Fourier space intermittency of the small-scale turbulent dynamo

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Abstract

The small-scale turbulent dynamo in the high Prandtl number regime is described in terms of the one-point Fourier space correlators. The second order correlator of this kind is the energy spectrum and it has been previously studied in detail. We examine the higher order k-space correlators which contain important information about the phases of the magnetic wavepackets and about the dominant structures of the magnetic turbulence which cause intermittency. In particular, the fourth-order correlators contain information about the mean-square phase difference between any two components of the magnetic field in a plane transverse to the wavevector. This can be viewed as a measure of the magnetic field's polarization. Examining this new quantity, the magnetic field is shown to become plane polarized in the Kazantsev-Kraichnan model at large time, corresponding to a strong deviation from Gaussianity. We derive a closed equation for the generating function of the Fourier correlators and find the large-time asymptotic solutions of these correlators at all orders. The time scaling of these solutions implies the magnetic field has log-normal statistics, whereas the wavenumber scaling indicates that the field is dominated by intermittent fluctuations at high k.

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I. INTRODUCTION

In some astrophysical applications such as the interstellar medium and protogalactic plasmas the kinematic viscosity is greater than the magnetic field diffusivity by the factor 10^{14} to 10^{22} [1,2]. In these situations, there is a vast scaling range where the magnetic field has a smaller characteristic length-scale than the velocity field. The dynamo process of the stochastic stretching and amplification of the magnetic field can be studied in this case in terms of the statistics of Lagrangian deformations [3,4]. It can be pictured as a collection of magnetic field wavepackets, each moving along a fluid particle path and being distorted by the local strain.[1] Such a regime of smooth velocity fields is similar to the Batchelor's regime found in the related problem of passive scalar advection. The Batchelor's regime has a long history of study, some of the more recent advances can be found, for example, in [2,4] (and references therein). A common further simplification of the problem is to assume that the local strain matrix is a Gaussian white noise process, this is commonly known as the Kazantsev-Kraichnan model [5,6]. However, this assumption is not always necessary, and some results have been shown to be universal for a broader class of stochastic flows [3].

The first analysis of the dynamo problem were based in Fourier space [5–7], with particular emphasis being placed on the second order moment, corresponding to the energy spectrum. It was realized, however, that the Fourier space and coordinate space equations have a similar structure. Recent studies of the turbulence intermittency (in both the passive scalar and dynamo problems) have focussed on coordinate space moments of second order or higher. This approach was motivated by a feeling that this description is more natural and can give more information than the k-space moments [3,4,8]. As a result, the only k-space correlator that has been seriously studied to date is the energy spectrum and there is no theory describing the higher Fourier space correlators.

In the present paper, we turn our attention back to Fourier space and consider the

[1]The second and higher derivatives of the velocity field can be ignored in this case.

one-point moments of the magnetic field Fourier transform. Although the problem under investigation here is the small-scale turbulent dynamo, we would like to give several reasons why the description of the Fourier space moments is important and why it should be developed in a broader turbulence context.

(i) The presence of singular structures in turbulence is known to affect the scaling of the structure functions. The most well-known examples of this being the β and multi-fractal models [9]. However, some coherent structures are singular in Fourier space rather than in coordinate space, and therefore can be detected by an investigation of the Fourier moments. A simple example of structures that are singular in Fourier space (and regular in coordinate space) is a sea of vortex filaments in 2D turbulence. The layered pattern of the vortices in coordinate space corresponds to a 1D curve in Fourier space.[2]

(ii) To date, the only Fourier moment that was studied in detail is the second order moment that describes the distribution of the turbulent energy over wavenumbers. However, there exist other Fourier moments that have a clear physical meaning and describe important properties of the turbulence. One of these objects is related to the fourth order moments and has a physical meaning of the mean turbulence polarization. It will be introduced and studied in this paper.

(iii) Phases of the Fourier modes can be dealt with directly, and therefore the validity of the random phase assumption can be examined.

(iv) Finally, in some cases a Fourier space analysis is the only way to have a treatable problem due to the introduction of non-localities, for example via a pressure or wave dispersion. Note that the dynamo and passive scalar systems do not fit into this class of problems. However, the methods developed in the present paper will be applied to the

[2]The Fourier transform should be taken over a local box, which is large enough to fit many layers of filaments, but is smaller than the large-scale vortices in this case. This will be discussed in the next section.

non-local Navier-Stokes equation[3] (which involves a pressure term) in our next paper [10].

II. ONE-POINT AND TWO-POINT CORRELATORS OF FOURIER AMPLITUDES

To avoid any confusion, we should state that the Fourier transforms in this paper are performed over a finite box of a size much greater than typical length-scale of the magnetic turbulence, but much less than the scale of the advecting velocity field. It is only for finite box Fourier transforms that one-point correlators are well defined objects. Secondly, the center of each finite box is not fixed. The box center coordinate dependence of the Fourier transforms is a useful measure of the slow variability of the large-scale magnetic turbulence. In this paper we make each box move in unison with a fluid particle, and thus any coordinate dependence is replaced by a time dependence along a given fluid trajectory.

One should be aware that in this case the finite box size does not produce a discrete spectrum as the magnetic field is in general not periodic within the box. On the other hand, the lack of periodicity can produce unphysical power-law tails in the Fourier transforms at (large) wavenumbers that are greater than one over the box length. Slow decay of these tails can be undesirable in numerical simulations and other applications. However, this problem can be easily cured by using a smooth filter instead of the box, for example by using Gabor transforms instead [11]. In this presentation the box (or filter) shape is actually not an issue. We will present the results in terms of finite box Fourier transforms. However, performing the analysis using Gabor transforms will in fact produce identical results.

Let us consider the general two-point correlator of the Fourier transformed magnetic field components

$$\langle B_{i_1}(\mathbf{k}_1)B_{i_2}(\mathbf{k}_1)\cdots B_{i_n}(\mathbf{k}_1)B_{j_1}(\mathbf{k}_2)B_{j_2}(\mathbf{k}_2)\cdots B_{j_n}(\mathbf{k}_2)\rangle, \tag{1}$$

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[3]Such an equation is the basis of Rapid Distortion theory which will be developed for the case of a stochastic strain.

where $i_1, i_2, \cdots, i_n$ and $j_1, j_2, \cdots, j_n$ take the values 1, 2 or 3 corresponding to the three components of the magnetic field, and n is an arbitrary natural number. Let us assume the turbulence is quasi-homogeneous. That is, the correlator $\langle B_i(\mathbf{x}_1)B_j(\mathbf{x}_2)\rangle$ (and similar) depend only on $\mathbf{x}_1 + \mathbf{x}_2$ as slowly as the advecting strain field. On the other hand, such correlators decay in $\mathbf{x}_1 - \mathbf{x}_2$ at distances much less than the box size L. In this case

$$\langle B_{i_1}(\mathbf{k}_1)B_{i_2}(\mathbf{k}_1)\cdots B_{i_n}(\mathbf{k}_1)B_{j_1}(\mathbf{k}_2)B_{j_2}(\mathbf{k}_2)\cdots B_{j_n}(\mathbf{k}_2)\rangle = \tag{2}$$
$$\langle B_{i_1}(\mathbf{k}_1)B_{i_2}(\mathbf{k}_1)\cdots B_{i_n}(\mathbf{k}_1)B_{j_1}(-\mathbf{k}_1)B_{j_2}(-\mathbf{k}_1)\cdots B_{j_n}(-\mathbf{k}_1)\rangle\ f(\mathbf{k}_1+\mathbf{k}_2),$$

where $f(\mathbf{k}) = \frac{1}{L^d}\Pi_{l=1}^{d}\frac{\sin(Lk_l)}{k_l}$ is the Fourier transform of the filter function and d is the number of space dimensions. Here we see that any two-point correlator is fully determined by the one-point correlators

$$I_{i_1,i_2,\cdots,i_n;j_1,j_2,\cdots,j_n}(\mathbf{k}) = \langle B_{i_1}(\mathbf{k})B_{i_2}(\mathbf{k})\cdots B_{i_n}(\mathbf{k})B_{j_1}(-\mathbf{k})B_{j_2}(-\mathbf{k})\cdots B_{j_n}(-\mathbf{k})\rangle. \tag{3}$$

This simple observation means we can avoid lengthy derivations dealing with two-point objects directly, and instead can obtain results from the one-point correlators that are themselves far easier to deal with. Turbulence isotropy and the divergence-free condition further narrow the class of possible one-point (and two-point) correlator tensors we need to consider (see for example [12])

$$I_{i_1,i_2,\cdots,i_n;j_1,j_2,\cdots,j_n}(\mathbf{k}) = \sum_{\text{pairs}} \Phi_s^n(k) D_{l_1,l_2} D_{l_3,l_4}\cdots D_{l_{n-1},l_n}, \tag{4}$$

where

$$D_{ij} \equiv \delta_{i,j} - \frac{k_i k_j}{k^2}, \tag{5}$$

and the summation is over the set of all possible pairings $(l_1, l_2), (l_3, l_4), \cdots, (l_{n-1}, l_n)$ of the indices i and j from the left hand-side of (4). For each pairing, the index s is equal to the number of pairs that consist only of i's, [4] for example (i_1, i_3) or (i_8, i_2). One can see that

[4] There are obviously going to be the same number of pairs that consist only of j's.

any correlator of order $2n$ can be expressed in terms of $N = \mathrm{Max}(s)+1 = \mathrm{Int}(n/2)+1$ independent functions $\Phi_s^n(k)$. There is only one such function for the second order correlators, two functions for the fourth and sixth orders, and three functions at the eighth and tenth orders.

Instead of $\Phi_s^n(k)$, we can choose another set of independent functions, in particular the following set of correlators

$$\Psi_s^n = \langle |\mathbf{B}(\mathbf{k})|^{(2n-4s)} |\mathbf{B}(\mathbf{k})^2|^{2s} \rangle. \tag{6}$$

These correlators are linear combinations of $\Phi_s^n(k)$'s, for example at the fourth order we have

$$\Psi_0^{(2)} = d(d-1)\Phi_0^{(2)} + (d-1)\Phi_1^{(2)},$$
$$\Psi_1^{(2)} = 2(d-1)\Phi_0^{(2)} + (d-1)^2\Phi_1^{(2)}.$$

Below, we will derive a technique that allows us to deal with the fundamental set of correlators (6).

III. THE MODEL

Let us start with the equation for the magnetic field $\mathbf{B}(\mathbf{x}, t)$,

$$\partial_t \mathbf{B} + (\mathbf{u}\cdot\nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{u} + \kappa\Delta\mathbf{B}, \tag{7}$$

where $\mathbf{u}$ is the velocity field, which we assume has a much slower spatial variation than the magnetic field $\mathbf{B}$, and κ is the magnetic diffusivity (determined by the fluid conductivity). We will make use of the previously discussed finite box Fourier transforms. Each box has sides of length L which is chosen to lie between the length-scales associated with $\mathbf{B}$ and $\mathbf{u}$, namely L_B and L_u. The box Fourier transform is defined as

$$\hat{\mathbf{B}}(\mathbf{k}, \mathbf{x}, t) \int_{box} \mathbf{B}(\mathbf{r}, t)\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{r})}\, d\mathbf{r}, \tag{8}$$

where $\mathbf{x}$ is the coordinate of the box center. Applying this Fourier transform to (7) we have (with accuracy up to the first order of the scale separation parameter $\epsilon = L_B/L_u \ll 1$)

$$\partial_t \hat{B}_m + u_i \nabla_i \hat{B}_m = \sigma_{ij} k_i \, \partial_j \hat{B}_m + \sigma_{mi} \hat{B}_i - \kappa k^2 \hat{B}_m, \tag{9}$$

where $\sigma_{ij} = \nabla_j u_i$ is the strain matrix and the operators ∇_i and ∂_i correspond to derivatives with respect to x_i and k_i respectively. In considering a fluid path determined by $\dot{\mathbf{x}}(t) = \mathbf{u}$, and thus noting that $\sigma_{ij}(\mathbf{x}, t) \to \sigma_{ij}(\mathbf{x}(t), t)$ and $\hat{\mathbf{B}}(\mathbf{k}, \mathbf{x}, t) \to \hat{\mathbf{B}}(\mathbf{k}, \mathbf{x}(t), t)$, this equation becomes[5]

$$\partial_t B_m = \sigma_{ij} k_i \, \partial_j B_m + \sigma_{mi} B_i - \kappa k^2 B_m. \tag{10}$$

It should also be noted that the strain σ_{ij}, taken along a fluid path, only enters this equation as a given function of time. To complete the model one has to specify this dependence or postulate the strain statistics. In what follows we choose the strain matrix to be Gaussian such that

$$\sigma_{ij} = \Omega \left(A_{ij} - \frac{A_{ll}}{d} \delta_{ij} \right), \tag{11}$$

where A_{ij} is a matrix with statistically independent elements that are white in time

$$\langle A_{ij}(t) A_{kl}(0) \rangle = \delta_{ij} \delta_{kl} \delta(t). \tag{12}$$

This choice of strain matrix ensures incompressibility and statistical isotropy. Indeed,

$$\langle \sigma_{ij}(t) \sigma_{kl}(0) \rangle = \Omega \left(\delta_{ik} \delta_{jl} - \frac{1}{d} \delta_{ij} \delta_{kl} \right) \delta(t). \tag{13}$$

However, one must be aware that this is not the only way to satisfy the incompressibility and isotropy conditions; for example [3,4,13] have chosen

$$\langle \sigma_{ij}(t) \sigma_{kl}(0) \rangle = \Omega \left[(d+1) \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl} \right] \delta(t). \tag{14}$$

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[5] Hereafter, we will the drop hat notation on $\hat{\mathbf{B}}$ because only Fourier components will be considered. Also, we will not mention explicitly dependence on the fluid path and simply write $\mathbf{B} \equiv \mathbf{B}(\mathbf{k}, t)$.

More generally, there is an infinite one-parametric family of possible strains. The strain matrices that are Gaussian and white in time correspond to the Kazantsev-Kraichnan model, a natural starting point for an analytical analysis because of its simplicity. However, it should be noted that some results remain universal in the case of other smooth velocity fields found in a much wider class of statistical models [3,4,13]. In our future work we will investigate the behaviour of the Fourier space correlators in the case of more general statistics.

IV. GENERATING FUNCTION

Let us consider the following generating function

$$Z(\lambda, \alpha, \beta, k) = \langle e^{\lambda|\mathbf{B}(\mathbf{k})|^2 + \alpha \mathbf{B}^2(\mathbf{k}) + \beta \overline{\mathbf{B}}^2(\mathbf{k})} \rangle, \tag{15}$$

where the overline denotes the complex conjugation. This function allows one to obtain any of the fundamental one-point correlators (6) via differentiation with respect to λ, α and β

$$\Psi_s^n = \langle |\mathbf{B}(\mathbf{k})|^{(2n-4s)} |\mathbf{B}^2(\mathbf{k})|^{2s} \rangle = \left[\partial_\lambda^{(2n-4s)} \partial_\alpha^s \partial_\beta^s Z \right]_{\lambda=\alpha=\beta=0}. \tag{16}$$

Differentiating (15) with respect to time and using the dynamical equation (10), we have

$$\begin{aligned} \dot{Z} = k_i \partial_j \langle \sigma_{ij} E \rangle + \lambda \langle \sigma_{ml} (\overline{B}_m B_l + \overline{B}_l B_m) E \rangle + 2\alpha \langle \sigma_{ml} B_m B_l E \rangle + 2\beta \langle \sigma_{ml} \overline{B}_m \overline{B}_l E \rangle \\ -2\kappa k^2 \langle [\lambda |\mathbf{B}(\mathbf{k})|^2 + \alpha \mathbf{B}^2(\mathbf{k}) + \beta \overline{\mathbf{B}}^2(\mathbf{k})] E \rangle, \end{aligned} \tag{17}$$

where

$$E = e^{\lambda|\mathbf{B}(\mathbf{k})|^2 + \alpha \mathbf{B}^2(\mathbf{k}) + \beta \overline{\mathbf{B}}^2(\mathbf{k})}. \tag{18}$$

To find the correlators on the right hand-side of (17), one needs to make use of the Gaussianity of the strain matrix σ_{ij} and perform a Gaussian integration by parts. We then use the whiteness of the strain field to find the response function (functional derivative of B_l with respect to σ_{ij}). Finally, one can use the statistical isotropy of the strain, so that the final equation involves only $k = |\mathbf{k}|$ and no angular dependence of the wavevector. This derivation is discussed in more detail in the Appendix. Here, we just write the final result

$$\dot{Z} = \frac{\Omega}{2}\left[\left(1-\frac{1}{d}\right)k^2 Z_{kk} + \frac{1}{d}(-4\mathcal{D}+d^2-1)kZ_k \right. \tag{19}$$
$$\left. +(2d-6)\mathcal{D}Z + 4\left(1-\frac{1}{d}\right)\mathcal{D}^2 Z + 2(\lambda^2-4\alpha\beta)(Z_{\alpha\beta}-Z_{\lambda\lambda})\right] - 2\kappa k^2 \mathcal{D}Z,$$

where the k, α, β and λ subscripts in Z denote differentiation with respect to k, α, β and λ respectively, and

$$\mathcal{D} = \lambda\partial_\lambda + \alpha\partial_\alpha + \beta\partial_\beta. \tag{20}$$

Actually, one can reduce the number of independent variables in this equation by taking into account that Z depends on α and β only in the combination $\eta = \alpha\beta$. The easiest way to see this is to consider a Taylor series expansion of Z from (15); any term that contains α and β in a different combination will be zero because of the quasi-homogeneity of the turbulence. Thus, we can write

$$\dot{Z} = \frac{\Omega}{2}\left[\left(1-\frac{1}{d}\right)k^2 Z_{kk} + \frac{1}{d}(-4\mathcal{D}+d^2-1)kZ_k \right. \tag{21}$$
$$\left. +(2d-6)\mathcal{D}Z + 4\left(1-\frac{1}{d}\right)\mathcal{D}^2 Z + 2(\lambda^2-4\eta)(Z_\eta + \eta Z_{\eta\eta}-Z_{\lambda\lambda})\right] - 2\kappa k^2 \mathcal{D}Z,$$

where

$$\mathcal{D} = \lambda\partial_\lambda + 2\eta\partial_\eta. \tag{22}$$

In what follows, we will restrict our consideration to the 3D case ($d = 3$) in which case the equation for Z reduces to

$$\dot{Z} = \frac{\Omega}{3}\left[k^2 Z_{kk} + (4-2\mathcal{D})kZ_k \right. \tag{23}$$
$$\left. +4\mathcal{D}^2 Z + 3(\lambda^2-4\eta)(Z_\eta + \eta Z_{\eta\eta} - Z_{\lambda\lambda})\right] - 2\kappa k^2 \mathcal{D}Z.$$

V. ENERGY SPECTRUM

Let us now consider the energy spectrum of the magnetic turbulence, given by the second order correlator

$$E(k,t) = \Psi_0^1 = \langle|\mathbf{B}(\mathbf{k})|^2\rangle = [\partial_\lambda Z]_{\lambda=\eta=0}\,. \tag{24}$$

Differentiating (23) with respect to λ and taking the result at $\lambda = \eta = 0$ we have

$$\dot{E} = \frac{\Omega}{3}(k^2 E_{kk} + 2kE_k + 4E) - 2\kappa k^2 E. \tag{25}$$

This equation for the evolution of the energy spectrum was first obtained by Kazantsev [5] and Kraichnan and Nagarajan [6]. Kazantsev analyzed an eigenvalue problem associated with this equation which allowed him to obtain the growth exponents of the total magnetic energy. Numerically, the energy spectrum was studied by Kulsrud and Anderson [1] who gave a detailed description of the k-space evolution of this spectrum. Recently, Schekochihin, Boldyrev and Kulsrud [2] presented the complete solution of (25) obtained by the use of Kontorovich-Lebedev transforms (KLT). Note that this integral transform approach has an advantage over the Kazantsev's eigenvalue analysis in that it allows us to obtain not only the growth exponents, but also the power-law prefactors, of the large time asymptotic solutions.

As they will be of use later in this presentation, let us briefly review the previous results for the energy spectrum before we discuss the higher order moments. Using the following substitution [2]

$$E = e^{5\Omega t/4}\, k^{-1/2}\, \phi(k/k_{d_1}, t) \quad k_{d_n} = \sqrt{\frac{\Omega}{6n\kappa}}, \tag{26}$$

one can reduce (25) to

$$\frac{3}{\Omega}\dot{\phi}(p,t) = p^2\phi_{pp} + p\phi_p - p^2\phi, \tag{27}$$

where $p = k/k_{d_1}$. At scales much greater than the dissipative one, $p \ll 1$, there is a perfect conductor regime for time $t \ll (\ln q)^2$ (where $q \ll 1$ is the mean wavenumber of the initial condition). Thus, in this regime the last term in (25) can be neglected. By changing to logarithmic coordinates and a moving frame of reference one can transform this equation into a heat equation. For $t \gg 1/\Omega$ the solution of which is just the Green's function, which gives [1,2,14,15]

$$\phi = \mathrm{const}\, t^{-1/2}\, e^{-\frac{3(\ln k/q)^2}{4\Omega t}}, \tag{28}$$

where the constant is fixed by the initial condition. This solution describes a spectrum with an expanding $k^{-1/2}$ scaling range. At $t \sim (\ln q)^2$ the front of this scaling range reaches the dissipative scales. To solve (25) in this case, we note that the right hand-side of this equation is just the modified Bessel operator and by using KLT one immediately obtains [2]

$$\phi(p,t) = \text{const} \int_0^\infty ds\, s \sinh(\pi s)\, K_{is}(p) K_{is}(q)\, e^{-s^2 t}, \tag{29}$$

where K_{is} is a MacDonald function of imaginary order. Again this solution is given by the Green's function only because the condition that $t \gg 1/\Omega$ is obviously satisfied if $q \ll 1$. For time $t \gg (\ln q)^2$, the function $e^{-s^2 t}$ is strongly peaked at $t = 0$ and integration of (29) gives

$$\phi = \text{const}\;\; t^{-3/2} K_0(p). \tag{30}$$

One should note that although the $K_0(p)$ shape is somehow predicted by the Kazantsev's eigenmode analysis [2,5,14,15], the $t^{3/2}$ factor can only be obtained by solving the full initial value problem. For $p \ll 1$, $K_0(p) \approx -\ln p$, which means that at large times the scales far larger than the dissipative one are affected by diffusion via a logarithmic correction

$$E(k) \sim k^{-1/2}\, \ln(k_{d_1}/k). \tag{31}$$

The energy evolution gives an important, but incomplete picture, of the dynamo process. In particular, it does not capture the existence of small-scale intermittency and does not allow us to predict the type of coherent structures dominating the turbulence at large time. To deal with these issues one has to study higher order correlators. Higher one-point correlators in coordinate space were studied in [3] and used to predict the shape of the dominant structures. Below, we proceed to study the higher k-space correlators. In particular, this will lead to the discovery of a new quantity of interest, corresponding to the mean polarization of the magnetic turbulence.

VI. 4TH-ORDER CORRELATORS, TURBULENCE POLARIZATION AND FLATNESS

There are two independent 4th order correlators

$$S(k,t) = \Psi_0^{(2)} = \langle |\mathbf{B}(\mathbf{k})|^4 \rangle = [Z_{\lambda\lambda}]_{\lambda=\eta=0}\,, \tag{32a}$$

$$T(k,t) = \Psi_1^{(2)} = \langle |\mathbf{B}^2(\mathbf{k})|^2 \rangle = [Z_{\eta}]_{\lambda=\eta=0}\,. \tag{32b}$$

Differentiating (23) twice with respect to λ and taking the result at $\lambda = \eta = 0$ we have

$$\dot{S} = \frac{\Omega}{3}(k^2 S_{kk} + 10S + 6T) - 4\kappa k^2 S. \tag{33}$$

Now, differentiating (23) with respect to η and taking the result at $\lambda = \eta = 0$ we get

$$\dot{T} = \frac{\Omega}{3}(k^2 T_{kk} + 4T + 12S) - 4\kappa k^2 T. \tag{34}$$

Equations (33) and (34) make up a complete system for S and T and can be solved exactly in the general case. Observe that there is a closed equation for $W = S - T$

$$\dot{W} = \frac{\Omega}{3}(k^2 W_{kk} - 2W) - 4\kappa k^2 W. \tag{35}$$

Before solving this equation, let us examine the physical meaning of W by writing it as

$$W = 4\sum_{j\neq l}^{3} \langle [\Im(B_j \overline{B}_l)]^2 \rangle = 4\sum_{j\neq l}^{3} \langle |B_j|^2 |B_l|^2 \sin^2(\phi_j - \phi_l) \rangle \geq 0, \tag{36}$$

where $\Im$ denotes the imaginary part and ϕ_j and ϕ_l are the phases of the components B_j and B_l respectively. We see therefore that W contains information not only about the amplitudes but also about the phases of the Fourier modes. In particular, $W \equiv 0$ corresponds to the case where all Fourier components of the magnetic field are plane polarization. If $W \neq 0$ then other polarizations (circular, elliptic) are present. This is the case for example for a Gaussian field where one finds $W = E^2/2 > 0$. On the other hand, the smallness of the phase differences in W can be overpowered by large amplitudes. Therefore, a better measure of the mean polarization would be a normalized W, for example

$$P = W/S. \tag{37}$$

Defined in this way, the mean turbulence polarization is an example of an important physical quantity which can be obtained from the one-point Fourier correlators, and that is unavailable from the (one-point or two-point) coordinate space correlators.

In the perfect conductor regime, when the diffusivity term in the equations (33) – (35) can be ignored, the solution for W can be obtained in a similar manner to the previous energy spectrum analysis. Namely, by reducing (35) into a heat equation via a logarithmic change of variables and passing into a moving frame of reference. The solution therefore is

$$W = W_0 \, t^{-1/2} e^{-3\Omega t/4} \, k^{1/2} e^{-\frac{3(\ln k/q)^2}{4\Omega t}}, \tag{38}$$

where W_0 is a constant which can be found from the initial conditions. We see that W develops a $k^{1/2}$ scaling range which is cut-off at low and high k by exponentially propagating fronts. Within this scaling range, W decays exponentially in time.

Given W, one can also find S by representing it as $S = V + cW$ and choosing the constant c such that the equation for V is closed. This gives $c = 3/7$ and

$$S = t^{-1/2} \, k^{1/2} e^{-\frac{3(\ln k/q)^2}{4\Omega t}} \left(V_0 \, e^{21\Omega t/4} + \frac{3}{7} W_0 \, e^{-3\Omega t/4} \right), \tag{39}$$

where V_0 is another constant which can be found from the initial conditions. For $t \gg 1/\Omega$, the second term in the parenthesis should be neglected, and we have the following solution for the mean turbulence polarization

$$P = W/S = \frac{W_0}{V_0} e^{-6\Omega t}. \tag{40}$$

In the perfect conductor regime we see the mean polarization tends to a value that is independent of k and decays exponentially in time. This means that all Fourier modes of the magnetic field eventually become plane polarized. Recall that such turbulence is very far from being Gaussian, remembering that the mean polarization of a Gaussian field remains finite.[6]

[6] That is, elliptic and circular polarized modes are present.

It is also easy to obtain solutions to W for the diffusive regime. Indeed, following the example of the energy spectrum we make use of the substitutions

$$W = e^{-3\Omega t/4} k^{1/2} \phi(k/k_{d_2}), \tag{41}$$

$$V = e^{21\Omega t/4} k^{1/2} \phi(k/k_{d_2}), \tag{42}$$

to transform the governing equations for W and V into a form similar to (27). In each case we can solve the equation for ϕ using KLT and find that for times $t \gg (\ln q)^2$

$$W \simeq W_0 k^{1/2} t^{-3/2} e^{-3\Omega t/4} K_0(k/k_{d_2}), \tag{43}$$

$$V \simeq V_0 k^{1/2} t^{-3/2} e^{21\Omega t/4} K_0(k/k_{d_2}). \tag{44}$$

Correspondingly, S has the solution

$$S \simeq k^{1/2} t^{-3/2} \left[V_0 e^{21\Omega t/4} + \frac{3}{7} W_0 e^{-3\Omega t/4} \right] K_0(k/k_{d_2}), \tag{45}$$

while importantly we find that the normalised polarisation $P = W/S$ behaves identically in both the diffusive and perfect conductor regimes.

Therefore, in the diffusive regime W continues to decrease in time with the same exponential rate as perfect conductor case and has the same $t^{-3/2}$ prefactor as the energy spectrum. Thus, by the time the diffusive regime is achieved W can be essentially put equal to zero. The fact that the polarization becomes plane has quite a simple physical explanation. Indeed, a magnetic field wavepacket of arbitrary polarization will be strongly distorted by stretching. The stretching being strongest along the direction of the dominant eigenvector of the Lagrangian deformation matrix (corresponding to the greatest Lyapunov exponent). Such a stretching will make any initial “spiral” structure flat at large time, with the dominant field component lying in a plane passing through the eigenvector stretching and wavevector directions.[7]

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[7] Of course, the incompressibility of $\mathbf{B}(\mathbf{k})$ ensures that it is perpendicular to $\mathbf{k}$.

Another measure of intermittency in turbulence is the flatness which can be defined in k-space as $F = S/E^2$. For large time, in the perfect conductor regime

$$F \sim t^{1/2}\, e^{11\Omega t/4}\, k^{3/2}. \tag{46}$$

We see that the flatness grows both in time and in k which indicates the presence of small-scale intermittency. Such an intermittency can be attributed to the presence of coherent structures in k-space. In the diffusive regime, in contrast to the polarisation the behaviour of the flatness is modified and at large time one finds

$$F \simeq k^{3/2} t^{3/2} e^{11\Omega t/4} \frac{K_0(k/k_{d_2})}{[K_0(k/k_{d_1})]^2}. \tag{47}$$

For small k, we again have a region of $k^{3/2}$ scaling but now with a log correction arising from the MacDonald functions. For large k, below the spectral cut-off, the additional MacDonald functions act to heighten the flatness. That is, the introduction of a finite diffusivity actually increases the small-scale intermittency. In the next section we will investigate the correlators of all orders.

VII. LARGE-TIME BEHAVIOR OF HIGHER CORRELATORS

The observation in the last section, that there is a dominant field component, allows us to predict that for large time $|\mathbf{B}|^4 \approx |\mathbf{B}^2|^2$ in each realization, that is $Z_{\lambda\lambda} \approx Z_{\alpha\beta}$. Therefore, the property $Z_{\lambda\lambda} = Z_{\alpha\beta}$, if valid initially, should be preserved by the equation for Z. Indeed, let us differentiate the equation (23) twice with respect to λ and subtract it from the same equation differentiated with respect to α and β. This gives the following closed equation for the combination $w = Z_{\lambda\lambda} - Z_{\alpha\beta}$

$$\begin{aligned}\dot{w} = \frac{\Omega}{3}\big[&k^2 w_{kk} - 2k\mathcal{D}w_k + 4\mathcal{D}^2 w \\ &+16(\mathcal{D}w + w) + 3(\partial_{\lambda\lambda} - \partial_{\alpha\beta})\{(\lambda^2 - 4\alpha\beta)w\}\big] - 2\kappa k^2(\mathcal{D}w + 2w).\end{aligned} \tag{48}$$

We see that if $w \equiv 0$ at $t = 0$, then it will remain identically zero for all time. Thus, we can consider a class of solutions of (23), corresponding to large-time asymptotics of the general solution, such that $Z_{\lambda\lambda} = Z_{\alpha\beta}$. Assuming this equality in (23) and putting $\eta = 0$ we have

$$\dot{Z} = \frac{\Omega}{3}[k^2 Z_{kk} + (-2\lambda\partial_\lambda + 4)kZ_k + 4(\lambda\partial_\lambda)^2 Z] - 2\kappa k^2 \lambda\partial_\lambda Z. \tag{49}$$

Let us consider a solution to this equation that is formally represented as a series in λ (for example via a Taylor series):

$$Z = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!}\,\Psi^{(n)}(k,t). \tag{50}$$

Here, the function $\Psi^{(n)}$ is the correlator of order $2n$. We omitted the lower index in $\Psi^{(n)}$ because the correlators corresponding to different lower indices are identical in this case. Substituting (50) into (49) we have the following equation for these correlators

$$\dot{\Psi}^{(n)} = \frac{\Omega}{3}[k^2\Psi^{(n)}_{kk} + (-2n+4)k\Psi^{(n)}_{k} + 4n^2\Psi^{(n)}] - 2\kappa k^2 n\Psi^{(n)}. \tag{51}$$

Note that for $n=1$ this equation agrees with the Kazantsev equation for the energy spectrum (25). Moreover, by the substitution

$$\Psi^{(n)} = e^{(n-1/2)(n+3/2)\Omega t}\,k^{n-3/2}\,\phi(k/k_{d_n},t), \tag{52}$$

one can reduce (51) to an equation for ϕ, similar to (27), that is independent of n. We can therefore immediately write down the general solution for the correlators at any order n. In particular, in the perfect conductor regime $1/\Omega \ll t \ll (\ln q)^2$ we have

$$\Psi^{(n)} = \mathrm{const}(n)\,t^{-1/2}\,e^{(n-1/2)(n+3/2)\Omega t}\,e^{-\frac{3(\ln k/q)^2}{4\Omega t}}\,k^{n-3/2}, \tag{53}$$

and in the diffusive regime $t \gg (\ln q)^2$

$$\Psi^{(n)} = \mathrm{const}(n)\,t^{-3/2}e^{(n-1/2)(n+3/2)\Omega t}\,k^{n-3/2}\,K_0(k/k_{d_n}). \tag{54}$$

We see that the main effect of the dissipation on all the moments (including the energy spectrum) is the prefactor changes from $t^{-1/2} \to t^{-3/2}$ (but without a change in the exponential growth rate), further the $K_0(k/k_{d_n})$ form-factor corresponds to a log-correction at $k \ll 1$ and an exponential cut-off at larger k.[8] It is this exponential cut-off that causes

[8]Note that the higher the order of the correlator the earlier the spectral cut-off.

the exponential growth of the mean magnetic energy [5] and higher x-space moments of the magnetic field [3] to change. Indeed, simply integrating $\Psi^{(n)}$ over k with a cut-off at $k = k_{d_n}$ (and ignoring the prefactor and log corrections) one recovers the Kazantsev growth rate of the magnetic energy in the dissipative regime [5]. Such an explanation was previously given in [1].

VIII. PHYSICAL INTERPRETATION OF THE SCALINGS

Let us analyze the physical origins of the scalings obtained in the previous section because this can give us an indication to whether the same scalings should be expected for more general strain statistics.

Let us consider the expression (26) for the n-th order correlator and re-write it in the form $C(n) \cdot exp((2n^2 + 2n)\Omega t)k^n \cdot P(k,t)$, where $P(k,t)$ is a universal function. The physical meaning of various terms in this expression can be easily analyzed. It follows from the Central Limit theorem that the large time statistics of the Lyapunov exponents is Gaussian with a dispersion $D(t) \sim \sqrt{t}$. It follows from time reversal invariance that the average values of the Lyapunov exponents are $\bar{\lambda}, 0, -\bar{\lambda}$. Hence, in the large time limit, $B(t) \sim exp(\lambda t)$, where $p.d.f.(t) \sim exp(-\frac{(\lambda-\bar{\lambda})^2}{2\Delta t^{-1}})$. Therefore, the statistics of the magnetic field is log-normal and $\langle B^{2n} \rangle \sim exp((2\bar{\lambda}n^2 + 2\Delta n)t)$. For the statistics chosen in the present paper $\bar{\lambda} = \Delta = \Omega$. Substituting these values into the last expression for $\langle B^{2n} \rangle$, we restore the correct n-dependence of the exponential growth rate of Ψ^n. Note that terms of order n^2 in the growth rate are due to the Gaussian nature of the fluctuations of λ around $\bar{\lambda}$, while the terms of order n are due to the fact that $\bar{\lambda} \neq 0$. It is interesting that the magnetic field has log-normal statistics in both the perfect conductor and the dissipative regimes. For the coordinate space moments of B, persistence of the log-normality was emphasized in [16], although the $e^{\lambda n^2 t}$ dependence was established earlier in [3]. An equivalent result for the random matrices can be traced back to [17] (see also [4]).

The k-dependence of the magnetic field correlators is also very important because it gives us information about the dominant structures in wavenumber space. Suppose that initially the magnetic turbulence is isotropic and concentrated in a ball centered at the origin in wavenumber space. For each realization such a ball will stretch into an ellipsoid with one large, one short and one neutral dimension. One can visualize this ellipsoid as an elongated flat cactus leaf with thorns aligned to the direction of the magnetic field. Note that in this picture one component of the magnetic field (transverse to the cactus leaf) is dominant, this is captured by the fact that the polarization W introduced in this paper tends to zero at large time. Another consequence of this picture is that the wavenumber space will be covered by the ellipsoids more and more sparsely at large k, this implies large intermittent fluctuations of the magnetic field Fourier transform. These fluctuations can be quantified by the flatness F which was shown in (46) to grow as $k^{3/2}$, a clear indication of the small-scale intermittency.

To investigate the Kraichnan-Kazantsev model based dynamo problem further, a set of numerical experiments have been performed to, among other things, investigate the sensitivity of these analytical results to changes in the strain statistics. The details of this investigation can be found in [14,18]. To help visualise the cactus leaf structures described above we will briefly include some snapshots of the configuration of a wavepacket ensemble in wavenumber space. As above each wavepacket is initially randomly distributed on a unit sphere in k-space with a randomly orientated magnetic field that lies in a plane perpendicular to the wavepacket's wavevector, tangent to the sphere. Figure 1 shows the real magnetic fields of a set of 500 wavepackets that have been subjected to two different realisations of the strain matrix.[9] In the left-hand figure it is clear the magnetic field in this realisation is far from being plane polarised, the magnetic vectors still being predominantly random in their orientation at this given point in time. In contrast, in the right-hand figure, which has also been taken at the same point in time, we see the ellipsoid has become very elongated and

[9]The imaginary magnetic fields are qualitatively similar and have therefore not been included.

the magnetic field appears plane polarised. In contrast, figure 2 shows the same two strain realisations but with a finite diffusivity. The right-hand figure in particular demonstrates why, in the diffusive regime, an ellipsoid will cover k-space more sparsely due to the decay of its magnetic field at its tips. This is the reason why spectral flatness increases in the diffusive regime.

IX. CONCLUSION

In this paper we introduced a description of the small-scale magnetic turbulence in terms of the one-point correlators of the Fourier amplitudes. From the classic works of [5–7] to more recent analysis (see for example [2] and references therein), the only correlator of this kind that was studied is the energy spectrum. The higher order correlators have only been considered in coordinate space [3,4]. In this paper we have considered the Fourier correlators of all orders and showed that they contain important additional information about the turbulence, which is unavailable from a similar analysis of one-point or two-point coordinate space correlators. In particular, the fourth order Fourier correlators carry information about the mean polarization of the magnetic field modes. We showed that this polarization becomes plane for the Kraichnan-Kazantsev model. The k-scaling of the higher correlators allows us to determine the structures in Fourier space responsible for the intermittency, which for the Kraichnan-Kazantsev dynamo turns out to be elongated ellipsoids centered at the origin. The time scaling of the higher correlators allows one to conclude that the magnetic field has a log-normal statistics, although the same information is contained, and was established before, in analysis of the coordinate space correlators [3,4,16].

Finally, we would like to discuss an interesting connection between this work and the recent work of Schekochihin *et al.* [8,16]. In particular the connection between the statistics of the magnetic field curvature studied by Schekochihin *et al.* and the magnetic polarization measure introduced in this present paper. Schekochihin *et al.* found that that the curvature of the magnetic field decreases, corresponding to folded and strongly stretched structures.

This agrees with our results that the Fourier modes of the magnetic field tend to a state of plane polarization. However, the polarization gives more information than the curvature statistics. Indeed, zero curvature allows any structure that is constant along the magnetic field, in particular a set of magnetic filaments parallel to each other or a set of layered sheets, such that the magnetic field is constant on each sheet but its direction may change arbitrarily when passing from one layer to another. On the other hand, the wavenumber scalings obtained in this paper, (53) and (54), indicate that the magnetic field structures are layers in coordinate space and this rules out any filamentary structures. Further, our result about the plane polarization inhibit any "twists" of the magnetic field between layers, i.e. the magnetic field direction stays the same (or reverses) when passing from one layer to another. In fact, the presence of one neutral direction in the Lagrangian deformations tells us that these layers have a finite width in one direction and thus look like ribbons with the magnetic field directed along these ribbons.

APPENDIX

Our aim here is to derive a closed equation for the generating function Z (21), starting with equation (17). The last term in this equation is the easiest one

$$-2\kappa k^2\langle(\lambda|\mathbf{B}(\mathbf{k})|^2+\alpha\mathbf{B}^2(\mathbf{k})+\beta\overline{\mathbf{B}}^2(\mathbf{k}))E\rangle=-2\kappa k^2\mathcal{D}Z, \tag{A1}$$

where $\mathcal{D}$ is a differential operator defined in (20). The correlators containing a factor of σ_{ij} can be found using Gaussian integration by parts. In particular

$$\begin{aligned}\langle\sigma_{ij}E\rangle=\Omega\langle\frac{\delta E}{\delta\sigma_{ij}}\rangle=\frac{\Omega}{2}[&\lambda\langle(\Gamma_{m,ij}\overline{B}_m+\overline{\Gamma}_{m,ij}B_m)E\rangle\\&+2\alpha\langle\Gamma_{m,ij}B_mE\rangle+2\beta\langle\overline{\Gamma}_{m,ij}\overline{B}_mE\rangle],\end{aligned} \tag{A2}$$

where we have used the definition (18). Here, $\Gamma_{m,ij}$ is a response function

$$\Gamma_{m,ij}=\frac{\delta B_m}{\delta\sigma_{ij}}. \tag{A3}$$

Differentiating (10) with respect to σ_{ij} and using the statistical whiteness of the strain tensor we get

$$\Gamma_{m,ij} = \left[k_i\,\partial_j - \frac{\delta_{ij}}{d}(1 + k_l\,\partial_l)\right] B_m + \delta_{mi} B_j. \tag{A4}$$

In what follows we will make use of the isotropy of the turbulence, in particular, expressions of the type

$$\langle(\overline{B}_i B_j + \overline{B}_j B_i)E\rangle = \frac{2}{d-1}\langle|B|^2 E\rangle\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{A5}$$

$$\langle B_i B_j E\rangle = \frac{1}{d-1}\langle B^2 E\rangle\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{A6}$$

$$\langle \overline{B}_i \overline{B}_j E\rangle = \frac{1}{d-1}\langle \overline{B}^2 E\rangle\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{A7}$$

Substituting (A4) into (A2) and using the above isotropy relations we have

$$\langle\sigma_{ij}E\rangle = \frac{\Omega}{2}\left(k_i\partial_j - \frac{\delta_{ij}}{d}k_l\partial_l\right)Z - \frac{\Omega}{d-1}\left(\frac{k_i k_j}{k^2} - \frac{\delta_{ij}}{d}\right)\mathcal{D}Z, \tag{A8}$$

where $\mathcal{D}$ is a differential operator defined in (20). This allows us to find the first term on the right hand-side of (17)

$$k_i\partial_j\langle\sigma_{ij}E\rangle = \frac{\Omega}{2}\left[\frac{d-1}{d}k^2 Z_{kk} + \frac{1}{d}\left(-2\mathcal{D} + d^2 - 1\right)kZ_k - 2\mathcal{D}Z\right]. \tag{A9}$$

Similarly, the other three terms on the right hand-side of (17) can be obtained via Gaussian integration by parts, and the use of the response function (A4) and isotropy condition. After some lengthy but straightforward algebra one obtains

$$\begin{aligned}\lambda\langle\sigma_{ml}(\overline{B}_m B_l + \overline{B}_l B_m)E\rangle = \lambda\Omega\Bigg[&\left(d - \frac{2}{d}\right)Z_\lambda + 2\left(1 - \frac{1}{d}\right)\mathcal{D}Z_\lambda \\ &+\lambda(Z_{\alpha\beta} - Z_{\lambda\lambda}) - \frac{1}{d}k_i\partial_i Z_\lambda\Bigg],\end{aligned} \tag{A10}$$

and

$$\begin{aligned}2\alpha\langle\sigma_{ml}B_m B_l E\rangle = \alpha\Omega\Bigg[&\left(d - \frac{2}{d}\right)Z_\alpha + 2\left(1 - \frac{1}{d}\right)\mathcal{D}Z_\alpha \\ &-2\beta Z_{\alpha\beta} + 2\beta Z_{\lambda\lambda} - \frac{1}{d}k_i\partial_i Z_\alpha\Bigg].\end{aligned} \tag{A11}$$

The fourth term can be obtained from (A11) via interchanging α with β and B with $\overline{B}$

$$2\beta\langle\sigma_{ml}\overline{B}_m\overline{B}_l E\rangle = \beta\Omega\Bigg[\left(d-\frac{2}{d}\right)Z_\beta + 2\left(1-\frac{1}{d}\right)\mathcal{D}Z_\beta \\ -2\alpha Z_{\alpha\beta} + 2\alpha Z_{\lambda\lambda} - \frac{1}{d}k_i\partial_i Z_\beta\Bigg]. \tag{A12}$$

Using the expressions (A9), (A10), (A11), (A12) and (A1), we obtain the final equation

$$\dot{Z} = \frac{\Omega}{2}\Bigg[\left(1-\frac{1}{d}\right)k^2 Z_{kk} + \frac{1}{d}(-4\mathcal{D} + d^2 - 1)kZ_k \\ +(2d-6)\mathcal{D}Z + 4\left(1-\frac{1}{d}\right)\mathcal{D}^2 Z + 2(\lambda^2 - 4\alpha\beta)(Z_{\alpha\beta} - Z_{\lambda\lambda})\Bigg] - 2\kappa k^2\mathcal{D}Z. \tag{A13}$$

REFERENCES

[1] R. Kulsrud and S. Anderson, *Astrophys. J.* **396**, 606 (1992).

[2] A. A. Schekochihin, S. A. Boldyrev and R. M. Kulsrud, *Astrophys. J.* **567**, 828 (2002).

[3] M. Chertkov, G. Falkovich, I. Kolokolov and M. Vergasolla, *Phys. Rev. Lett.* **83**, 4065 (1999).

[4] G. Falkovich, K. Gawedzki and M. Vergasolla, *Rev. Mod. Phys.* **73**, 913 (2001).

[5] A. P. Kazantsev, *Sov. Phys. JETP* **26**, 1031 (1968).

[6] R. Kraichnan and S. Nagarajan, *Phys. Fluids* **10**, 859 (1967).

[7] G. K. Batchelor, *Proc. R. Soc. London, Ser. A* **202**, 405 (1950).

[8] A. Schekochihin, S. Cowley, J. Maron and L. Malyshkin, *Phys. Rev. E* **65**, 016305 (2002).

[9] U. Frisch, *Turbulence - the legacy of A. N. Kolmogorov* (CUP, Cambridge, 1995).

[10] S. Nazarenko and O. Zaboronski, 2003 (to be published).

[11] S. Nazarenko, N. Kevlahan and B. Dubrulle, *J. Fluid Mech.* **390**, 325 (1999).

[12] W. D. McComb, *The physics of fluid turbulence* (Clarendon Press, Oxford, 1990).

[13] E. Balkovsky and A. Fouxon, *Phys. Rev. E* **60**, 4164 (1999).

[14] R. J. West, Ph.D. thesis, University of Warwick, 2002.

[15] R. J. West, S. Nazarenko and S. Galtier, in *Proceedings of the European Turbulence Conference 9, Southampton, 2002*, edited by I. Castro, P. E. Hancock and T. G. Thomas (University of Southampton, Southampton, 2002).

[16] A. A. Schekochihin, J. L. Maron, S. Cowley, J. C. McWilliams, *Astophys. J.* to be published.

[17] H. Furstenberg, *Trans. Am. Math. Soc.* **108**, 377 (1963).

[18] R. J. West, S. Nazarenko, J.P laval and S. Galtier, 2003 (to be published).

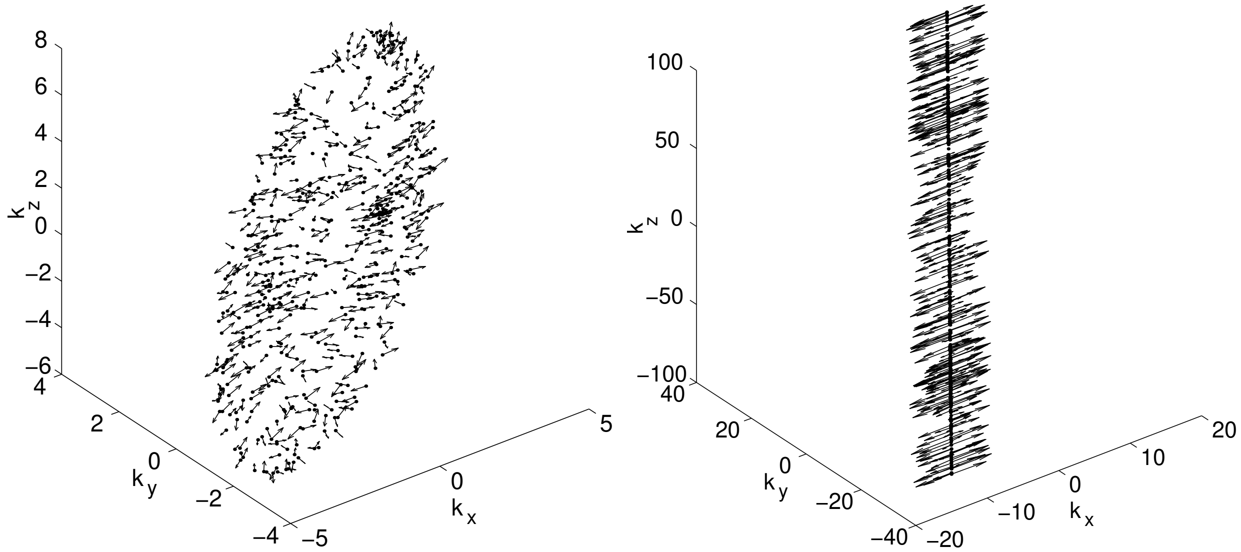

FIG. 1. The magnetic field of 500 wavepackets in a numerical simulation with $\Omega = 0.36$ and $\kappa = 0$. The figures show the particle's positions in k-space at $t = 6$ and their corresponding real magnetic fields. The left and right figures correspond to different realisations of the strain matrix.

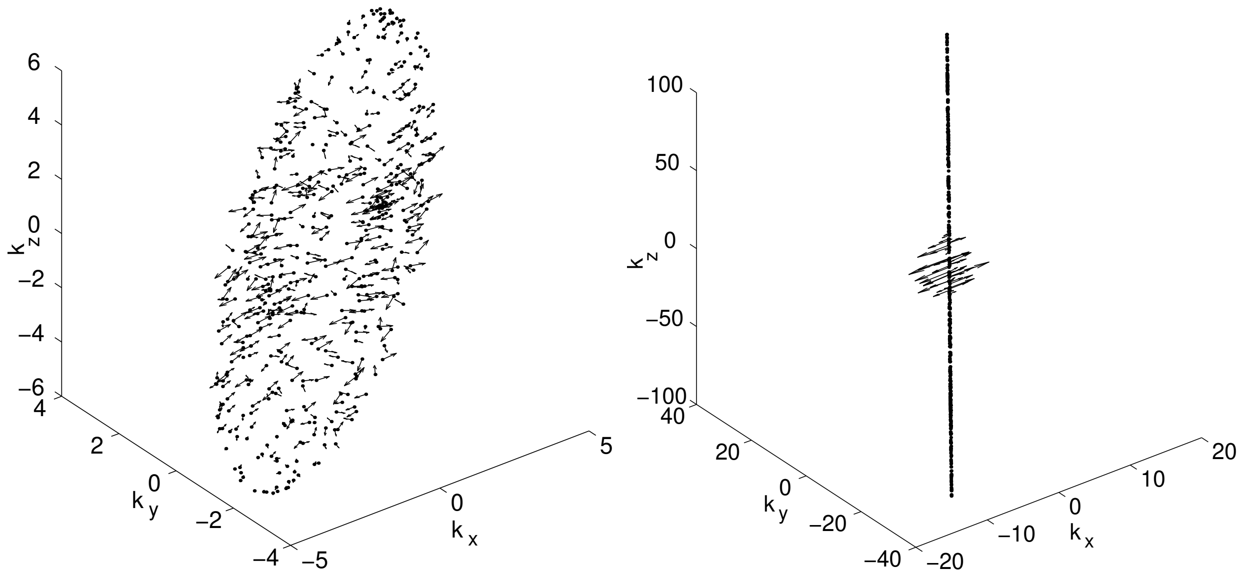

FIG. 2. The magnetic field of 500 wavepackets in a numerical simulation with $\Omega = 0.36$ and $\kappa = 0.005$. The figures show the particle's positions in k-space at $t = 6$ and their corresponding real magnetic fields for two different realisations of the strain matrix. The left figure corresponds to the same strain field as the left-hand graph of figure 1 and similarly for the right-hand graphs.