Dimensional Analysis and Weak Turbulence

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Abstract

In the study of weakly turbulent wave systems possessing incomplete self-similarity it is possible to use dimensional arguments to derive the scaling exponents of the Kolmogorov-Zakharov spectra, provided the order of the resonant wave interactions responsible for nonlinear energy transfer is known. Furthermore one can easily derive conditions for the breakdown of the weak turbulence approximation. It is found that for incompletely self-similar systems dominated by three wave interactions, the weak turbulence approximation usually cannot break down at small scales. It follows that such systems cannot exhibit small scale intermittency. For systems dominated by four wave interactions, the incomplete self-similarity property implies that the scaling of the interaction coefficient depends only on the physical dimension of the system. These results are used to build a complete picture of the scaling properties of the surface wave problem where both gravity and surface tension play a role. We argue that, for large values of the energy flux, there should be two weakly turbulent scaling regions matched together via a region of strongly nonlinear turbulence.

1 Introduction

The time evolution of the average spectral wave-action density, $n_{\mathbf{k}}$, of an ensemble of weakly interacting dispersive waves is governed by the so-called kinetic equation. For a system with dispersion law $\omega_{\mathbf{k}}$, dominated by 3-wave interactions with interaction coefficient $L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$, the kinetic equation is :

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 4\pi \int \left|L_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\right|^2 n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \mathcal{F}_3\left[n\right] \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\; d\mathbf{k}_1 d\mathbf{k}_2 \qquad (1)$$

where

$$\mathcal{F}_3[n] = \left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right)\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})$$
$$+ \left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} + \frac{1}{n_{\mathbf{k}_2}}\right)\delta(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_2}) \tag{2}$$
$$+ \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}}\right)\delta(\omega_{\mathbf{k}_2} - \omega_{\mathbf{k}} - \omega_{\mathbf{k}_1}).$$

More generally, if the dominant nonlinear interaction is N-wave, the *schematic* form of the kinetic equation is

$$\frac{\partial n_{\mathbf{k}}}{\partial t} \sim \int L_N^2\, n_{\mathbf{k}}^{N-1}\, \delta(\omega_{\mathbf{k}})\, \delta(\mathbf{k})\, (d\mathbf{k})^{N-1}. \tag{3}$$

Let us consider homogeneous, isotropic systems in physical dimension d. Let us further assume scale invariance with $\omega_{\mathbf{k}}$ and L_N homogeneous functions of their arguments. Denote the degrees of homogeneity of the dispersion, $\omega_{\mathbf{k}}$, and the N-wave interaction coefficient, L_N, by α and γ_N respectively. Under these assumptions, the kinetic equation possesses exact stationary solutions, found originally by Zakharov in the early 70's, which carry constant fluxes of conserved quantities, such as energy or wave-action. These solutions are called Kolmogorov-Zakharov (KZ) spectra.

The 3-wave kinetic equation admits a single KZ spectrum carrying a constant flux, P, of energy :

$$n_{\mathbf{k}} = c^{(3)} P^{\frac{1}{2}} k^{-(\gamma_3+d)}. \tag{4}$$

The 4-wave kinetic equation conserves wave-action in addition to energy and thus admits a pair of KZ spectra, one carrying an energy flux, P, the other carrying a wave-action flux, Q. They are

$$n_{\mathbf{k}} = c_1^{(4)} P^{\frac{1}{3}} k^{-\frac{1}{3}(2\gamma_4+3d)}, \tag{5}$$
$$n_{\mathbf{k}} = c_2^{(4)} Q^{\frac{1}{3}} k^{-\frac{1}{3}(2\gamma_4+3d-\alpha)}. \tag{6}$$

The dimensional constants, $C^{(N)}$, can be explicitly calculated.

Suppose we know that the physical system under consideration depends on only one dimensional constant. Such a wave system is said to possess *incomplete self-similarity* (ISS). Zakharov, Lvov and Falkovich have pointed out [1, chap. 3] that for such systems the scaling of the KZ spectra can be obtained from a dimensional argument. The dimensional argument for ISS systems uses only the scaling of the dispersion relation and not the scaling of the interaction coefficients required for the more general scale invariant systems considered above. This fact has not been fully appreciated and is rarely used, despite the fact that most of the known wave turbulence systems are ISS, as will be shown in this paper.

The dimensional analysis determines the scaling of the interaction coefficients for systems possessing ISS, a point that has been mostly overlooked before. The fact that the scaling exponent of the interaction coefficient is not an independent quantity may have some consequences for the practical applicability of some theoretical results on weak turbulence, where the scaling exponents of the interaction coefficients are regarded as arbitrary. In particular, Biven, Nazarenko and Newell[2; 3] have recently pointed out that the weak turbulence approximation is almost never uniformly valid in k but rather breaks down either at large or small scales. The breakdown of weak turbulence at small scales is presumed to signal the onset of small scale intermittency. It is possible to use a simple dimensional argument to recover the criteria obtained in [2] in the ISS case. One finds that the condition for breakdown at small scales is inconsistent for three wave systems. As a result one would not expect such systems to exhibit small scale intermittency.

The goal of this paper is to use the ISS dimensional argument, for the first time in some examples, to derive the KZ spectra and the scaling of the interaction coefficients for a large number of commonly considered applications of weak turbulence. We then use our results to discuss the uniformity of the weak turbulence approximation in $\mathbf{k}$ for these physical systems. In the final section we consider the water wave system in more detail. It is shown that by considering the effect of both the gravity dominated and surface tension dominated parts of the spectrum together, one can build a consistent picture of energy transfer in the system, even when the flux is sufficiently large to cause breakdown of the weak turbulence approximation.

2 Dimensional Derivation of Kolmogorov-Zakharov Spectra

Before we begin, let us clarify a point of notation. We deal with isotropic systems. Physical quantities such as spectral wave-action density, $n_{\mathbf{k}}$, or spectral energy density, $E_{\mathbf{k}} = \omega_{\mathbf{k}} n_{\mathbf{k}}$, only depend on the modulus, k, of the wave-vector, $\mathbf{k}$. It is often convenient to integrate over angles in $\mathbf{k}$-space. We need to make a distinction between a spectral quantity which has been averaged over angle and one which has not. To do this, we use a regular type argument to denote a quantity which has been integrated over angles, as in n_k, and a bold type argument to denote one which has not, as in $n_{\mathbf{k}}$. The two are easily related. Consider for example, the wave-action density:

$$\int n_k\,dk = \int n_{\mathbf{k}}\,d\mathbf{k}$$
$$\Rightarrow n_k = \Omega_D\, n_{\mathbf{k}} k^{D-1}, \tag{7}$$

where Ω_D is the solid angle coming from the integration over angles in D-dimensional wave-vector space. We shall use C to denote a generic dimensionless constant whose value cannot be determined from dimensional arguments.

2.1 Constant Energy Flux Spectra

Suppose we have a wave system characterised by a single additional dimensional parameter, λ, which appears in the dispersion relation in the form

$$\omega_{\mathbf{k}} = \lambda k^{\alpha}. \tag{8}$$

It is convenient to set the density of the medium to 1. Our unit of mass then has dimension L^3 and energy has dimension $\mathrm{L}^5\mathrm{T}^{-2}$. We suppose that the d-dimensional energy density, $\mathcal{E}$, is finite in physical space. For example, $d = 2$ for water waves while $d = 3$ for acoustic waves. We denote the dimension of the Fourier transform used to go to a spectral description of the theory by D. Usually $D = d$ but not always (see, for example, section 6). The spectral energy density, E_k, is defined by

$$\mathcal{E} = \int E_{\mathbf{k}}\,d\mathbf{k} = \int_0^{\infty} E_k\,dk. \tag{9}$$

E_k clearly has dimension $\mathrm{L}^{6-d}\mathrm{T}^{-2}$. The energy flux, P, has dimension $\mathrm{L}^{5-d}\mathrm{T}^{-3}$ and λ has dimension $\mathrm{L}^{\alpha}\mathrm{T}^{-1}$. Let us now consider the constant energy flux spectrum for this system. For 3-wave processes, the energy flux is proportional to the square of the spectral energy density so we can write

$$E_k = C\sqrt{P}\lambda^{\mathrm{X}}k^{\mathrm{Y}}, \tag{10}$$

where C is a dimensionless constant and the exponents X and Y are to be determined by dimensional analysis. This yields

$$\mathrm{X} = \frac{1}{2}, \qquad \mathrm{Y} = \frac{1}{2}\left(d + \alpha - 7\right). \tag{11}$$

This argument, used by Kraichnan [4] in the context of Alfvén waves, can be generalised to N-wave systems. In a system dominated by N wave processes, the energy flux is proportional to the $N-1^{\mathrm{th}}$ power of the spectral energy density and a similar argument yields the scaling law

$$E_k = CP^{\frac{1}{N-1}} \lambda^{\mathrm{X}} k^{\mathrm{Y}}, \tag{12}$$

with X and Y given by

$$\begin{aligned} \mathrm{X} &= \frac{2N-5}{N-1}, \\ \mathrm{Y} &= (2\alpha + d - 6) + \frac{5 - 3\alpha - d}{N-1}. \end{aligned} \tag{13}$$

Associated with each constant energy flux spectrum, we have a particle number (wave action) spectrum, $n_{\mathbf{k}}$. One can be obtained from the other via the relation

$$\int_0^\infty E_k \, dk = \Omega_D \int_0^\infty \omega_{\mathbf{k}} \, n_{\mathbf{k}} \, k^{D-1} \, dk, \tag{14}$$

where Ω_D is the D-dimensional solid angle. The resulting scaling law for $n_{\mathbf{k}}$ for an N wave system is

$$n_{\mathbf{k}} = CP^{\frac{1}{N-1}} \lambda^{\mathrm{X}} k^{\mathrm{Y}}, \tag{15}$$

where

$$\begin{aligned} \mathrm{X} &= \frac{N-4}{N-1}, \\ \mathrm{Y} &= (\alpha + d - D - 5) + \frac{5 - 3\alpha - d}{N-1}. \end{aligned} \tag{16}$$

2.2 Constant Particle Flux Spectra

In the case of a system with 4-wave interactions, the total particle number, $N = \int n_k \, dk$ is also a conserved quantity. As a result, there can also exist a constant-flux spectrum carrying a flux of particles rather than energy. Such behaviour is associated with a continuity equation of the form

$$\frac{\partial n_k}{\partial t} + \frac{\partial Q_k}{\partial k} = 0, \tag{17}$$

where Q_k is the particle flux. One can perform the same dimensional analysis for this spectrum, bearing in mind that dimensionally, $P = \omega_{\mathbf{k}} Q$. One obtains the following spectrum describing a constant flux of particles:

$$n_k = C Q^{\frac{1}{3}} \lambda^{\frac{1}{3}} k^{-\frac{1}{3}(-2d-\alpha+13)}, \tag{18}$$

or

$$n_{\mathbf{k}} = C Q^{\frac{1}{3}} \lambda^{\frac{1}{3}} k^{-\frac{1}{3}(3D-2d-\alpha+10)}. \tag{19}$$

2.3 Scaling of the Interaction Coefficients

In the regime where the system is scale invariant, the nonlinear interaction coefficients, V_{ijk} (3-wave) and T_{ijkl} (4-wave) often possess nontrivial scaling properties. For the 3-wave case we have :

$$V_{h\mathbf{k}h\mathbf{k}_1 h\mathbf{k}_2} = h^{\beta} V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}, \tag{20}$$

and for the 4-wave case :

$$T_{h\mathbf{k}h\mathbf{k}_1 h\mathbf{k}_2 h\mathbf{k}_3} = h^{\gamma} T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}. \tag{21}$$

In fact β and γ cannot be arbitrary. They may be determined from dimensional analysis of the dynamical equations. Schematically, the dynamical equations for an N wave system look like

$$\frac{\partial a_{\mathbf{k}}}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = \int L_N \, a_{\mathbf{k}}^{N-1} \, \delta(\mathbf{k}) \, (d\mathbf{k})^{N-1}. \tag{22}$$

Recalling that dimensionally, $[\mathbf{k}] = \mathrm{L}^{-D}$ and $[\delta(\mathbf{k})] = \mathrm{L}^{D}$, we see that:

$$[L_N] = \frac{[\omega_{\mathbf{k}}]}{[a_{\mathbf{k}}]^{N-2} \, [\mathrm{L}^{-D}]^{N-2}}.$$

Determine the dimension of $a_{\mathbf{k}}$ as follows,

$$< a_{\mathbf{k}} a^*_{\mathbf{k}'} >= \delta(\mathbf{k} - \mathbf{k}') \, n_{\mathbf{k}}.$$

So

$$[a_{\mathbf{k}}]^2 = [\delta(\mathbf{k} - \mathbf{k}')] \, \frac{[E_k]}{[\omega_{\mathbf{k}}] \, \mathrm{L}^{1-D}} = \frac{\mathrm{L}^{6-d}\mathrm{T}^{-2}}{\mathrm{T}^{-1}\mathrm{L}^{1-2D}}.$$

This results in the following expression for the dimension of the interaction coefficient

$$[L_N] = \mathrm{T}^{\frac{1}{2}(N-4)}\mathrm{L}^{\frac{1}{2}(N-2)(d-5)}, \tag{23}$$

and dimensional analysis then yields,

$$L_{N\,\mathbf{k}_1\cdots\mathbf{k}_N} = \lambda^{\frac{1}{2}(4-N)}k^{\gamma_N}\, f_{\mathbf{k}_1\cdots\mathbf{k}_N}, \tag{24}$$

where

$$\gamma_N = -\frac{1}{2}\left\{(N-2)(d-5) + (N-4)\alpha\right\}. \tag{25}$$

Here $f_{\mathbf{k}_1\cdots\mathbf{k}_N}$ is a dimensionless function of $\mathbf{k}_1\cdots\mathbf{k}_N$. Interestingly, for 4-wave systems, $N=4$, the scaling of the interaction coefficients depends only on the dimension, d, of the system and is independent of any dimensional parameter, including λ. We see that *all* incompletely self-similar 4-wave systems exhibit the same scaling behaviour of their interaction coefficients,

$$\gamma_4 = 5 - d. \tag{26}$$

Applying our analysis to the 3-wave case yields $L_{3\,\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim k^{\gamma_3}$, where

$$\gamma_3 = \frac{1}{2}\left(5 + \alpha - d\right). \tag{27}$$

The scaling in this case depends on the dispersion index, α, but we see that γ_3 and α are not independent quantities. This fact, while obvious from this point of view, is possibly not fully appreciated. We shall see that the class of incompletely self-similar systems for which this analysis is valid includes most of the common physical applications of weak turbulence.

3 Breakdown of the Weak Turbulence Approximation

By analysing the scaling behaviour of the kinetic equations describing non-linear energy transfer in weak turbulence, Biven, Nazarenko and Newell [2] have, under certain assumptions, given a set of criteria for the breakdown of the weak turbulence approximation. These assumptions are that the turbulent transfer is sufficiently local that after using homogeneity properties to remove the k-dependence of the collision integral and the other integrals arising in the expression for the frequency renormalisation, the remaining integrals converge. We discuss in the conclusion and in [5] how this may not always be the case when the coefficient of long-wave short-wave interaction is too strong.

In this section we shall adopt the commonly used notation $\gamma_3 = \beta$ and $\gamma_4 = \gamma$. For three wave systems, breakdown occurs at small scales for $\beta - 2\alpha > 0$ and at large scales for $\beta - 2\alpha < 0$. For four wave systems breakdown occurs at small scales for $\gamma - 3\alpha > 0$ in the presence of a pure energy flux and for $\gamma - 2\alpha > 0$ in the case of a pure particle flux. The breakdown at large scales can be masked by the large scale forcing but breakdown at small scales, in the absence of dissipation, is taken to signal the onset of small scale intermittency. In the case of ISS systems we can construct a characteristic scale, k_{NL}, from the flux and the parameter λ. From our previous discussion of dimensions, we see that the quantity, $(P/\lambda^3)^{1/5-d-3\alpha}$ has the dimension of a length. Thus in the case of a finite energy flux, we define

$$k_{\mathrm{NL}} = \left(\frac{P}{\lambda^3}\right)^{-\frac{1}{5-d-3\alpha}}. \tag{28}$$

Likewise, in the case of a finite particle number flux, Q, in a four wave system we define

$$k_{\mathrm{NL}} = \left(\frac{Q}{\lambda^3}\right)^{-\frac{1}{5-d-2\alpha}}. \tag{29}$$

For small fluxes, $P \to 0$, we see that the breakdown occurs at small scales for $5-d-3\alpha > 0$ for finite P and for $5-d-2\alpha > 0$ for finite Q. Upon substitution of (23) into these expressions, we recover the criteria of [2] in terms of the scaling exponents of the interaction coefficients. It is interesting to note that for finite energy flux, the breakdown criterion is $\alpha < 2/3$ in 3 dimensions and $\alpha < 1$ in 2 dimensions. However it is known from the work of Krasitskii [6] that for $\alpha < 1$, 3-wave terms in the interaction Hamiltonian are nonresonant and can be removed by an appropriate change of canonical variables to give an effective description in terms of four wave interactions. Thus the small scale breakdown criterion can never be realised for three wave systems in two or three dimensions. This means that a significant number of physical systems cannot be hoped to exhibit intermittency at small scales as discussed in the following section. Conversely, these systems can always exhibit intermittency at large scales, provided the forcing is sufficiently strong, without affecting the validity of the weak turbulence approximation at small scales. At this point, it is worth mentioning that the case $\alpha = 1$ is borderline in two dimensions. Such three wave systems, 2-d sound being an example, are known to be rather special and must be carefully treated separately.

4 Examples of 3-Wave Systems

Sound and magnetic sound in 3 dimensions

Acoustic turbulence [7; 8; 9] corresponds to the almost linear dispersion $\omega_{\mathbf{k}} \approx ck$, where c is the sound speed or magnetic sound speed, so that $\alpha = 1$ and $d = 3$. We thus obtain the following pair of spectra for the energy and wave action

$$E_k = C\sqrt{Pc}k^{-\frac{3}{2}}, \qquad n_{\mathbf{k}} = C'\sqrt{\frac{P}{c}}k^{-\frac{9}{2}}. \tag{30}$$

These are the original spectra obtained by Zakharov and Sagdeev. According to our analysis, this spectrum remains uniformly valid at small scales.

3-D Alfvén waves

3-D Alfvén wave turbulence was originally considered by Iroshnikov [10] and Kraichnan [4] in the 60's . Such waves are also weakly dispersive and from the point of view of the dimensional analysis are identical to acoustic waves discussed above. The resulting $-\frac{3}{2}$ and $-\frac{9}{2}$ spectra are not actually realised in real plasmas because the true Alfvén wave turbulence is anisotropic.

Quasi 2-D Alfvén waves

In reality, the Alfvén turbulence is strongly anisotropic and is described by quasi-2D rather than 3D spectra. For this system we have again $\alpha = 1$ but $d = 2$. This yields the stationary spectra

$$E_k = C\sqrt{Pc}k^{-2}, \qquad n_{\mathbf{k}} = C'\sqrt{\frac{P}{c}}k^{-4}, \tag{31}$$

which are the spectra obtained using a dimensional analysis by Ng and Bhatachargee [11] and analytically derived by Galtier, Nazarenko and Newell [12]. As mentioned already, the case $\alpha = 1$ is borderline in 2-d so that our argument concerning the breakdown of the weak turbulence approximation remains inconclusive. In fact, unlike typical three wave systems, this system does exhibit breakdown at small scales as shown in [12].

Capillary waves on deep water

$\lambda = \sqrt{\sigma}$, where σ is the coefficient of surface tension. In this case, $\alpha = \frac{3}{2}$, $d = 2$ and the Kolmogorov spectrum is

$$E_k = C\sqrt{P}\sigma^{\frac{1}{4}}k^{-\frac{7}{4}}, \qquad n_{\mathbf{k}} = C'\sqrt{P}\sigma^{-\frac{1}{4}}k^{-\frac{17}{4}}. \tag{32}$$

This spectrum was first derived by Zakharov and Filonenko [13]. There is no small scale intermittency in this system.

5 Examples of 4-Wave Systems

Gravity waves on deep water

For this system, $\alpha = \frac{1}{2}$, $d = 2$ and $\lambda = \sqrt{g}$, where g is the gravitational constant. The Kolmogorov spectrum corresponding to a constant flux of energy is then

$$E_k = CP^{\frac{1}{3}}g^{\frac{1}{2}}k^{-\frac{5}{2}}, \qquad n_{\mathbf{k}} = C'P^{\frac{1}{3}}k^{-4}. \tag{33}$$

There is also a second spectrum corresponding to a constant flux of wave action,

$$n_{\mathbf{k}} = CQ^{\frac{1}{3}}g^{\frac{1}{6}}k^{-\frac{23}{6}}. \tag{34}$$

These spectra were obtained by Zakharov and Filonenko [14]. In this case, the energy spectrum breaks down at small scales.

This is one of the cases where certain integrals appearing in the frequency renormalisation series diverge on the K-Z spectrum. The problem is that the interaction coefficient between the high k mode and long wave partners in its resonant quartet is too strong. This leads to a modification of the breakdown criterion and means that the breakdown can occur for values of k less than that value calculated when local interactions dominate.

Langmuir waves are described by the dispersion relation

$$\omega_{\mathbf{k}}^2 = \omega_p^2 \left(1 + 3r_D^2 k^2\right), \tag{35}$$

where ω_p and r_D are the plasma frequency and Debye length respectively. Magnetic spin waves in solids also obey a dispersion relation of this type but the physical meaning of the dimensional parameters is different. [15], [1] For long Langmuir waves we can Taylor expand $\omega_{\mathbf{k}}$ as $\omega_{\mathbf{k}} = \omega_p + \frac{3}{2}\omega_p r_D^2 k^2$. The constant factor ω_p cancels out of both sides of the 4-wave resonance condition so that the effective dispersion is $\omega \sim k^2$. Thus taking $\lambda = \omega_p r_D^2$, $\alpha = 2$, $d = 3$ we obtain the energy spectrum

$$E_k = C\, P^{\frac{1}{3}}\, (\omega_p r_D^2)\, k^{-\frac{1}{3}}, \qquad n_{\mathbf{k}} = C'\, P^{\frac{1}{3}}\, k^{-\frac{13}{3}}. \tag{36}$$

Using equation (28), $k_{\mathrm{NL}} \sim (P/\omega_p^3 r_D^6)^{1/4} \to 0$ as $P \to 0$ so this spectrum should break down at large scales. The second spectrum carrying the wave action flux is

$$n_{\mathbf{k}} = C\, Q^{\frac{1}{3}}\, (\omega_p r_D^2)^{\frac{1}{3}}\, k^{-\frac{11}{3}}, \tag{37}$$

which should also break down at large scales since $k_{\mathrm{NL}} \sim (Q/\omega_p^3 r_D^6)^{1/2} \to 0$ as $Q \to 0$ from equation (29). These spectra were originally derived by Zakharov [16] (1972).

6 A 5-Wave Example

One-dimensional gravity waves were considered by Dyachenko et al [17]. They found that the 4-wave interaction coefficient is identically zero on the resonant manifolds so that the nonlinear exchange of energy in the system is, in fact, due to 5-wave interactions. This system is an example of a case where the dimension of the physical energy density differs from the dimension of the Fourier space. In this case, $d = 2$ but $D = 1$. For this system, $\lambda = \sqrt{g}$ and $\alpha = \frac{1}{2}$. Applying (15) and (12) we obtain the finite energy flux spectrum

$$\begin{aligned} E_k &= C P^{\frac{1}{4}} g^{\frac{5}{8}} k^{\frac{-21}{8}}, \\ n_{\mathbf{k}} &= C'(P\sqrt{g})^{\frac{1}{4}} k^{\frac{-25}{8}}, \end{aligned} \tag{38}$$

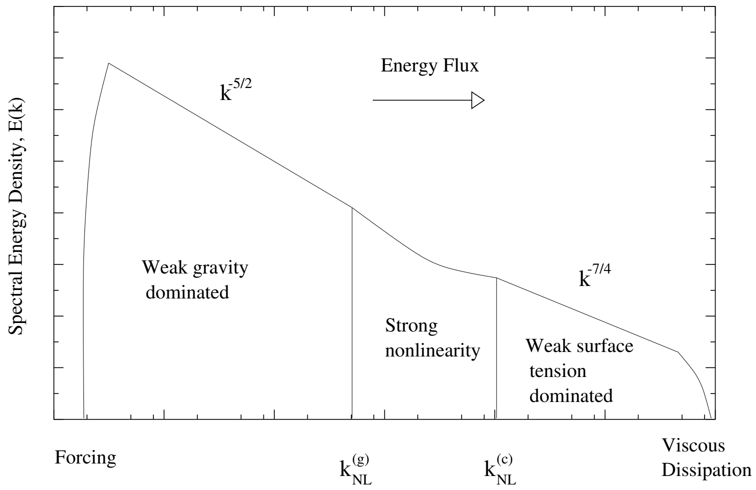

Fig. 1. Schematic representation of the different scaling regimes in the surface wave problem. The width of the window of fully nonlinear turbulence depends on the energy flux input at the forcing scale.

as found in [17]. From equation (28), $k_{\rm NL} \sim P^{-2/3} g$ goes to large $\mathbf{k}$ for small P. Therefore this spectrum should break down at small scales.

7 Matching the Gravity and Capillary Wave Spectra

Let us now consider the complete surface wave problem including both gravity and surface tension effects. At large scales the system is entirely gravity dominated. We assume that the forcing is at large scales only. At small scales the system is entirely surface tension dominated down to the viscous scale where the wave energy is finally dissipated. The characteristic scale, k_0, where surface tension and gravity are comparable can be estimated from the dispersion relation,

$$\omega(k) = \sqrt{gk + \sigma k^3}. \tag{39}$$

The gravity and surface tension effects are of comparable order when

$$k \approx k_0 = \sqrt{\frac{g}{\sigma}}. \tag{40}$$

We expect that for $k << k_0$ the system is well described by the gravity wave spectrum (33), for $k >> k_0$ the system is well described by the capillary wave spectrum, (32). In between there is a non-scale invariant cross-over regime. Let us consider the question of whether the weak turbulence approximation remains consistent through this cross-over regime. In order for the turbulence to remain weak as we approach k_0 from the left, the gravity wave spectrum must remain valid at least to the scale k_0 where surface tension effects can start carrying the flux. Thus we require

$$k_{\mathrm{NL}}^{(g)} > k_0 \tag{41}$$

where $k_{\mathrm{NL}}^{(g)}$ is the breakdown scale for pure gravity waves which we calculate from (28) :

$$k_{\mathrm{NL}}^{(g)} \approx P^{-\frac{2}{3}} g. \tag{42}$$

Using expressions (42) and (40) this gives us a condition on the flux,

$$P < (g\sigma)^{\frac{3}{4}} \tag{43}$$

In order for the turbulence to remain weak as we approach k_0 from the right, the capillary wave spectrum should be valid by the time we reach scale k_0 so that it can connect to the gravity wave spectrum. Thus we require

$$k_{\mathrm{NL}}^{(c)} < k_0 \tag{44}$$

where $k_{\mathrm{NL}}^{(c)}$ is the breakdown scale for pure capillary waves,

$$k_{\mathrm{NL}}^{(c)} \approx P^{\frac{2}{3}} \sigma^{-1}. \tag{45}$$

Inserting expressions (45) and (40) this gives us the same condition, (43), on the flux! We see that there is a critical energy flux, $P_c = (g\sigma)^{3/4}$ which can be carried by the weak turbulence spectra. The issue of what happens if $P > P_c$ is of paramount interest. It is clear that in this case there is a window in k space corresponding roughly to $\left[k_{\mathrm{NL}}^{(c)}, k_{\mathrm{NL}}^{(g)}\right]$ where the nonlinearity is not weak and the dynamics is presumably dominated by fully nonlinear structures. This situation is illustrated schematically in figure 1. It is suggestive that the value of P_c, if expressed in terms of the wind speed, corresponds roughly to the threshold for the formation of whitecaps on the ocean surface[18].

The phenomenon of intermittency is thought to be associated with the generation of such strongly nonlinear structures and would manifest itself in a deviation of the structure functions, $S_N(\mathbf{r}_1, \ldots \mathbf{r}_{N-1})$, of the wave field from joint Gaussianity. If one assumes that the statistics are dominated by whitecaps then one can estimate the scaling behaviour of field gradients. However, the support of the set of singularities need not be simple set. There are reasons to expect that whitecaps are supported on a fractal set of dimension

$0 \leq D \leq 1$ [1] although we will consider fractal sets up to dimension 2. In this case the n^{th} moment of the field gradients scales as

$$S_N(\mathbf{r}) \sim (\Delta\theta)^N \left(\frac{r}{L}\right)^{(2-D)}, \tag{46}$$

where $\Delta\theta$ is a characteristic size of the jump discontinuities in the derivative and L is the integral scale. It then follows (see [19] sec. 8.5) that,

$$\frac{S_{2N}(\mathbf{r})}{(S_2(\mathbf{r}))^N} \sim \left(\frac{r}{L}\right)^{(1-N)(2-D)}. \tag{47}$$

For $D < 2$ the system deviates from joint Gaussianity. Such behaviour is generally thought to be beyond the standard picture of weak turbulence. Nonetheless, Biven, Nazarenko and Newell [2; 3] have calculated the first correction to joint Gaussian statistics in the case where the weak turbulence approximation breaks down. For N even,

$$\frac{S_{2N}(\mathbf{r})}{(S_2(\mathbf{r}))^N} = 1 + \sum_{i=1}^{N/2} C_{N_i} \left(P^{1/3} r^{\alpha - \frac{\gamma}{3}}\right)^{2i-1} + \ldots \tag{48}$$

Breakdown occurs for $\gamma > 3\alpha$ in which case, the second term in (48) scales like $r^{(\alpha-\gamma/3)(N-1)}$ as $r \to 0$. If we wish to attribute this breakdown to the emergence of whitecap-dominated behaviour, we observe that it is possible to match the scalings (47) and (48) for *all* N if we choose

$$D = \alpha - \frac{\gamma}{3} + 2. \tag{49}$$

For gravity waves, $\alpha = 1/2$ and $\gamma = 3$ so the dimension of the set of whitecaps would be $3/2$. It would be nice if one obtained a value of D less than one but there are several reasons why this argument is an oversimplification. In particular, expression (48) represents only the first terms in an infinite series. It is highly likely that in the regime where weak turbulence breaks down, the higher order corrections which are neglected here actually contribute strongly. Nonetheless it is a nontrivial fact that this matching can be done consistently for all values of N simultaneously, even if the actual value of the fractal dimension obtained here must be considered with caution.

[1] The authors would like to thank Prof. V.E. Zakharov for sharing his observations on this topic.

8 Summary and Conclusion

Our aim in this article was to show that many of the commonly considered applications of weak turbulence possess the incomplete self-similarity property, which can be exploited to obtain core results using a simple dimensional argument without resorting to the more complex methods required in general. For such systems, recent results on the breakdown and range of applicability of weak turbulence can also be obtained in a simple way. It was found that dimensional considerations rule out the development of small scale intermittency in most physically relevant three-wave systems.

We considered the gravity-capillary surface wave system in more detail and discussed, from the point of view of the dimensional quantities present, how the validity of the weak turbulence approximation depends only on the energy flux input at the largest scales. Even in the case where this flux is large enough to cause breakdown of the gravity dominated part of the spectrum, we still have a consistent mechanism to transfer energy to the viscous scale which consists of two different weakly turbulent regimes connected by a window of scales where the nonlinearity is strong. We made some speculative observations about the relationship between the breakdown of weak turbulence and the emergence of whitecaps in this window of strong turbulence.

It is appropriate that we finish with some balancing remarks about situations in which the simple approach outlined here does not work. Firstly, there are obviously cases of physical interest which are self-similar. Some important examples are provided by optical waves of diffraction in nonlinear dielectrics and the turbulence of waves on Bose-Einstein condensates, both of which are described by the Nonlinear Schrodinger equation [20]. In these cases, there are two relevant dimensional parameters.

Secondly, even in the case of incompletely self-similar systems, a cautionary note should be sounded. The long time behaviour of these systems is determined by the kinetic equation for the spectral wave action density,

$$\frac{\partial n_k}{\partial t} = T_2\left[n_k\right] + T_4\left[n_k\right] + \dots, \tag{50}$$

and a nonlinear frequency modulation,

$$\omega_k \to \omega_k + \Omega_2\left[n_k\right] + \Omega_4\left[n_k\right] + \dots. \tag{51}$$

For 4-wave systems with an interaction coefficient, $T_{kk_1k_2k_3}$,

$$T_2\left[n_k\right] = 0, \tag{52}$$

$$T_4\left[n_k\right]=\int\left|T_{kk_1k_2k_3}\right|^2 n_1n_2n_3\delta(\omega_{01,23})\delta(\mathbf{k}_{01,23})d\mathbf{k}_{123}+n_k\mathrm{Im}\ \Omega_4\left[n_k\right], \quad (53)$$

with the frequency modulation integrals given by

$$\Omega_2\left[n_k\right]=\int T_{kk_1kk_1}n_1\ dk_1, \quad (54)$$
$$\mathrm{Im}\ \Omega_4\left[n_k\right]=\int\left|T_{kk_1k_2k_3}\right|^2(n_1n_3+n_1n_2-n_2n_3)\delta(\omega_{01,23})\delta(\mathbf{k}_{01,23})d\mathbf{k}_{123}. \quad (55)$$

Notice that the imaginary part of the frequency modulation enters the collision integral. It has been pointed out by Zakharov and others that if the interaction coefficient is not uniformly homogeneous in its arguments,

$$T_{kk_1k_2k_3}\sim(kk_2)^{(1-\chi)\gamma}(k_1k_3)^{\chi\gamma} \qquad 0\le\chi\le1, \quad (56)$$

for example, the frequency correction, Im $\Omega_2\left[n_k\right]$ can be divergent at low $\mathbf{k}$, even though T is still homogeneous of the same degree, γ. Luckily, this divergence cancels in the kinetic equation to lowest order, $T_4\left[n_k\right]$. However, the divergences in the Ω's may resurface at higher orders in the full collision integral, (50). It is not clear whether we are rescued by such cancellations as occur in $T_4\left[n_k\right]$.

If these divergences persist then the dimensional argument applied in section 3 to estimate the range of validity and breakdown scales would be not valid and would require modification to include some non-universal dependence on χ and a low $\mathbf{k}$ cutoff. Thus wave turbulence may possess a mechanism for breaking incomplete self-similarity even without the need to introduce additional dimensional parameters. Whether this occurs in reality is, at present, an open question.

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