore but where the average of $z_j^s(\mathbf{x}_\perp, x_\parallel)$ over $x_\parallel$ is nonzero. The total average of z_j^s is still zero when one averages $\bar{z}_j^s(\mathbf{x}_\perp)$ over $\mathbf{x}_\perp$. In this case, it is not hard to show that correlations

$$\langle z_j^s(\mathbf{x}_\perp, x_\parallel)\, z_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + r_\parallel)\rangle$$

divide into two parts

$$\langle z_j^s(\mathbf{x}_\perp)\, z_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp)\rangle + \langle \hat{z}_j^s(\mathbf{x}_\perp, x_\parallel)\, \hat{z}_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp, x_\parallel + r_\parallel)\rangle$$

with Fourier transforms,

$$q_{jj'}^{ss'}(\mathbf{k}) = \delta(k_\parallel)\, \bar{q}_{jj'}^{ss'}(k_\perp) + \hat{q}_{jj'}^{ss'}(k_\perp, k_\parallel)\,, \tag{8}$$

when $\bar{q}$ is smooth in $\mathbf{k}_\perp$ and $\hat{q}$ is smooth in both $\mathbf{k}_\perp$ and $k_\parallel$. The former is simply the transverse Fourier of the two point correlations of the $x_\parallel$ averaged field. Likewise all higher order cumulants have delta function multipliers $\delta(k_\parallel)$ for each $\mathbf{k}$ dependence. For example

$$\bar{q}_{jj'j''}^{ss's''}(\mathbf{k}, \mathbf{k}') = \delta(k_\parallel)\, \delta(k'_\parallel)\, \bar{q}_{jj'j''}^{ss's''}(\mathbf{k}_\perp, \mathbf{k}'_\perp)$$

is the Fourier transform of

$$\langle \bar{z}_j^s(\mathbf{x}_\perp)\, \bar{z}_{j'}^{s'}(\mathbf{x}_\perp + \mathbf{r}_\perp)\, \bar{z}_{j''}^{s''}(\mathbf{x}_\perp + \mathbf{r}'_\perp)\rangle \,.$$

Such singular dependence of the Fourier space cumulants has a dramatic effect on the dynamics especially since the singularity is supported precisely on the resonant manifold. Indeed the hierarchy of cumulant equations for $\bar{q}^{(n)}$ simply loses the fast (Alfvén) time dependence altogether and becomes fully nonlinear MHD turbulence in two dimensions with time t replaced by ϵt. Let us imagine, then, that the initial fields are dominated by this two dimensional component and that the fields have relaxed on the time scale $t \sim \epsilon^{-1}$ to their equilibrium solutions of finite energy flux for which $\bar{E}(k_\perp)$ is the initial Kolmogorov finite energy flux spectrum $k_\perp^{-5/3}$ for $k_\perp > k_0$, k_0 some input wavenumber and $\bar{E}(k_\perp) \sim k_\perp^{-1/3}$ corresponding to the inverse flux of the squared magnetic vector potential $(\mathcal{A}\,; \mathbf{B} = \nabla \times \mathcal{A})$. $\hat{\mathcal{A}}(\mathbf{k})$, the spectral density of $\langle \mathcal{A}^2\rangle$, bahaves as $k_\perp^{-7/3}$. These are predicted from phenomenological arguments and supported by numerical simulations.

Let us then ask : how do Alfvén waves (Bragg) scatter off this two dimensional turbulent field ? To answer this question, one should of course redo all the analysis taking proper account of the $\delta(k_\|)$ factors in $\bar{q}^{(n)}$. However, there is a simpler way. Let us imagine that the power spectra for the $\hat{z}_j^s$ fields are supported at finite $k_\|$ and have much smaller integrated power over an interval $0 \le k_\| < \beta \ll 1$ than do the two dimensional fields. Let us replace the $\delta(k_\|)$ multiplying $q_{jj'}^{ss'}(\mathbf{k}_\perp)$ by a function of finite width β and height β^{-1}. Then the kinetic equation is linear and describes how the power spectra, and in particular $\hat{e}^s(\mathbf{k})$, of the $\hat{z}_j^s$ fields interact with the power spectra of the two dimensional field $\bar{z}_j^s$. Namely, the $Q^{-s}(\mathbf{k}_\perp, 0)$ field in the kinetic equation is determined by the two dimensional field and taken as known. The time scale of the interaction is now $\beta\epsilon^{-2}$, because the strength of the interaction is increased by β^{-1} and, is faster than that of pure Alfvén wave scattering. But the equilibrium of the kinetic equation will retain the property that $n_{-s} + n_s = -4$ where now n_{-s} is the phenomenological exponent associated with two dimensional MHD turbulence and n_s the exponent of the Alfvén waves. Note that when $n_{-s} = -5/3$, n_s is $-7/3$, which would agree with the temporary spectrum observed in the finite time transition to the $k_\perp^{-2}$ spectrum.

We now proceed to a detailed presentation of our results.

2 The derivation of the kinetic equations

The purpose in this section is to obtain closed equations for the energy and helicity spectra of weak MHD turbulence, using the fact that, in the presence of a strong uniform magnetic field, only Alfvén waves of opposite polarities propagating in opposite directions interact.

2.1 The basic equations

We will use the weak turbulence approach, the ideas of which are described in great detail in the book of Zakharov, L'vov and Falkovich (1992). There are several different ways to derive the weak turbulence kinetic equations. We follow here the technique that can be found in Benney and Newell (1969). We write the 3D incompressible MHD equations for the velocity $\mathbf{v}$ and the Alfvén velocity $\mathbf{b}$

$$(\partial_t + \mathbf{v} \cdot \nabla)\, \mathbf{v} = -\nabla P_* + \mathbf{b} \cdot \nabla\, \mathbf{b} + \nu \nabla^2 \mathbf{v}\,, \tag{9}$$

$$(\partial_t + \mathbf{v} \cdot \nabla)\, \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{b}\,, \tag{10}$$

where P_* is the total pressure, ν the viscosity, η the magnetic diffusivity and $\nabla \cdot \mathbf{v} = 0$, $\nabla \cdot \mathbf{b} = 0$. In the absence of dissipation, these equations have three quadratic invariants in dimension three, namely the total energy $E^T = \frac{1}{2}\langle v^2 + b^2 \rangle$, the cross–correlation $E^C = \langle \mathbf{v} \cdot \mathbf{b} \rangle$ and the magnetic helicity $H^M = \langle \mathcal{A} \cdot \mathbf{B} \rangle$ [47].

The Elsässer variables $\mathbf{z}^s = \mathbf{v} + s\mathbf{b}$ with $s = \pm 1$ give these equations a more symmetrized form, namely :

$$(\partial_t + \mathbf{z}^{-s} \cdot \nabla)\, \mathbf{z}^s = -\nabla P_*\ , \tag{11}$$

where we have dropped the dissipative terms which pose no particular closure problems. The first two invariants are then simply written as $2E^s = \langle |\mathbf{z}^s|^2 \rangle$.

We now assume that there is a strong uniform magnetic induction field $\mathbf{B_0}$ along the unit vector $\hat{\mathbf{e}}_\parallel$ and non dimensionalize the equations with the corresponding magnetic induction $\mathbf{B_0}$, where the z^s fields have an amplitude proportional to ϵ ($\epsilon \ll 1$) assumed small compared to b_0. Linearizing the equations leads to

$$(\partial_t - s b_0 \partial_\parallel) z_j^s = -\epsilon \partial_{x_m} z_m^{-s} z_j^s - \partial_{x_j} P_*\,, \tag{12}$$

where $\partial_\parallel$ is the derivative along $\hat{\mathbf{e}}_\parallel$. The frequency of the modes at a wavevector $\mathbf{k}$ is $\omega(\mathbf{k}) = \omega_k = \mathbf{b}_0 \cdot \mathbf{k} = b_0 k_\parallel$. We Fourier transform the wave fields using the interaction representation,

$$z_j^s(\mathbf{x},t) = \int A_j^s(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k} = \int a_j^s(\mathbf{k},t)\, e^{i(\mathbf{k}\cdot\mathbf{x}+s\omega_k t)}\, d\mathbf{k}\,, \tag{13}$$

where $a_j^s(\mathbf{k},t)$ varies slowly in time because of the weak nonlinearities ; hence

$$\partial_t a_j^s(\mathbf{k},t) = -i\epsilon k_m P_{jn} \int a_m^{-s}(\boldsymbol{\kappa})\, a_n^s(\mathbf{L})\, e^{i(-s\omega_k - s\omega_\kappa + s\omega_L)t} \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}} \tag{14}$$

with $d_{\boldsymbol{\kappa}\mathbf{L}} = d\boldsymbol{\kappa} d\mathbf{L}$ and $\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} = \delta(\mathbf{k}-\boldsymbol{\kappa}-\mathbf{L})$; finally, $P_{jn}(k) = \delta_{jn} - k_j k_n k^{-2}$ is the usual projection operator keeping the $\mathbf{A^s}(\mathbf{k})$ fields transverse to $\mathbf{k}$ because of incompressibility. The exponentially oscillating term in (14) is essential : its exponent should not vanish when $(\mathbf{k}-\boldsymbol{\kappa}-\mathbf{L}) = \mathbf{0}$, *i.e.* the waves should be dispersive for the closure procedure to work. In that sense, incompressible MHD can be coined "pseudo"–dispersive because although $\omega_k \sim k$, the fact that waves of one s–polarity interact *only* with the opposite polarity has the consequence that the oscillating factor is non–zero except at resonance ; indeed with $\omega_k = b_0 k_\parallel$, one immediately sees that $-s\omega_k - s\omega_\kappa + s\omega_L = s(-k_\parallel - \kappa_\parallel + L_\parallel) = -2s\kappa_\parallel$ using the convolution constraint between the three waves in interaction. In fact, Alfvén waves may have a particularly weak form of interactions since such interactions take place only when two waves propagating in opposite directions along the lines of the uniform magnetic field meet. As will be seen later (see §3), this has the consequence that the transfer in the direction parallel to $\mathbf{B}_0$ is zero, rendering the dynamics two–dimensional, as is well known (see *e.g.* [29] [39]). Technically, we note that there are two types of waves that propagate in opposite directions, so that the classical criterion [48] for resonance to occur, *viz.* $\omega'' > 0$ does not apply here.

2.2 Toroidal and poloidal fields

The divergence–free condition implies that only two scalar fields are needed to describe the dynamics ; following classical works in anisotropic turbulence, they are taken as [10] [15] [36]

$$\mathbf{z}^s = \mathbf{z}_1^s + \mathbf{z}_2^s = \nabla \times (\psi^s \hat{\mathbf{e}}_\parallel) + \nabla \times (\nabla \times (\phi^s \hat{\mathbf{e}}_\parallel))\,, \tag{15}$$

which in Fourier space gives

$$A_j^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel\, \hat{\psi}^s(\mathbf{k}) - \mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}}_\parallel)\, \hat{\phi}^s(\mathbf{k})\ . \tag{16}$$

We elaborate somewhat on the significance of the $\hat{\psi}^s$ and $\hat{\phi}^s$ fields since they are the basic fields with which we shall deal. Note that $\mathbf{z}_1^s$ are two–dimensional fields with no parallel component and thus with only a vertical vorticity component (vertical means parallel to $\mathbf{B_0}$), whereas the $\mathbf{z}_2^s$ fields have zero vertical vorticity ; such a decomposition is used as well for stratified flows (see [22] and references therein). Indeed, rewriting the double cross product in (16) leads to :

$$\mathbf{A}^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel \, \hat{\psi}^s(\mathbf{k}) - \mathbf{k}\, k_\parallel \hat{\phi}^s(\mathbf{k}) + \hat{\mathbf{e}}_\parallel \, k^2 \hat{\phi}^s(\mathbf{k}) \tag{17}$$

or using $\mathbf{k} = \mathbf{k}_\perp + k_\parallel \hat{\mathbf{e}}_\parallel$:

$$\mathbf{A}^s(\mathbf{k}) = i\mathbf{k} \times \hat{\mathbf{e}}_\parallel \, \hat{\psi}^s(\mathbf{k}) - \mathbf{k}_\perp k_\parallel \, \hat{\phi}^s(\mathbf{k}) + \hat{\mathbf{e}}_\parallel \, k_\perp^2 \hat{\phi}^s(\mathbf{k})\,. \tag{18}$$

The above equations indicate the relationships between the two orthogonal systems (with $\mathbf{p} = \mathbf{k} \times \hat{\mathbf{e}}_\parallel$ and $\mathbf{q} = \mathbf{k} \times \mathbf{p}$) made of the triads $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, $(\hat{\mathbf{e}}_\parallel, \mathbf{p}, \mathbf{k}_\perp)$ and the system $(\hat{\mathbf{e}}_\parallel, \mathbf{p}, \mathbf{k})$. In terms of the decomposition used in [45] with

$$\mathbf{h}_\pm = \mathbf{p} \times \mathbf{k} \pm i\mathbf{p} \tag{19}$$

and writing $\mathbf{z}^s = A_+^s \mathbf{h}_+ + A_-^s \mathbf{h}_-$, it can be shown easily that $\psi^s = A_+^s - A_-^s$ and $\phi^s = A_+^s + A_-^s$. In these latter variables, the s–energies E^s are proportional to $\langle |A_+^s|^2 + |A_-^s|^2 \rangle$ and the s–helicities $\langle \mathbf{z}^s \cdot \nabla \times \mathbf{z}^s \rangle$ are proportional to $\langle |A_+^s|^2 - |A_-^s|^2 \rangle$. Note that E^s are not scalars : when going from a right-handed to a left-handed frame of reference, E^s changes into E^{-s}.

2.3 Moments and cumulants

We now seek a closure for the energy tensor $q_{jj'}^{ss'}(\mathbf{k})$ defined as

$$\langle a_j^s(\mathbf{k})\, a_{j'}^{s'}(\mathbf{k}') \rangle \equiv q_{jj'}^{ss'}(\mathbf{k}')\, \delta(\mathbf{k} + \mathbf{k}') \tag{20}$$

in terms of second order moments of the two scalars fields $\hat{\psi}^s(\mathbf{k})$ and $\hat{\phi}^s(\mathbf{k})$. Simple manipulations lead, with the restriction $s = s'$ (it can be shown that correlations with $s' = -s$ have no long time influence and therefore are, for convenience of exposition, omitted), to :

$$
\begin{aligned}
q^{ss}_{11}(\mathbf{k}') &= k_2^2\Psi^s(\mathbf{k}) - k_1k_2k_\parallel I^s(\mathbf{k}) + k_\parallel^2 k_1^2\Phi^s(\mathbf{k})\,,\\
q^{ss}_{22}(\mathbf{k}') &= k_1^2\Psi^s(\mathbf{k}) + k_1k_2k_\parallel I^s(\mathbf{k}) + k_\parallel^2 k_2^2\Phi^s(\mathbf{k})\,,\\
q^{ss}_{12}(\mathbf{k}') + q^{ss}_{21}(\mathbf{k}') &= -2k_1k_2\Psi^s(\mathbf{k}) + k_\parallel(k_1^2-k_2^2)I^s(\mathbf{k}) + 2k_1k_2k_\parallel^2\Phi^s(\mathbf{k})\,,\\
q^{ss}_{1\parallel}(\mathbf{k}') + q^{ss}_{\parallel 1}(\mathbf{k}') &= k_2k_\perp^2 I^s(\mathbf{k}) - 2k_1k_\parallel k_\perp^2\Phi^s(\mathbf{k})\,,\\
q^{ss}_{2\parallel}(\mathbf{k}') + q^{ss}_{\parallel 2}(\mathbf{k}') &= -k_1k_\perp^2 I^s(\mathbf{k}) - 2k_2k_\parallel k_\perp^2\Phi^s(\mathbf{k})\,,\\
q^{ss}_{\parallel\ \parallel}(\mathbf{k}') &= k_\perp^4\Phi^s(\mathbf{k})\,,\\
\tfrac{1}{k_1}[q^{ss}_{2\parallel}(\mathbf{k}') - q^{ss}_{\parallel 2}(\mathbf{k}')] &= \tfrac{1}{k_2}[q^{ss}_{\parallel 1}(\mathbf{k}') - q^{ss}_{1\parallel}(\mathbf{k}')] = \tfrac{1}{k_\parallel}[q^{ss}_{12}(\mathbf{k}') - q^{ss}_{21}(\mathbf{k}')]\\
&= -ik_\perp^2 R^s(\mathbf{k})\,,
\end{aligned}
\tag{21}
$$

where the following correlators involving the toroidal and poloidal fields have been introduced :

$$
\begin{aligned}
\langle\hat\psi^s(\mathbf{k})\hat\psi^s(\mathbf{k}')\rangle &= \delta(\mathbf{k}+\mathbf{k}')\Psi^s(\mathbf{k}')\,,\\
\langle\hat\phi^s(\mathbf{k})\hat\phi^s(\mathbf{k}')\rangle &= \delta(\mathbf{k}+\mathbf{k}')\Phi^s(\mathbf{k}')\,,\\
\langle\hat\psi^s(\mathbf{k})\hat\phi^s(\mathbf{k}')\rangle &= \delta(\mathbf{k}+\mathbf{k}')\Pi^s(-\mathbf{k})\,,\\
\langle\hat\phi^s(\mathbf{k})\hat\psi^s(\mathbf{k}')\rangle &= \delta(\mathbf{k}+\mathbf{k}')\Pi^s(\mathbf{k})\,,\\
R^s(\mathbf{k}) &= \Pi^s(-\mathbf{k}) + \Pi^s(\mathbf{k})\,,\\
I^s(\mathbf{k}) &= i[\Pi^s(-\mathbf{k}) - \Pi^s(\mathbf{k})]\,,
\end{aligned}
\tag{22}
$$

and where $k_\perp^2 = k_1^2 + k_2^2$, $k^2 = k_\perp^2 + k_\parallel^2$. Note that $\Sigma_s R^s$ is the only pseudo–scalar, linked to the possible lack of symmetry of the equations under plane reversal, *i.e.* to a non–zero helicity.

The density energy spectrum writes

$$
e^s(\mathbf{k}) = \Sigma_j\, q^{ss}_{jj}(\mathbf{k}) = \mathbf{k}_\perp^2(\Psi^s(\mathbf{k}) + k^2\Phi^s(\mathbf{k}))\ . \tag{23}
$$

Note that it can be shown easily that the kinetic and magnetic energies $\frac{1}{2}\langle u^2\rangle$ and $\frac{1}{2}\langle b^2\rangle$ are equal in the context of the weak turbulence approximation. Similarly, expressing the magnetic induction as a combination of $\mathbf{z}^\pm$ and thus of $\hat\psi^\pm$ and $\hat\phi^\pm$, the following symmetrized cross–correlator of magnetic helicity (where the Alfvén velocity is used for convenience) and its Fourier transform are found to be

$$
\frac{1}{2}\langle\hat{\mathcal{A}}_j(\mathbf{k})\,b_j(\mathbf{k}')\rangle + \frac{1}{2}\langle\hat{\mathcal{A}}_j(\mathbf{k}')\,b_j(\mathbf{k})\rangle = \frac{1}{4}\mathbf{k}_\perp^2\,\Sigma_s R^s(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}')\,, \tag{24}
$$

where the correlations between the + and − variables are ignored because they are exponentially damped in the approximation of weak turbulence. Similarly to the case of energy, there is equivalence between the kinetic and magnetic helical variables in that approximation, hence the kinetic helicity defined as $\langle \mathbf{u} \cdot \omega \rangle$ writes simply in terms of its spectral density $H^V(\mathbf{k})$:

$$H^V(\mathbf{k}) = k^2 H^M(\mathbf{k}) = \frac{1}{4} k^2 \mathbf{k}_\perp^2 \Sigma_s R^s(\mathbf{k}) \, . \tag{25}$$

In summary, the eight fundamental spectral density variables for which we seek a weak turbulence closure are the energy $e^s(\mathbf{k})$ of the three components of the $\mathbf{z}^s$ fields, the energy density along the direction of the uniform magnetic field $\Phi^s(\mathbf{k})$, the correlators related to the off–diagonal terms of the spectral energy density tensor $I^s(\mathbf{k})$ and finally the only helicity–related pseudo–scalar correlators, namely $R^s(\mathbf{k})$.

The main procedure that leads to a closure of weak turbulence for incompressible MHD is outlined in the Appendix. It leads to the equations (83) giving the temporal evolution of the components of the spectral tensor $q_{jj'}^{ss'}(\mathbf{k})$ just defined. The last technical step consists in transforming equations (83) of the Appendix in terms of the eight correlators we defined above. This leads us to the final set of equations, constituting the kinetic equations for weak MHD turbulence.

2.4 The kinetic equations

In the general case the kinetic equations for weak MHD turbulence are

$$\partial_t e^s(\mathbf{k}) = \tag{26}$$

$$\frac{\pi\varepsilon^2}{b_0} \int \Bigg[\left(L_\perp^2 - \frac{X^2}{k^2} \right) \Psi^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L^2} \right) \Psi^s(\mathbf{k}) + \left(L_\perp^2 L^2 - \frac{k_\parallel^2 W^2}{k^2} \right) \Phi^s(\mathbf{L})$$

$$- \left(k_\perp^2 k^2 - \frac{k_\parallel^2 Y^2}{L^2} \right) \Phi^s(\mathbf{k}) + \left(\frac{k_\parallel XY}{L^2} \right) I^s(\mathbf{k}) - \left(\frac{k_\parallel XW}{k^2} \right) I^s(\mathbf{L}) \Bigg] Q_k^{-s}(\boldsymbol{\kappa})$$

$$\delta(\kappa_\parallel) \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} \, d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$\partial_t \left[k_\perp^2 k^2 \Phi^s(\mathbf{k})\right] = \tag{27}$$

$$\frac{\pi\varepsilon^2}{b_0}\int\left[k_\parallel^2 X^2\left(\frac{\Psi^s(\mathbf{L})}{k_\perp^2 k^2}-\frac{\Phi^s(\mathbf{k})}{L_\perp^2}\right)+\left(k_\parallel^2 Z+k_\perp^2 L_\perp^2\right)^2\left(\frac{\Phi^s(\mathbf{L})}{k_\perp^2 k^2}-\frac{\Phi^s(\mathbf{k})}{L_\perp^2 L^2}\right)\right.$$

$$\left.+\frac{k_\parallel X}{k_\perp^2 k^2}\left(k_\parallel^2 Z+k_\perp^2 L_\perp^2\right) I^s(\mathbf{L})+\left(\frac{k_\parallel XY}{2L^2}\right) I^s(\mathbf{k})\right] Q_k^{-s}(\boldsymbol{\kappa})\ \delta(\kappa_\parallel)\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$-\frac{\varepsilon^2}{b_0} s R^s(\mathbf{k}) \mathcal{P}\int \frac{X}{2\kappa_\parallel L^2}\left(k_\parallel Z-L_\parallel k_\perp^2\right) Q_k^{-s}(\boldsymbol{\kappa})\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\ d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$\partial_t \left[k_\perp^2 R^s(\mathbf{k})\right] = \tag{28}$$

$$-\frac{\pi\varepsilon}{b_0}\int\left[L_\perp^2\left(\frac{Z+k_\parallel^2}{k^2}\right) R^s(\mathbf{L})+\frac{k_\perp^2}{2}\left(1+\frac{(Z+k_\parallel^2)^2}{k^2 L^2}\right) R^s(\mathbf{k})\right] Q_k^{-s}(\boldsymbol{\kappa})$$

$$\delta(\kappa_\parallel)\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ +\ \frac{\varepsilon^2}{b_0} s\mathcal{P}\int\left[2X\left(k_\parallel Z-L_\parallel k_\perp^2\right)\left(\Psi^s(\mathbf{k})+k^2\Phi^s(\mathbf{k})\right)\right.$$

$$\left.+\left(\left(k_\parallel Z-L_\parallel k_\perp^2\right)^2-k^2X^2\right) I^s(\mathbf{k})\right]\frac{Q_k^{-s}(\boldsymbol{\kappa})}{2\kappa_\parallel k^2 L^2}\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$\partial_t \left[k_\perp^2 k^2 I^s(\mathbf{k})\right] = \tag{29}$$

$$\frac{\pi\varepsilon^2}{b_0}\int\left[\left(L_\perp^2 Z+\frac{k_\parallel^2}{k_\perp^2}(Z^2-X^2)\right) I^s(\mathbf{L})+\left(\frac{k_\parallel^2 Y^2}{2L^2}-k_\perp^2 k^2+\frac{k^2X^2}{2L^2}\right) I^s(\mathbf{k})\right.$$

$$\left.+\left(\frac{k_\parallel XY}{L^2}\right)\left(\Psi^s(\mathbf{k})+k^2\Phi^s(\mathbf{k})\right)+\frac{2k_\parallel X}{k_\perp^2}\left(Z\Psi^s(\mathbf{L})-\left(k_\parallel^2 Z+k_\perp^2 L_\perp^2\right)\Phi^s(\mathbf{L})\right)\right]$$

$$Q_k^{-s}(\boldsymbol{\kappa})\delta(\kappa_\parallel)\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ -\ \frac{\varepsilon^2}{b_0} sR^s(\mathbf{k})\mathcal{P}\int\frac{1}{2\kappa_\parallel L^2}\left(\left(k_\parallel Z-L_\parallel k_\perp^2\right)^2-k^2X^2\right)$$

$$Q_k^{-s}(\boldsymbol{\kappa})\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\ d_{\boldsymbol{\kappa}\mathbf{L}}$$

with

$$\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} = \delta(\mathbf{L}+\boldsymbol{\kappa}-\mathbf{k}),$$

$$d_{\boldsymbol{\kappa}\mathbf{L}} = d\boldsymbol{\kappa}\, d\mathbf{L},$$

and

$$\begin{aligned} Q_k^{-s}(\boldsymbol{\kappa}) &= k_m k_p\, q_p^{-s}{}_m^{-s}(\boldsymbol{\kappa}) \\ &= X^2\Psi^{-s}(\boldsymbol{\kappa}) + X(k_3\kappa_\perp^2 - \kappa_3 Y)I^{-s}(\boldsymbol{\kappa}) + (\kappa_3 Y - k_3\kappa_\perp^2)^2\phi^{-s}(\boldsymbol{\kappa})\,. \end{aligned} \tag{30}$$

Note that Q_k^{-s} does not involve the spectral densities $R^s(\mathbf{k})$, because of symmetry properties of the equations. The geometrical coefficients appearing in the kinetic equations are

$$\begin{aligned} X &= (\mathbf{k}_\perp \wedge \boldsymbol{\kappa}_\perp)_z = k_\perp\kappa_\perp \sin\theta\,, \\ Y &= \mathbf{k}_\perp \cdot \boldsymbol{\kappa}_\perp = k_\perp\kappa_\perp\cos\theta\,, \\ Z &= \mathbf{k}_\perp \cdot \mathbf{L}_\perp = k_\perp^2 - k_\perp\kappa_\perp\cos\theta = k_\perp^2 - Y\,, \\ W &= \boldsymbol{\kappa}_\perp \cdot \mathbf{L}_\perp = k_\perp^2 - L_\perp^2 - k_\perp\kappa_\perp\cos\theta = Z - L_\perp^2\,, \end{aligned} \tag{31}$$

where θ is the angle between $\mathbf{k}_\perp$ and $\boldsymbol{\kappa}_\perp$, and with

$$d\boldsymbol{\kappa}_\perp = \kappa_\perp d\kappa_\perp d\theta = \frac{L_\perp}{k_\perp \sin\theta}\, d\kappa_\perp dL_\perp\,, \tag{32}$$

$$\cos\theta = \frac{\kappa_\perp^2 + k_\perp^2 - L_\perp^2}{2\kappa_\perp k_\perp}\,. \tag{33}$$

In (27), (28) and (29) $\mathcal{P}\int$ means the Cauchy Principal value of the integral in question.

3 General properties of the kinetic equations

3.1 Dynamical decoupling in the direction parallel to $\mathbf{B_0}$

The integral on the right-hand side of the kinetic equation (26) contains a delta function of the form $\delta(\kappa_\parallel)$, the integration variable corresponding to the parallel component of one of the wavenumbers in the interacting triad. This delta function arises because of the three-wave frequency resonance condition. Thus, in any resonantly interacting wave triad $(\mathbf{k},\ \boldsymbol{\kappa},\ \mathbf{L})$, there is always one wave that corresponds to a purely 2D motion – having no dependence on the direction parallel to the uniform magnetic field – whereas the other two waves have equal parallel components of their corresponding wavenumbers, *viz.* $L_\parallel = k_\parallel$. This means that the parallel components of the wavenumber enter in the kinetic equation of the total energy $e^s(\mathbf{k})$ as an external parameter and that the dynamics is decoupled at each level of $k_\parallel$. In other words, there is no transfer associated with the three-wave resonant interaction along the $k_\parallel$-direction in $\mathbf{k}$-space for the total energy. This result, using the exact kinetic equations developed here, corroborates what has already been found in [29] using a phenomenological analysis of the basic MHD equations, and in [31] [32] in the framework of a model of weak MHD turbulence using individual wave packets.

As for the kinetic equation (26), the other kinetic equations (27) to (29) have integrals containing delta functions of the form $\delta(\kappa_\parallel)$. But, in addition, they have PVIs which can, *a priori*, contribute to a transfer in the parallel direction. The eventual contributions of these PVIs are discussed in §3.4.

3.2 Detailed energy conservation

Detailed conservation of energy for each interacting triad of waves is a usual property in weak turbulence theory. This property is closely related with the frequency resonance condition

$$\omega_{\mathbf{k}} = \omega_{\mathbf{L}} + \omega_{\boldsymbol{\kappa}},$$

because ω can be interpreted as the energy of one wave "quantum". For Alfvén waves, the detailed energy conservation property is even stronger because one of the waves in any resonant triad belongs to the 2D state with frequency equal to zero,

$$\omega_{\boldsymbol{\kappa}} \propto \kappa_\parallel = 0.$$

Thus, for every triad of Alfvén waves $\mathbf{k}, \mathbf{L}$ and $\boldsymbol{\kappa}$ (such that $\kappa_\parallel = 0$) the energy is conserved within two co-propagating waves having wavevectors $\mathbf{k}$ and $\mathbf{L}$. Mathematically, this corresponds to the symmetry of the integrand in the equation for e^s with respect to changing $\mathbf{k} \leftrightarrow \mathbf{L}$ (and correspondingly $\boldsymbol{\kappa} = \mathbf{k} - \mathbf{L} \to -\boldsymbol{\kappa}$).

As we have said, energy is conserved $k_\parallel$ plane by $k_\parallel$ plane so that, for each $k_\parallel$, it can be shown from (26)

$$\frac{\partial}{\partial t} \int e^s(\mathbf{k}_\perp, k_\parallel)\, d\mathbf{k}_\perp = 0\,. \tag{34}$$

3.3 Properties of R^s and I^s

Although R^s and I^s evolve according to their own kinetic equations (29) and (28), the range of values they can take on is bounded by Ψ^s and Φ^s, with the bounds being a simple consequence of the definition of these quantities. The correlators $R^s(\mathbf{k})$ and $I^s(\mathbf{k})$ have been defined in the previous section as the real part and the imaginary part of $\Pi^s(\mathbf{k})$, the cross–correlator of the toroidal field $\hat{\psi}^s(\mathbf{k})$ and of the poloidal field $\hat{\phi}^s(\mathbf{k})$. Then $\mathbf{k}_\perp^2 \Sigma_s R^s(\mathbf{k})$ appears as the spectral density of the magnetic helicity, the integral of which is a conserved quantity in three dimensions (see [42] for the case with a mean magnetic field). On the other hand $I^s(\mathbf{k})$, which we will call the anisotropy correlator (or pseudo-helicity), is not a conserved quantity. Neither is it positive definite.

Two realizability conditions (see also [8]) between the four correlators Ψ^s, Φ^s, I^s and R^s can be obtained from

$$\langle\, |\hat{\psi}^s(\mathbf{k}) \pm k\hat{\phi}^s(\mathbf{k})|^2\,\rangle \geq 0\,, \tag{35}$$

and

$$\langle\, |\hat{\psi}^s(\mathbf{k})|^2\,\rangle\langle\, |\hat{\phi}^s(\mathbf{k})|^2\,\rangle \geq |\langle\, \hat{\psi}^s(\mathbf{k})\hat{\phi}^s(-\mathbf{k})\,\rangle|^2\,. \tag{36}$$

These conditions are found to be respectively

$$\Psi^s(\mathbf{k}) + k^2\Phi^s(\mathbf{k}) \geq |kR^s(\mathbf{k})|\,, \tag{37}$$

and

$$4\Psi^s(\mathbf{k})\Phi^s(\mathbf{k}) \geq R^{s2}(\mathbf{k}) + I^{s2}(\mathbf{k})\,. \tag{38}$$

Note that the combination

$$Z = (1/2)k_\perp^2[k^2\Phi(\mathbf{k}) - \Psi(\mathbf{k}) - i|k|I(\mathbf{k})]$$

is named polarization anisotropy in [8].

3.4 Purely 2D modes and two-dimensionalisation of 3D spectra

The first consequence of the fact that there is no transfer of the total energy in the $k_\parallel$ direction in $\mathbf{k}$-space is an asymptotic two-dimensionalisation of the energy spectrum $e^s(\mathbf{k})$. Namely, the 3D initial spectrum spreads over the transverse wavenumbers, $\mathbf{k}_\perp$, but remains of the same size in the $k_\parallel$ direction, and the support of the spectrum becomes very flat (pancake-like) for large time. The two-dimensionalisation of weak MHD turbulence has been observed in laboratory experiments [37] [49], and in the the solar wind data [4] [3] [16] [7] and in many direct numerical simulations of the three–dimensional MHD equations [33].

From the mathematical point of view, the two-dimensionalisation of the total energy means that, for large time, the energy spectrum $e^s(\mathbf{k})$ is supported on a volume of wavenumbers such that for most of them $k_\perp \gg k_\parallel$. This implies $\Psi^s(\mathbf{k})$ and $\Phi^s(\mathbf{k})$ are also supported on the same anisotropic region of wavenumbers because both of them are non-negative. This, in turn, implies that both R^s and I^s will also be non-zero only for the same region in the $\mathbf{k}$-space as $e^s(\mathbf{k})$, $\Psi^s(\mathbf{k})$ and $\Phi^s(\mathbf{k})$, as it follows from the bound (37) and (38). This fact allows one to expand the integrands in the kinetic equations in powers of small $k_\parallel/k_\perp$. At the leading order in $k_\parallel/k_\perp$, one obtains

$$\partial_t \left[k_\perp^2 \Psi^s(\mathbf{k})\right] = \tag{39}$$

$$\frac{\pi\varepsilon^2}{b_0} \int \left[\left(L_\perp^2 - \frac{X^2}{k_\perp^2}\right) \Psi^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L_\perp^2}\right) \Psi^s(\mathbf{k}) \right] X^2 \Psi^{-s}(\boldsymbol{\kappa})\, \delta(\kappa_\parallel)\, \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ ,$$

$$\partial_t \left[k_\perp^4 \Phi^s(\mathbf{k})\right] = \tag{40}$$

$$\frac{\pi\varepsilon^2}{b_0} \int \left[L_\perp^4 \Phi^s(\mathbf{L}) - k_\perp^4 \Phi^s(\mathbf{k}) \right] X^2 \Psi^{-s}(\boldsymbol{\kappa})\ \delta(\kappa_\parallel)\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ ,$$

$$\partial_t \left[k_\perp^2 R^s(\mathbf{k})\right] = \tag{41}$$

$$-\frac{\pi\varepsilon^2}{b_0} \int \left[\left(\frac{L_\perp^2 Z}{k_\perp^2}\right) R^s(\mathbf{L}) + \left(\frac{k_\perp^2}{2} + \frac{Z^2}{2L_\perp^2}\right) R^s(\mathbf{k}) \right] X^2 \Psi^{-s}(\boldsymbol{\kappa})\, \delta(\kappa_\parallel)\, \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ ,$$

$$\partial_t \left[k_\perp^2 I^s(\mathbf{k})\right] = \tag{42}$$

$$\frac{\pi\varepsilon^2}{b_0} \int \left[\left(\frac{L_\perp^2 Z}{k_\perp^2}\right) I^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{2L_\perp^2}\right) I^s(\mathbf{k}) \right] X^2 \Psi^{-s}(\boldsymbol{\kappa})\ \delta(\kappa_\parallel)\ \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\ .$$

Note that the principal value terms drop out of the kinetic equations at leading order. This property means that there is no transfer of any of the eight correlators in the $k_\parallel$ direction in $\mathbf{k}$-space.

One can see from the above that the equations for the transverse and parallel energies decouple for large time. These equations describe the shear-Alfvén and pseudo-Alfvén waves respectively. Moreover, in the large time limit, the magnetic helicity $\Sigma_s R^s$ and the polarisation correlator I^s are also described by equations which are decoupled from each other and from the parallel and transverse energies. It is interesting that the kinetic equation for the shear-Alfvén waves (*i.e.* for the transverse energy) can be obtained also from the reduced MHD equations (or RMHD) which have been derived under the same conditions of quasi two-dimensionality (see *e.g.* [41]).

An important consequence of the dynamical decoupling at different $k_\parallel$'s within the kinetic equation formalism is that the set of purely 2D modes (corresponding to $k_\parallel = 0$) evolve independently of the 3D part of the spectrum (with $k_\parallel \neq 0$) and can be studied separately. One can interpret this fact as a neutral stability of the purely 2D state with respect to 3D perturbations. As we mentioned in the Introduction, the kinetic equations themselves are applicable to a description of $k_\parallel = 0$ modes only if the correlations of the dynamical fields decay in all directions, so that their spectra are sufficiently smooth for all wavenumbers including the ones with $k_\parallel = 0$. To be precise, the characteristic $k_\parallel$ over which the spectra can experience significant changes must be greater than ϵ^2. Study of such 2D limits of 3D spectrum will be presented in the next section. It is possible, however, that in some physical situations the correlations decay slowly along the magnetic field due to a (hypothetical) energy condensation at the $k_\parallel = 0$ modes. In this case, the modes with $k_\parallel = 0$ should be treated as a separate component, a condensate, which modifies the dynamics of the 3D modes in a manner somewhat similar to the superfluid condensate, as described by Bogoliubov [20]. We leave this problem for future study, but we give some thoughts about it in the conclusion.

4 Asymptotic solution of the 3D kinetic equations

The parallel wavenumber $k_\parallel$ enters equations (39)-(42) only as an external parameter. In other words, the wavenumber space is foliated into the dynamically decoupled planes $k_\parallel = 0$. Thus, the large-time asymptotic solution can be found in the following form,

$$\Phi^s(\mathbf{k}_\perp, \mathbf{k}_\parallel) = f_1(k_\parallel)\Phi^s(\mathbf{k}_\perp, 0), \tag{43}$$

$$\Psi^s(\mathbf{k}_\perp, \mathbf{k}_\parallel) = f_2(k_\parallel)\Psi^s(\mathbf{k}_\perp, 0), \tag{44}$$

$$R^s(\mathbf{k}_\perp, \mathbf{k}_\parallel) = f_3(k_\parallel)R^s(\mathbf{k}_\perp, 0), \tag{45}$$

$$I^s(\mathbf{k}_\perp, \mathbf{k}_\parallel) = f_4(k_\parallel)I^s(\mathbf{k}_\perp, 0), \tag{46}$$

where f_i, $(i = 1,2,3,4)$ are some arbitrary functions of $k_\parallel$ satisfying the conditions $f_i(0) = 1$ (and such that the bounds (38) and (37) are satisfied). Substituting these formulae into (39)-(42), one can readily see that the functions f_i drop out of the problem, and the solution of the 3D equations is reduced to solving a 2D problem for $\Phi^s(\mathbf{k}_\perp,0), \Psi^s(\mathbf{k}_\perp,0), R^s(\mathbf{k}_\perp,0)$ and $I^s(\mathbf{k}_\perp,0)$, which will be described in the next section.

5 Two–dimensional problem

Let us consider Alfven wave turbulence which is axially symmetric with respect to the external magnetic field. Then $I^s(k_\perp,0) = 0$ because of the condition $I^s(-\mathbf{k}) = I^s(\mathbf{k})$. In the following, we will consider only solutions with $R^s = 0$. (One can easily see that R^s will remain zero if it is zero initially.) The remaining equations to be solved are

$$\frac{\partial E_\perp^s(k_\perp,0)}{\partial t} = \tag{47}$$

$$\frac{\pi\varepsilon^2}{b_0}\int (\hat{\mathbf{e}}_L\cdot\hat{\mathbf{e}}_k)^2\, sin\theta\, \frac{k_\perp}{\kappa_\perp}\, E_\perp^{-s}(\kappa_\perp,0)\left[k_\perp E_\perp^s(L_\perp,0) - L_\perp E_\perp^s(k_\perp,0)\right] d\kappa_\perp dL_\perp\,,$$

$$\frac{\partial E_\parallel^s(k_\perp,0)}{\partial t} = \tag{48}$$

$$\frac{\pi\varepsilon^2}{b_0}\int sin\theta\, \frac{k_\perp}{\kappa_\perp}\, E_\perp^{-s}(\kappa_\perp,0)\left[k_\perp E_\parallel^s(L_\perp,0) - L_\perp E_\parallel^s(k_\perp,0)\right] d\kappa_\perp dL_\perp\,,$$

where $\hat{\mathbf{e}}_k$ and $\hat{\mathbf{e}}_L$ are the unit vectors along $\mathbf{k}_\perp$ and $\mathbf{L}_\perp$ respectively and

$$E_\perp^s(k_\perp,0) = k_\perp^3 \Psi^s(k_\perp,0)\,, \tag{49}$$

$$E_\parallel^s(k_\perp,0) = k_\perp^5 \Phi^s(k_\perp,0)\,, \tag{50}$$

are the horizontal and the vertical components of the energy density. The equation (47) corresponds to the evolution of the shear-Alfvén waves for which the energy fluctuations are transverse to $\mathbf{B_0}$ whereas equation (48) describes the pseudo-Alfvén waves for which the fluctuations are along $\mathbf{B_0}$. Both waves propagate along $\mathbf{B_0}$ at the same Alfvén speed. Equation (47) describes interaction between two shear-Alfvén waves, $E_\perp^s$ and $E_\perp^{-s}$, propagating in opposite directions. On the other hand, the evolution of the pseudo-Alfvén waves depend on their interactions with the shear-Alfvén waves. The detailed energy conservation of the equation (47) implies that there is no exchange of energy between the two different kinds of waves. The physical picture in this case is that the shear-Alfvén waves interact only among themselves and evolve independently of the pseudo-Alfvén waves. The pseudo-Alfvén waves scatter from the shear-Alfvén waves without amplification or damping and they do not interact with each other.

Using a standard two–point closure of turbulence (see *e.g.* [22]) in which the characteristic time of transfer of energy is assumed known and written *a priori*, namely the EDQNM closure, Goldreich and Sridhar [13] derived a variant of the kinetic equation (47) but for strong MHD turbulence. In their analysis, the ensuing energy spectrum, which depends (as it is well known) on the phenomenological evaluation of the characteristic transfer time, thus differs from our result where the dynamics is self–consistent, closure being obtained through the assumption of weak turbulence.

It can be easily verified that the geometrical coefficient appearing in the closure equation in [13] is identical to the one we find for the $E^s_\perp(\mathbf{k}_\perp, k_\parallel)$ spectrum in the two–dimensional case. However, the two formulations, beyond the above discussion on characteristic time scales, differ in a number of ways: (i) We choose to let the flow variables to be non mirror–symmetric, whereas helicity is not taken into account in [13] where they have implicitly assumed $R^s \equiv 0$; (ii) However, because of the anisotropy introduced by the presence of a uniform magnetic field, one must take into account the coupled dynamics of the energy of the shear Alfvén waves, the pseudo–Alfvén wave and the anisotropy correlator I^s; indeed, even if initially $I^s \equiv 0$, it is produced by wave coupling and is part of the dynamics. (iii) In three dimensions, all geometrical coefficients that depend on $k^2 = k_\perp^2 + k_\parallel^2$ have a $k_\parallel$–dependence which is a function of initial conditions and again is part of the dynamics.

5.1 Kolmogorov spectra

The Zakharov transformation. The symmetry of the previous equations allows us to perform a conformal transformation, called the Zakharov transformation (also used in modeling of strong turbulence, see [22]), in order to find the exact stationary solutions of the kinetic equations as power laws [48]. This operation (see Figure 1) consists of writing the kinetic equations in dimensionless variables $\omega_1 = \kappa_\perp/k_\perp$ and $\omega_2 = L_\perp/k_\perp$, setting $E^+_\perp$ and $E^-_\perp$ by $k_\perp^{n_s}$ and $k_\perp^{n_{-s}}$ respectively, and then rearranging the collision integral by the transformation

$$\omega_1' = \frac{\omega_1}{\omega_2} \, , \tag{51}$$

and

$$\omega_2' = \frac{1}{\omega_2} \, . \tag{52}$$

The new form of the collision integral, resulting from the summation of the integrand in its primary form and after the Zakharov transformation, is

$$\partial_t E_\perp^s \sim \int \left(\frac{\omega_2^2 + 1 - \omega_1^2}{2\omega_2}\right)^2 \left(1 - (\frac{\omega_1^2 + 1 - \omega_2^2}{2\omega_1})^2\right)^{1/2} \omega_1^{n_{-s}-1}\omega_2$$

$$(\omega_2^{n_s-1} - 1)(1 - \omega_2^{-n_s-n_{-s}-4}) \, d\omega_1 \, d\omega_2 \, .$$

The collision integral can be null for specific values of n_s and n_{-s}. The exact solutions, called the Kolmogorov spectra, correspond to these values which satisfy

$$n_s + n_{-s} = -4 \, . \tag{53}$$

It is important to understand that Zakharov transformation is not an identity transformation, and it can lead to spurious solutions. The necessary and sufficient condition for a spectrum obtained by Zakharov transformation to be a solution of the kinetic equation is that the right hand side integral in (47) (*i.e.* before the Zakharov transformation) equation converges. This condition is called the locality of the spectrum and leads to the following restriction on the spectral indices in our case:

$$-3 < n_s \, , \; n_{-s} < -1 \, . \tag{54}$$

In the particular case of a zero cross–correlation one has $E_\perp^+ = E_\perp^- = E_\perp \sim k_\perp^n$ with only one solution

$$n = -2 \, .$$

Note that the thermodynamic equilibrium, corresponding to the equipartition state for which the flux of energy is zero instead of being finite as in the above spectral forms, corresponds to the choices $n_s = n_{-s} = 1$ for both the perpendicular and the parallel components of the energy.

The Kolmogorov constants $C_K(n_s)$ and $C_K'(n_s)$. The final expression of the Kolmogorov–like spectra found above as a function of the Kolmogorov constant (generalised to MHD) $C_K(n_s)$ and of the flux of energy $P_\perp^s(k_\perp)$ can

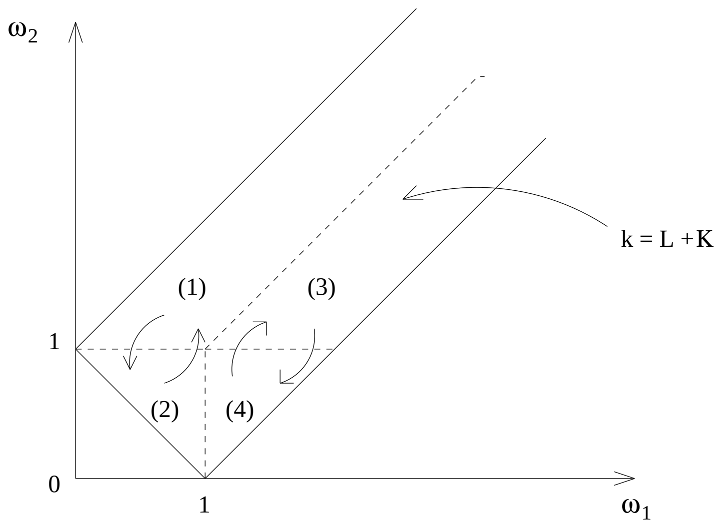

Fig. 1. Geometrical representation of the Zakharov transformation. The rectangular region, corresponding to the triad interaction $\mathbf{k}_\perp = \mathbf{L} + \boldsymbol{\kappa}$, is decomposed into four different regions called (1), (2), (3) and (4); ω_1 and ω_2 are respectively the dimensionless variables $\kappa_\perp/k_\perp$ and $L_\perp/k_\perp$. The Zakharov transformation applied to the collision integral consists to exchange regions (1) and (2), and regions (3) and (4).

be obtained in the following way. For a better understanding, the demonstration will be done in the simplified case of a zero cross–correlation. The generalization to the correlated case ($E^+ \neq E^-$) is straightforward. Using the definition of the flux,

$$\partial_t E_\perp(k_\perp, 0) = -\partial_{k_\perp} P_\perp(k_\perp) \ , \tag{55}$$

one can write the flux of energy as a function of the collision integral (with the new form of the integrand) depending on n_s. Then the limit $n_s \to -2$ is taken in order to have a constant flux $P_\perp$ with no more dependence in $k_\perp$, as it is expected for a stationary spectrum in the inertial range. Here we have considered an infinite inertial range to use the Zakharov transformation. Whereas the collision integral tends to zero when $n_s \to -2$, the limit with which we are concerned is not zero because of the presence of a denominator proportional to $2n_s + 4$, and which is a signature of the dimension in wavenumber of the flux. Finally the “L'Hospital's rule” gives the value of $P_\perp$ from which it is possible to write the Kolmogorov spectrum of the shear-Alfvén waves

$$E_\perp(k_\perp, 0) = P_\perp^{1/2}\; C_K(-2)\; k_\perp^{-2}\;, \tag{56}$$

with the Kolmogorov constant

$$C_K(n_s) = \sqrt{\frac{-2b_0}{\pi\epsilon^2 J_1(n_s)}}\;, \tag{57}$$

and with the following form for the integral $J_1(n_s)$

$$J_1(n_s) = 2^{n_s+3} \int_{x=1}^{+\infty} \int_{y=-1}^{1} \frac{\sqrt{(x^2-1)(1-y^2)}\;(xy+1)^2}{(x-y)^{n_s+6}\;(x+y)^{2-n_s}} \tag{58}$$
$$\left[2^{1-n_s} - (x+y)^{1-n_s}\right] ln\left(\frac{x+y}{2}\right)\; dxdy\;.$$

As expected, the calculation gives a negative value for the integral $J_1(n_s)$ and for the particular value $n_s = -2$, we obtain $C_K(-2) \simeq 0.585$. Note that the integral $J_1(n_s)$ converges only for $-3 < n_s < -1$.

The generalization to the case of non–zero cross–correlation gives the relations

$$\begin{aligned} E_\perp^+(k_\perp, 0)\; E_\perp^-(k_\perp, 0) &= P_\perp^+\; C_K^2(n_s)\; k_\perp^{-4} = P_\perp^-\; C_K^2(-n_s-4)\; k_\perp^{-4} \\ &= \sqrt{P_\perp^+ P_\perp^-}\, C_K(n_s)\, C_K(-n_s-4)\, k_\perp^{-4}\;, \end{aligned} \tag{59}$$

where the second formulation is useful to show the symmetry with respect to s. The computation of the Kolmogorov constant C_K as a function of $-n_s$ is given in Figure 2. An asymmetric form is observed which means the ratio $P_\perp^+/P_\perp^-$ is not constant, as we can see in Figure 3 where we plot this ratio as a function of $-n_s$. We see that for any ratio $P_\perp^+/P_\perp^-$ there corresponds a unique value of n_s, between the singular ratios $P_\perp^+/P_\perp^- = +\infty$ for $n_s = -3$ and $P_\perp^+/P_\perp^- = 0$ for $n_s = -1$. Thus, a larger flux of energy P^+ corresponds to a steeper slope of the energy spectra $E_\perp^+(k_\perp, 0)$ in agreement with the physical image that a larger flux of energy implies a faster energy cascade.

In the zero cross–correlation case, a similar demonstration for the pseudo-Alfvén waves $E_\parallel^s(k_\perp, 0)$ leads to the relation

$$E_\parallel(k_\perp, 0) = P_\parallel P_\perp^{-1/2}\; C_K'(-2)\; k_\perp^{-2}\;, \tag{60}$$

with the general form of the Kolmogorov constant

$$C_K'(n_s) = \sqrt{\frac{-2b_0 J_1(n_s)}{\pi\epsilon^2 J_2(n_s) J_2(-n_s-4)}}\;, \tag{61}$$

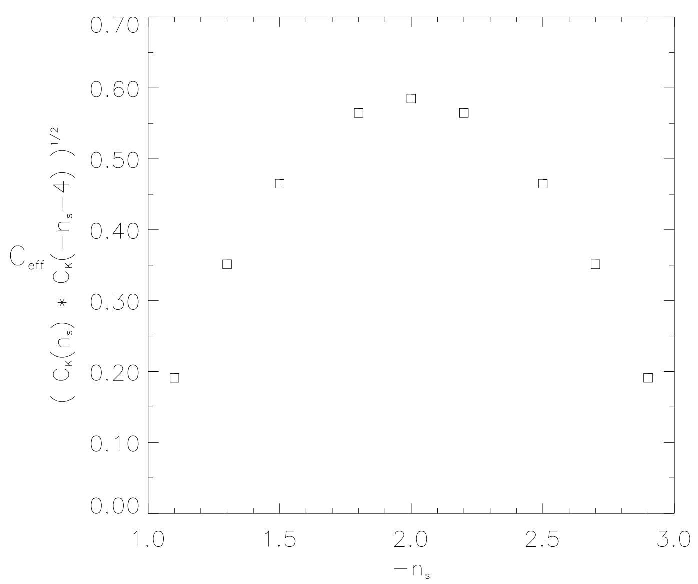

Fig. 2. Variation of $\sqrt{C_K(n_s)\, C_K(-n_s-4)}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s$ corresponding to the case of zero velocity-magnetic field correlation.

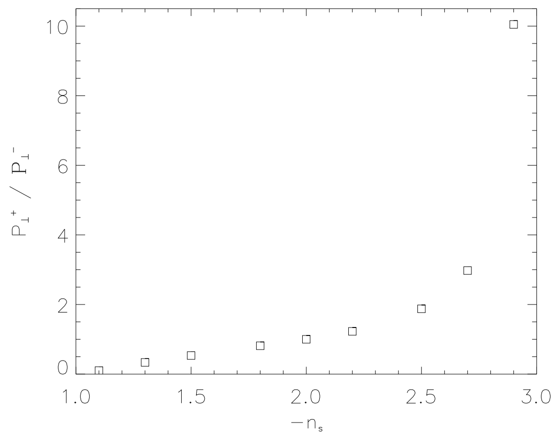

Fig. 3. Variation of $P_\perp^+/P_\perp^-$, the ratio of fluxes of energy, as a function of $-n_s$. For the zero cross correlation case the ratio is 1.

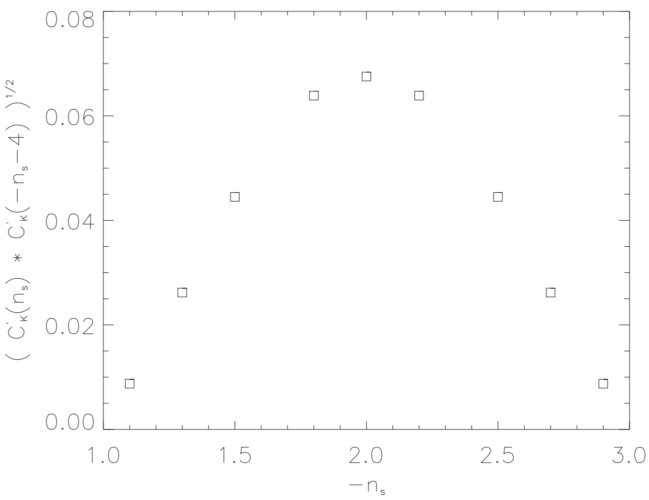

Fig. 4. Variation of $\sqrt{C'_K(n_s)\ C'_K(-n_s-4)}$ as a function of $-n_s$. Notice the symmetry around the value $-n_s$ corresponding to the zero cross correlation case.

where the integral $J_2(n_s)$ is

$$J_2(n_s) = 2^{n_s+3} \int_{x=1}^{+\infty} \int_{y=-1}^{1} \frac{\sqrt{(x^2-1)(1-y^2)}}{(x-y)^{n_s+6}\ (x+y)^{2-n_s}} \tag{62}$$

$$\left[2^{1-n_s} - (x+y)^{1-n_s}\right] ln \left(\frac{x+y}{2}\right)\ dxdy\ .$$

Note that the integral $J_2(n_s)$ converges only for $-3 < n_s < -1$. The presence of the flux $P_\perp$ in the Kolmogorov spectrum is linked to the presence of $E_\perp$ in the kinetic equation of $E_\parallel$. A numerical evaluation gives $C'_K(-2) \simeq 0.0675$ whereas the generalization for the non–zero cross–correlation is

$$E^+_\parallel(k_\perp,0)\ E^-_\parallel(k_\perp,0) = \frac{P^+_\parallel P^-_\parallel}{P^+_\perp}\ C'^2_K(n_s)\ k_\perp^{-4} = \frac{P^+_\parallel P^-_\parallel}{P^-_\perp}\ C'^2_K(-n_s-4)\ k_\perp^{-4}$$

$$= \frac{P^+_\parallel P^-_\parallel}{\sqrt{P^+_\perp P^-_\perp}}\ C'_K(n_s)\ C'_K(-n_s-4)\ k_\perp^{-4}\ , \tag{63}$$

where the last formulation shows the symmetry with respect to s. The power laws of the spectra $E^s_\parallel$ have the same indices than those of $E^s_\perp$ and the

Kolmogorov constant C' is in fact related to C by the relation

$$\frac{C'_K(n_s)}{C'_K(-n_s-4)} = \frac{C_K(-n_s-4)}{C_K(n_s)} \,. \tag{64}$$

Therefore the choice of the ratio $P_\perp^+/P_\perp^-$ determines not only $C_K(n_s)$ but also $C'_K(n_s)$, allowing for free choices of the dissipative rates of energy $P_\parallel^\pm$.

The result of the numerical evaluation of $C'_K(n_s)$ is shown in Figure 4. An asymmetrical form form is also visible ; notice also that the values of $C'_K(n_s)$ (*i.e.* the constant in front of the parallel energy spectra) are smaller by an order of magnitude than those of $C_K(n_s)$ for the perpendicular spectra.

5.2 Temporal evolution of the kinetic equations

Numerical method. Equations (47) and (48) can be integrated numerically with a standard method, as for example presented in [21]. Since that the energy spectrum varies smoothly with k, it is convenient to use a logarithmic subdivision of the k axis

$$k_i = \delta k \; 2^{i/F} \ , \tag{65}$$

where i is a non–negative integer ; δk is the minimum wave number in the computation and F is the number of wave numbers per octave. F defines the refinement of the "grid", and in particular it is easily seen that a given value of F introduces a cut–off in the degree of non–locality of the nonlinear interactions included in the numerical computation of the kinetic equations. But since the solutions are local, a moderate value of F can be used (namely, we take $F = 4$). Tests have nevertheless been performed with $F = 8$ and we show that no significant changes occur in the results to be described below.

This technique allows us to reach Reynolds numbers much greater than in direct numerical simulations. In order to regularize the equations at large k, we have introduced dissipative terms which were omitted in the derivation of the kinetic equations. We take the magnetic Prandtl number (ν/η) to be unity. For example, with $\delta k = 2^{-3}$, $F = 8$, $i_{max} = 225$; this corresponds to a ratio of scales $2^{28}/2^{-3}$. Taking a wave energy U_0^2 and an integral scale L_0 both of order one initially, and a kinematic viscosity of $\nu = 3.3 \times 10^{-8}$, the Reynolds number of such a computation is $\mathcal{R}_e = U_0 \; L_0/\nu \sim 10^8$. All numerical simulations to be reported here have been computed on an Alpha Server 8200 located at the Observatoire de Nice (SIVAM).

Shear-Alfvén waves. In this paper, we only consider decaying turbulence. As a first numerical simulation we have integrated the equation (47) in the zero cross–correlation case ($E^+ = E^-$) and without forcing. Figure 5 (top) shows the temporal evolution of the total energy $E_\perp(t)$ with by definition

$$E_\perp(t) = \int_{k_{min}}^{k_{max}} E_\perp(k_\perp, 0)\, dk_\perp \ , \tag{66}$$

where k_{min} and k_{max} have the values given in the previous section. The total

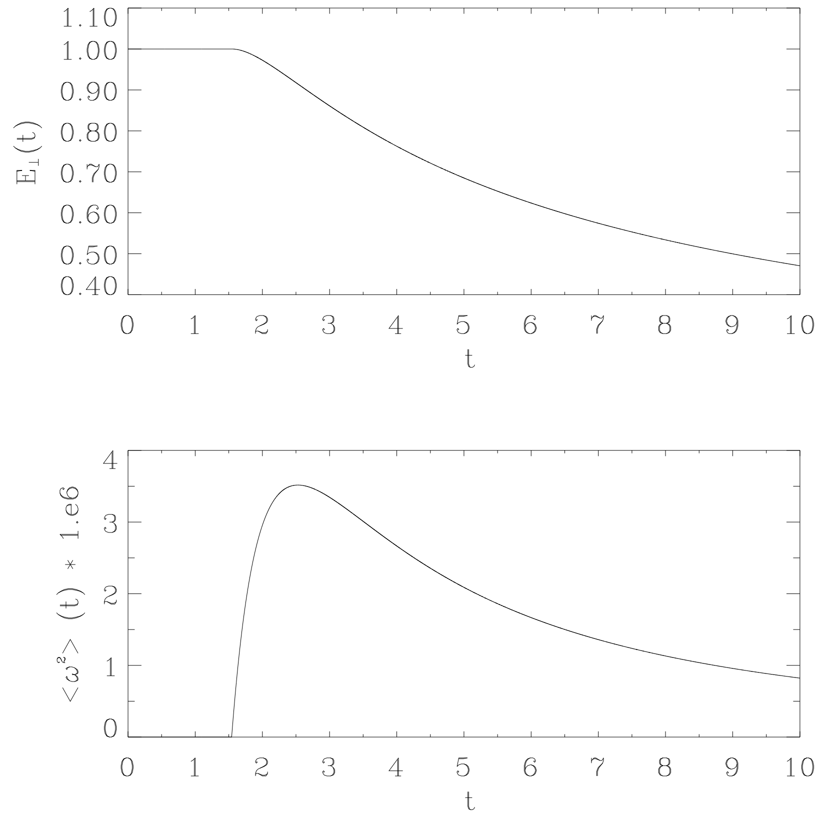

Fig. 5. Temporal evolution of the energy $E_\perp(t)$ (top) and the enstrophy $< \omega^2 > (t)$ normalized to 1.10^6. Notice the conservation of the energy up to the time t_*.

energy is conserved up to a time $t_* \simeq 1.6$ after which it decreases because of the dissipative effects linked to mode coupling, whereas the enstrophy

$\int k^2 E_\perp(k)\, d^2\mathbf{k}$ increases sharply (bottom of Figure 5). The energy spectra at different times are displayed in Figure 6. As we approach the time t_*, the spectra spread out to reach the smallest scales (*i.e.* the largest wavenumbers). For $t > t_*$, constant energy flux spectrum $k_\perp^{-2}$ is obtained (indicated by the straight line). For times t significantly greater than t_*, we have a self-similar energy decay, in what constitutes the turbulent regime.

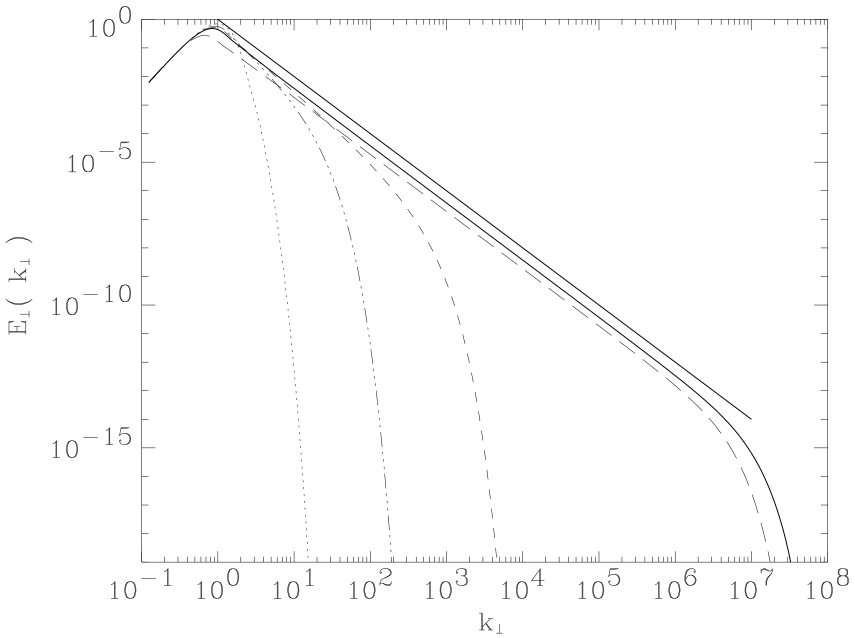

Fig. 6. Energy spectra $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves in the zero cross correlation case for the times $t = 0$ (dot), $t = 1.0$ (dash-dot), $t = 1.5$ (small-dash), $t = 1.6$ (solid) and $t = 10.0$ (long-dash) ; the straight line follows a $k_\perp^{-2}$.

Shear-Alfvén *versus* **pseudo-Alfvén waves.** In a second numerical computation we have studied the system (47) (48) with an initial normalised cross–correlation of 80%. The following parameters have been used : $\delta k = 2^{-3}$, $F = 4, i_{max} = 105$ and $\nu = 6.4\,10^{-8}$. Figure 7 (top) shows the temporal evolution of energies for the four different waves ($E_\perp^\pm$ and $E_\parallel^\pm$). The same behavior as that of Figure 5 (top) is observed, with a conservation of energy up to the

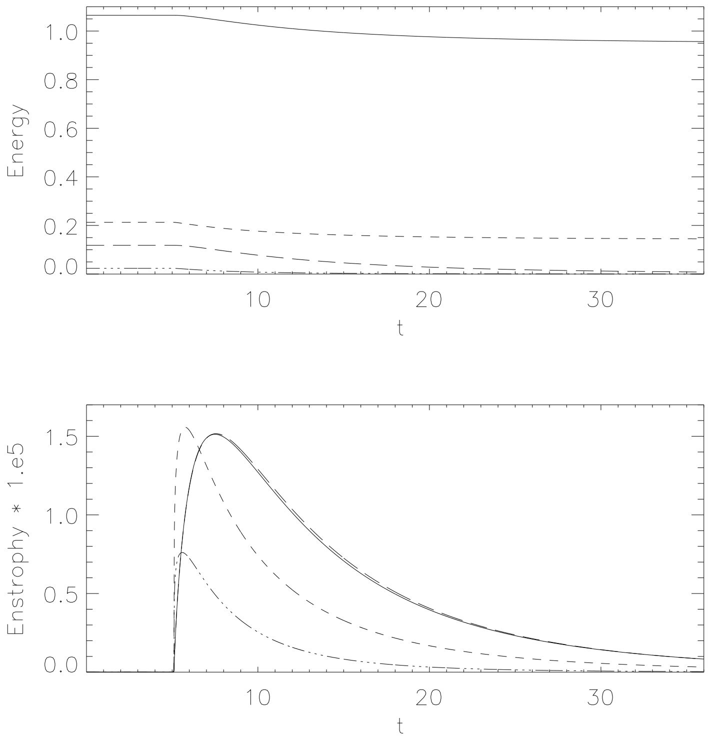

Fig. 7. Temporal evolution of energies (top) $E_\perp^+$ (solid), $E_\perp^-$ (long-dash), $E_\parallel^+$ (small-dash) and $E_\parallel^-$ (dash-dot) ; the same notation is used for the enstrophies (bottom) which are normalized to $1.\,10^5$. Notice that energies are conserved till the time t'_*.

time $t'_* \simeq 5$, and a decay afterwards ; this decay is nevertheless substantially weaker than when the correlation is zero, since in the presence of a significant amount of correlation between the velocity and the magnetic field. It is easily seen from the primitive MHD equations that the nonlinearities are strongly reduced. On the other hand the temporal evolution of enstrophies (bottom) displays that the maxima for these four types of waves are reached at different times : the pseudo-Alfvén waves are the fastest to reach their maxima at $t \simeq 5.5$ *vs.* $t \simeq 7.5$ for the shear-Alfvén waves. Figure 8 corresponds to the temporal evolution of another conserved quantity, the cross correlation ρ_x defined as

$$\rho_x = \frac{E_x^+ - E_x^-}{E_x^+ + E_x^-} \ , \tag{67}$$

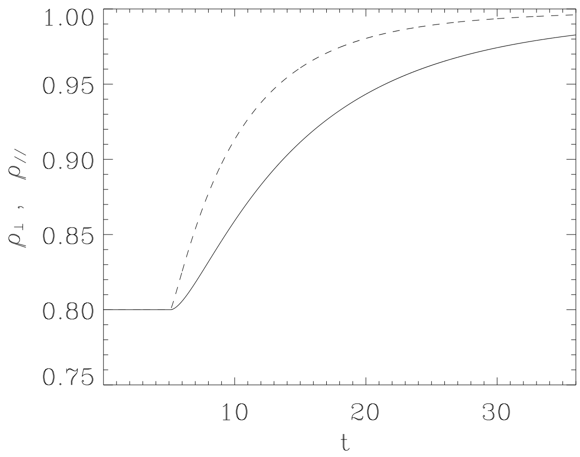

Fig. 8. Temporal evolution of the cross correlations $\rho_\perp$ (solid) and $\rho_\parallel$ (dash). These quantities are conserved up to the time t'_*.

where x symbolizes either $\perp$ or $\|$. As expected, ρ_x is constant during an initial period (till $t = t'_*$) and then tends asymptotically to one, but in a faster way for the pseudo-Alfvén waves. This growth of correlation is well documented in the isotropic case [24] and is seen to hold as well here in the weak turbulence regime. Figures 9 and 10 give the compensated spectra $E^+_\perp E^-_\perp k^4_\perp$ and $E^+_\parallel E^-_\parallel k^4_\perp$ respectively at different times. In both cases, from $t = 6$ onward, a plateau is observed over almost four decades and remains flat for long times; this confirms nicely the theoretical predictions (59) and (63).

As a final note, we remark that the analysis of the complete set of eight equations in dimension three as derived in section §2.4 remains to be done and is left for future work.

6 Front propagation

The numerical study of the transition between the initial state and the final state, where the $k_\perp^{-2}$–spectrum is reached, shows two remarkable properties illustrated by Figure 11 and 12.

We show in Figure 11 (top), in lin-log coordinates, the progression with time of the front of energy propagating to small scales; more precisely, we

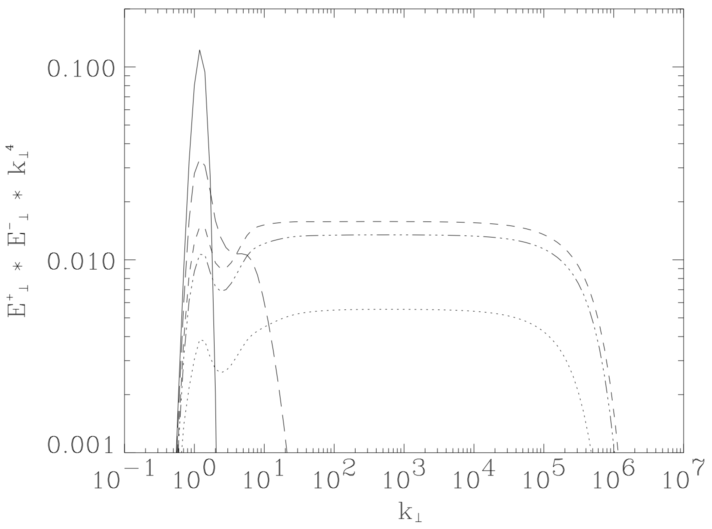

Fig. 9. Compensated spectra $E_\perp^+ E_\perp^- k_\perp^4$ at times $t = 0$ (solid), $t = 4$ (long-dash), $t = 6$ (small-dash), $t = 8$ (dash-dot) and $t = 20$ (dot).

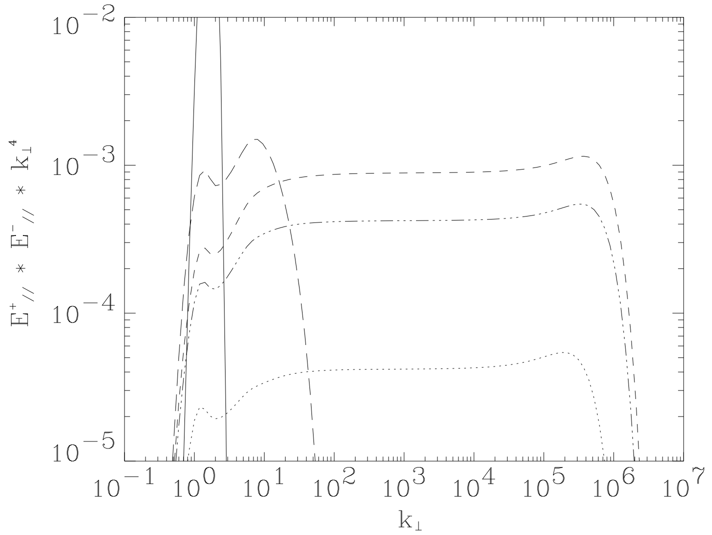

Fig. 10. Compensated spectra $E_\parallel^+ E_\parallel^- k_\perp^4$ for the same times and with the same legend as in Figure 9.

give the wavenumber at time t with an energy of, respectively, 10^{-25} (dash-dot line) and 10^{-16} (solid line). Note that all curves display an abrupt change at $t_0 \simeq 1.55$, after which the growth is considerably slowed down. Using this data, Figure 11 (bottom) gives $log(k_\perp)$ as a function of $log(1.55-t)$; the lines having the same meaning as in Figure 11 (top); the large dash represents a power law $k_\perp \sim (1.55-t)^{-1.5}$. Hence, the small scales, in this weak turbulence formalism, are reached in a finite time *i.e.* in a catastrophic way. This is also seen on the temporal evolution of the enstrophy (see bottom of Figure 5), with a catastrophic growth ending at $t \simeq 2.5$, after which the decay of energy begins.

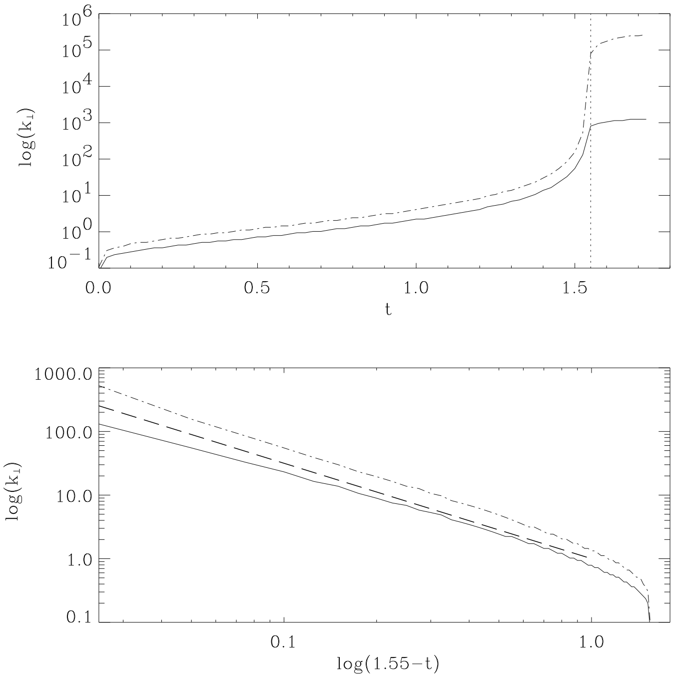

Fig. 11. Temporal evolution (top), in lin-log coordinates, of the front of energy propagating to small scales. The solid line and the dash-dot line correspond respectively to an energy of 10^{-16} and 10^{-25}. An abrupt change is visible at time $t_0 \simeq 1.55$ (vertical dotted line). The $log(k_\perp)$ as a function of $log(1.55-t)$ (bottom) displays a power law in $k_\perp \sim (1.55-t)^{-1.5}$ (large dash line).

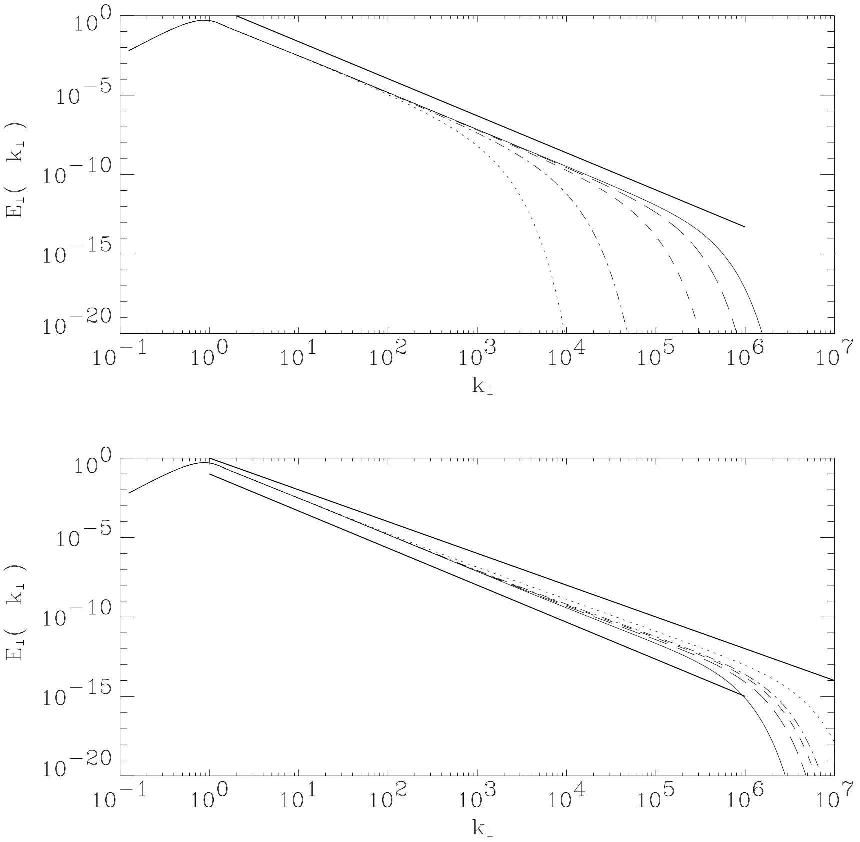

Fig. 12. Temporal evolution of the energy spectrum $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves around the catastrophic time $t_0 \simeq 1.544$. For $t < t_0$ (top) ($t = 1.50$ (dot), $t = 1.53$ (dash-dot), $t = 1.54$ (dash), $t = 1.542$ (long dash) and $t = 1.543$ (solid)) a $k_\perp^{-7/3}$– spectrum is observed. For $t \geq t_0$ (bottom) ($t = 1.544$ (solid), $t = 1.546$ (long dash), $t = 1.548$ (dash), $t = 1.55$ (dash-dot) and $t = 1.58$ (dot)) a fast change of the slope appears to give finally a $k_\perp^{-2}$– spectrum. Note that this change propagates from small scales to large scales. In both cases straight lines follow either a $k_\perp^{-7/3}$ or a $k_\perp^{-2}$.

Figure 12 shows the temporal evolution of the energy spectrum $E_\perp(k_\perp, 0)$ of the shear-Alfvén waves around the catastrophic time t_0. We see that before t_0, evaluated here with a better precision to 1.544, the energy spectrum propagates to small scales following a stationary $k_\perp^{-7/3}$– spectrum and not a $k_\perp^{-2}$– spectrum. It is only when the dissipative scale is reached, at t_0, that a remarkable effect is observed: in a very fast time the $k_\perp^{-7/3}$ solution turns into the finite energy flux spectrum $k_\perp^{-2}$ with a change of the slope propagating from small scales to large scales.

Note that this picture is different from the scenario proposed by Falkovich and Shafarenko (FS) (1991) for the finite capacity spectra. According to FS, the Kolmogorov spectrum forms right behing the propagating front, whereas in our case it forms only after the front reaches infinite wavenumbers (*i.e.* dissipative region). The front propagation can be described in terms of self-similar solutions having a form [11] [48]

$$E_\perp(k_\perp, 0) = \frac{1}{\tau^a} E_0(\frac{k_\perp}{\tau^b}) \,, \tag{68}$$

where $\tau = t_0 - t$. Substituting (68) into the kinetic equation (47) we have

$$\partial_\tau \left(\frac{1}{\tau^a} E_0(\frac{k_\perp}{\tau^b}) \right) \sim \tau^{-a} E_0(\frac{\kappa_\perp}{\tau^b}) \left(\tau^{b-a} E_0(\frac{L_\perp}{\tau^b}) - \tau^{b-a} E_0(\frac{k_\perp}{\tau^b}) \right) \tau^{2b} \,.$$

which leads to the relation

$$1 + 3b = a \,. \tag{69}$$

If E_0 is stationary and has a power-law form $E_0 \sim k^m$, then we have another relation between a and b

$$a + mb = 0 \tag{70}$$

Excluding a from (69) and (70) we have $1 + (3 + m)b = 0$. In our case this condition is satisfied because $b = -3/2$ and $m = -7/3$ which confirms that the front solution is of self-similar type.

7 Nonlocal isotropic 3D MHD turbulence

We have considered until now a turbulence of Alfvén waves that arises in the presence of a strong uniform magnetic field. Following Kraichnan [19], one can assume that the results obtained for turbulence in a strong external magnetic field are applicable to MHD turbulence at small scales which experience the magnetic field of the large-scale component as a quasi-uniform external field.

Furthermore, the large-scale magnetic field is much stronger than the one produced by the small-scales themselves because most of the MHD energy is condensed at large scales due to the decreasing distribution of energy among modes as the wavenumbers grow. In this case therefore, the small-scale dynamics consists again of a large number of weakly interacting Alfvén waves. Using such a nonlocality hypothesis and applying a dimensional argument, Kraichnan derived the $k^{-3/2}$ energy spectrum for MHD turbulence. However, Kraichnan did not take into account the local anisotropy associated with the presence of this external field. In [32], the dimensional argument of Kraichnan is modified in order to take into account the anisotropic dependence of the characteristic time associated with Alfvén waves on the wavevector by simply writing

$$\tau \sim \frac{1}{b_0 k_\|} \ . \tag{71}$$

In that way, one obtains a $k_\perp^{-2}$ energy spectrum, which agrees with the analytical and numerical results of the present paper for the spectral dependence on $k_\perp$. On the other hand, the dependence of the spectra on $k_\|$, as we showed before in this paper, is not universal because of the absence of energy transfer in the $k_\|$ direction, although it is shown in [18] that for a quasi–uniform field as considered in this section, there is some transfer in the quasi–parallel direction. In the strictly uniform case, this spectral dependence is determined only by the dependence on $k_\|$ of the driving and/or initial conditions.

For large time, the spectrum is almost two-dimensional. The characteristic width of the spectrum in $k_\|$ (described by the function $f_1(k_\|)$) is much less than its width in $k_\perp$, so that approximately one can write

$$e^s(\mathbf{k}) = C\, k_\perp^{-3}\, \delta(k_\|), \tag{72}$$

where C is a constant. The $k_\perp^{-3}$ factor corresponds to the $E_\perp \propto k_\perp^{-2}$ Kolmogorov–like spectrum found in this paper (the physical dimensions of e^s and $E_\perp^s$ being different). In the context of MHD turbulence, this spectrum is valid only locally, that is for distances smaller than the length–scale of the magnetic field associated with the energy–containing part of MHD turbulence. Let us average this spectrum over the large energy containing scales, that is over all possible directions of $\mathbf{B_0}$. Writing $k_\perp = |\mathbf{k} \times \mathbf{B_0}|/|B_0|$ and $k_\| = |\mathbf{k} \cdot B_0|/|B_0|$ and assuming that $\mathbf{B_0}$ takes all possible directions in 3D space with equal probability, we have for the averaged spectrum

$$< e^s >= \int e^s(\mathbf{k}, \mathbf{x})\, d\sigma(\zeta) = \int C\delta(\zeta \cdot \mathbf{k})\, |\zeta \times \mathbf{k}|^{-3} d\sigma(\zeta), \tag{73}$$

where $\zeta = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ is a unit vector in the coordinate space and $d\sigma = \sin\theta \, d\theta \, d\phi$ is the surface element on the unit sphere. Choosing θ to be the angle between $\mathbf{k}$ and $\mathbf{B_0}$ and ϕ to be the angle in the transverse to $\mathbf{k}$ plane, we have

$$< e^s >= C \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\delta(\cos\theta)}{|k|} \frac{(\sin\theta)^{-3}}{|k|^3} \sin\theta \, d\theta = 2\pi C k^{-4}. \qquad (74)$$

This isotropic spectrum represents the averaged energy density in 3D wavevector space. By averaging over all possible directions of the wavevector, we get the following density of the energy distribution over the absolute value of the wavevector,

$$E_k = 8\pi^2 C \, k^{-2}. \qquad (75)$$

As we see, taking into account the local anisotropy and subsequent averaging over the isotropic energy containing scales results in an isotropic energy spectrum k^{-2}. This result is different from the $k^{-3/2}$ spectrum derived by Kraichnan without taking into account the local anisotropy of small scales.

The difference in spectral indices may also arise from the fact that the approach here is that of weak turbulence, whereas in the strong turbulence case, isotropy is recovered on average and a different spectrum – that of Kraichnan – obtains. Solar wind data [25] indicates that the isotropic spectrum scales as $k^{-\alpha}$ with $\alpha \sim 1.67$, close to the Kolmogorov value for neutral fluids (without intermittency corrections which are known to occur) ; hence, it could be interpreted as well as being a Kraichnan–like spectrum steepened by intermittency effects which are known to take place in strong MHD turbulence as well in the form of current and vorticity filaments and magnetic flux tubes.

8 Conclusion

We have obtained in this paper the kinetic equations for weak Alfvénic turbulence in the presence of correlations between the velocity and the magnetic field, and taking into account the non–mirror invariance of the MHD equations leading to non–zero helical terms. These equations, contrary to what is stated in [40], obtain at the level of three–wave interactions.

In this anisotropic medium, a new spectral tensor must be taken into account in the formalism when compared to the isotropic case (which can include terms proportional to the helicity) ; this new spectral tensor I^s is linked to the anisotropy induced by the presence of a strong uniform magnetic field, and we can also study its dynamics. This purely anisotropic correlator was also analysed in the case of neutral fluids in the presence of rotation [8].

We obtain an asymptotic two-dimensionalisation of the spectra: indeed, the evolution of the turbulent spectra at each $k_\parallel$ is determined only by the spectra at the same $k_\parallel$ and by the purely 2D state characterized by $k_\parallel = 0$. This property of bi–dimensionalisation was previously obtained theoretically from an analysis of the linearised MHD equations [29] and using phenomenological models [31], and numerically as well [33], whereas they are obtained in our paper from the rigorously derived kinetic equations. Note that the strong field $\mathbf{B_0}$ has no structure (it is a $\mathbf{k} = 0$ field), whereas the analysis performed in [18] considers a strong quasi–uniform magnetic field of characteristic wavenumber $k_L \neq 0$, in which case the authors find that bi–dimensionalisation obtains as well for large enough wavenumbers.

A $k_\perp^{-2}$ energy spectrum is obtained for the inertial range turbulence, well verified numerically, and reached in a singular fashion with small scales developing in a finite time. We also obtain a family of Kolmogorov solutions with different values of spectra for different wave polarities and we show that the sum of the spectral exponents of these spectra is equal to -4. The dynamics of both the shear-Alfvén waves and the pseudo-Alfvén waves is obtained. Finally, the small-scale spectrum of nonlocal 3D MHD turbulence in the case when there is no external field is also derived.

In ongoing work, we are considering three–dimensional turbulence in the asymptotic regime of large time when the spectrum tends to a quasi-2D form. This is the same regime for which RMHD approach is valid [41]. However, in addition to the shear-Alfvén waves described by the RMHD equations, the kinetic equations describe also the dynamics of the so-called pseudo-Alfvén waves which are decoupled from the shear-Alfvén waves in this case, from the magnetic helicity and from the polarisation covariance. Finding the Kolmogorov solution for the 3D case is technically very similar to the case of 2D turbulence. Preliminary results indicate that we obtain for the energy spectrum, $f(k_\parallel)\, k_\perp^{-2}$ where $f(k_\parallel)$ is an arbitrary function which is to be fixed by matching in the forcing region at small wavenumbers. The $k_\parallel$ dependence is non-universal and depends on the form of the forcing because of the property that there is no energy transfer between different $k_\parallel$'s.

The weak turbulence regime remains valid as long as the Alfvén characteristic time $(k_\parallel b_0)^{-1}$ remains small compared to the eddy turn–over time of the waves $(k_\perp z_l)^{-1}$ where z_l is the characteristic Elsässer field at scale l.

Using the exact scaling law found in this paper ($E(\mathbf{k}) \sim k_\perp^{-2}$) together with a local analysis ($z_l^2 \sim kE(\mathbf{k})$) and recalling that $k_\parallel$ is negligible compared to $k_\perp$, one arrives at a scaling $k_\perp \sim B_0^2$ for this to happen.

The dynamo problem in the present formalism reduces to its simplest expression: in the presence of a strong uniform magnetic field $\mathbf{B_0}$, to a first approximation (closing the equations at the level of second–order correlation tensors), one obtains immediate equipartition between the kinetic and the magnetic wave energies, corresponding to an instantaneous efficiency of the dynamo. Of course, one may ask about the origin of $\mathbf{B_0}$ itself, in which case one may resort to standard dynamo theories (see *e.g.* [34]). We see no tendency towards condensation.

In view of the ubiquity of turbulent conducting flows embedded in strong quasi–uniform magnetic fields, the present derivation should be of some use when studying the dynamics of such media, even though compressibility effects have been ignored. However, it can be argued [13] that this incompressible approximation may be sufficient (see also [6]), because of the damping of the fast magnetosonic wave by plasma kinetic effects. Finally, the wave energy may not remain negligible for all times, in which case resort to phenomenological models for strong MHD turbulence is required. Is desirable as well an exploration of such complex flows through analysis of laboratory and numerical experiments, and through detailed observations like those stemming from satellite data for the solar wind, from the THEMIS instrument for the Sun looking at the small–scale magnetic structures of the photosphere, and the planned large array instrumentation (LSA) to observe in detail the interstellar medium.

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9 Appendix

From the dynamical equations (33) one writes successively for the second and third–order moments of the $\mathbf{z}^s$ fields:

$$\partial_t \left\{ a^s_j(\mathbf{k}) a^{s'}_{j'}(\mathbf{k}') \right\} = -i\epsilon k_m P_{jn}(\mathbf{k}) \int \left\{ a^s_m(\boldsymbol{\kappa}) a^s_n(\mathbf{L}) a^{s'}_{j'}(\mathbf{k}') \right\} e^{-2is\omega_\kappa t} \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}} d_{\boldsymbol{\kappa}\mathbf{L}}$$
$$-i\epsilon k'_m P_{j'm}(\mathbf{k}') \int \left\{ a^{-s'}_m(\boldsymbol{\kappa}) a^{s'}_n(\mathbf{L}) a^s_j(\mathbf{k}) \right\} e^{-2is'\omega_\kappa t} \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'} d_{\boldsymbol{\kappa}\mathbf{L}} \tag{76}$$

and

$$\partial_t \left\{ a^s_j(\mathbf{k}) a^{s'}_{j'}(\mathbf{k}') a^{s''}_{j''}(\mathbf{k}'') \right\} = \tag{77}$$
$$-i\epsilon k_m P_{jn}(\mathbf{k}) \int \left\{ a^{-s}_m(\boldsymbol{\kappa}) a^s_n(\mathbf{L}) a^{s'}_{j'}(\mathbf{k}') a^{s''}_{j''}(\mathbf{k}'') \right\} e^{-2is\omega_\kappa t} \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}} d_{\boldsymbol{\kappa}\mathbf{L}}$$
$$+ \{[\mathbf{k}, s, j] \to [\mathbf{k}', s', j'] \to [\mathbf{k}'', s'', j''] \to [\mathbf{k}, s, j]\}$$

The fourth–order moment in the above equation, $\langle \boldsymbol{\kappa}\mathbf{L}\mathbf{k}'\mathbf{k}'' \rangle$ in short–hand notation, decomposes into the sum of three products of second–order moments, and a fourth–order cumulant $\{mnj'j''\}$. The latter does not contribute to secular behavior, and of the remaining terms one is absent as well in the kinetic equations because it involves the combination of wavenumbers $\langle \boldsymbol{\kappa}\mathbf{L} \rangle \langle \mathbf{k}'\mathbf{k}'' \rangle$: it introduces, because of homogeneity, a $\delta(\boldsymbol{\kappa} + \mathbf{L})$ factor which combined with the convolution integral leads to a zero contribution for $\mathbf{k} = 0$. Hence, the third–order cumulant leads to six terms that read:

$$\partial_t \left\{ a^s_j(\mathbf{k}) a^{s'}_{j'}(\mathbf{k}') a^{s''}_{j''}(\mathbf{k}'') \right\} = -i\epsilon k_m P_{jn}(\mathbf{k}) q^{-ss'}_{mj'}(\mathbf{k}') q^{ss''}_{nj''}(\mathbf{k}'') e^{2is\omega' t}$$
$$-i\epsilon k_m P_{jn}(\mathbf{k}) q^{-ss''}_{mj''}(\mathbf{k}'') q^{ss'}_{nj'}(\mathbf{k}') e^{2is\omega'' t} - i\epsilon k'_m P_{j'n}(\mathbf{k}') q^{-s's''}_{mj''}(\mathbf{k}'') q^{s's}_{nj}(\mathbf{k}) e^{2is'\omega'' t}$$
$$-i\epsilon k'_m P_{j'n}(\mathbf{k}') q^{-s's}_{mj}(\mathbf{k}) q^{s's''}_{nj''}(\mathbf{k}'') e^{2is'\omega t} - i\epsilon k''_m P_{j''n}(\mathbf{k}'') q^{-s''s}_{mj}(\mathbf{k}) q^{s''s'}_{nj'}(\mathbf{k}') e^{2is''\omega t}$$
$$-i\epsilon k''_m P_{j''n}(\mathbf{k}'') q^{-s''s'}_{mj'}(\mathbf{k}') q^{s''s}_{nj}(\mathbf{k}) e^{2is''\omega' t} \; . \tag{78}$$

It can be shown that, of these six terms, only the fourth and fifth ones give non–zero contributions to the kinetic equations. Defining

$$\omega_{k,\kappa L} = \omega_k - \omega_\kappa - \omega_L \tag{79}$$

and integrating equation (78) over time, the exponential terms will lead to

$$\Delta(\omega_{k,\kappa L}) = \int_0^t \exp\left[it\omega_{k,\kappa L}\right] dt = \frac{\exp\left[i\omega_{k,\kappa L}\right] - 1}{i\omega_{k,\kappa L}} \; . \tag{80}$$

Substituting these expressions in (78), only the terms which have an argument in the Δ functions that cancel exactly with the arguments in the exponential appearing in (76) will contribute. We then obtain the fundamental kinetic equations for the energy tensor, *viz.* :

$$\partial_t q_{jj'}^{ss'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}') = \tag{81}$$

$$-\epsilon^2 k_m P_{jn}(\mathbf{k}) \int k_{2p} P_{nq}(\mathbf{L}) q_{pm}^{-s-s}(\boldsymbol{\kappa}) q_{qj'}^{ss'}(\mathbf{k}') \Delta(-2s\omega_1) \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}} d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$-\epsilon^2 k_m P_{jn}(\mathbf{k}) \delta_{ss'} \int k'_p P_{j'q}(\mathbf{k}') q_{pm}^{-s'-s}(\boldsymbol{\kappa}) q_{qn}^{s's}(\mathbf{L}) \Delta(-2s\omega_1) \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}} d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$-\epsilon^2 k'_m P_{j'n}(\mathbf{k}') \int k_{2p} P_{nq}(\mathbf{L}) q_{pm}^{-s'-s'}(\boldsymbol{\kappa}) q_{qj}^{s's}(\mathbf{k}) \Delta(-2s'\omega_1) \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'} d_{\boldsymbol{\kappa}\mathbf{L}}$$

$$-\epsilon^2 k'_m P_{j'n}(\mathbf{k}') \delta_{ss'} \int k_p P_{nq}(\mathbf{k}) q_{pm}^{-s-s'}(\boldsymbol{\kappa}) q_{qn}^{ss'}(\mathbf{L}) \Delta(-2s\omega_1) \delta_{\boldsymbol{\kappa}\mathbf{L},\mathbf{k}'} d_{\boldsymbol{\kappa}\mathbf{L}}$$

We now perform an integration over the delta and taking the limit $t \to +\infty$ we find

$$\partial_t [q_{jj'}^{ss'}(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}')] = \tag{82}$$

$$-\epsilon^2 \int\int\int \Bigg\{ Q_k^{-s}(\boldsymbol{\kappa}) P_{jn}(\mathbf{k}) P_{nl}(\mathbf{L}) [q_{jj'}^{ss'}(\mathbf{k}') \left[\frac{\pi}{2}\delta(\kappa_\parallel) - i\mathcal{P}(\frac{1}{2s\kappa_\parallel})\right]$$

$$+ Q_k^{-s'}(\boldsymbol{\kappa}) P_{j'n}(\mathbf{k}) P_{nl}(\mathbf{L}) [q_{lj}^{s's}(\mathbf{k}) \left[\frac{\pi}{2}\delta(\kappa_\parallel) + i\mathcal{P}(\frac{1}{2s'\kappa_\parallel})\right] K$$

$$-\pi \delta_{ss'} Q_k^{-s}(\boldsymbol{\kappa}) P_{j'l}(\mathbf{k}) P_{jn}(\mathbf{k}') q_{ln}^{ss}(\mathbf{L}) \delta(\kappa_\parallel) \Bigg\} d\kappa_1 d\kappa_2 d\kappa_\parallel \,.$$

where $\mathcal{P}$ stands for the principal value of the integral.

In the case where $s = s'$ of interest here because the cross–correlators between z–fields of opposite polarities decay to zero in that approximation, the above equations simplify to :

$$\tfrac{2}{\pi}\partial_t [q_{jj'}^{ss}(\mathbf{k}') \pm q_{j'j}^{ss}(\mathbf{k}')] = \tag{83}$$

$$2 \int P_{jn}(\mathbf{k}) P_{j'q}(\mathbf{k}) [q_{qn}^{ss}(\mathbf{L}) \pm q_{nq}^{ss}(\mathbf{L})] Q_k^{-s}(\boldsymbol{\kappa}) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel$$

$$- \int P_{jn}(\mathbf{k}) P_{nq}(\mathbf{L}) [q_{qj'}^{ss}(\mathbf{k}') \pm q_{j'q}^{ss}(\mathbf{k}')] Q_k^{-s}(\boldsymbol{\kappa}) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel$$

$$- \int P_{j'n}(\mathbf{k}) P_{nq}(\mathbf{L}) [q_{jq}^{ss}(\mathbf{k}') \pm q_{qj}^{ss}(\mathbf{k}')] Q_k^{-s}(\boldsymbol{\kappa}) \delta(\kappa_\parallel) d\kappa_1 d\kappa_2 d\kappa_\parallel \,.$$

When developping the above expression in terms of the correlators defined in (22), one arrives at the kinetic equations for weak MHD turbulence given in Section 2.4.

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