On wave turbulence in MHD

S. Galtier, S.V. Nazarenko, and A.C. Newell

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, U.K.

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Abstract

We describe the fundamental differences between weak (wave) turbulence in incompressible and weakly compressible MHD. The main difference is in the structure of the resonant manifolds and the mechanisms of redistribution of spetral densities along the applied magnetic field $\mathbf{B_0}$. Like for pure acoustic waves, a three-wave resonance between collinear wavevectors is observed but, in addition, we also have a resonance through tilted planes and spheres. The properties of resonances and their consequences for the asymptotics are also discussed.

1 Introduction

It is widely recognised that magnetohydrodynamics (MHD) provides a first satisfactory description of many turbulent plasmas encountered in astrophysics (Priest, 1982; Heiles et al., 1993; Pouquet, 1993; Marsch and Tu, 1994; Parker, 1994), as well as in laboratory devices such as tokamaks (Taylor, 1986; Gekelman and Pfister, 1988). MHD turbulence differs significantly from hydrodynamic turbulence in the fact that a strong magnetic field has a non-trivial effect on the dynamics. It was Iroshnikov (1963) and Kraichnan (1965) (hereafter IK) who first recognised that presence of Alfvén waves travelling in opposite directions along a local large magnetic fields leads to the weakening of energy transfer to small scales and therefore to a modification of the scaling of the (Kolmogorov) energy spectrum from $k^{-5/3}$, for neutral fluids, to $k^{-3/2}$.

In the IK phenomenology MHD turbulence is assumed to be 3D isotropic. However in many realistic situations the presence of strong magnetic fields is observed which makes MHD turbulence strongly anisotropic. Anisotropy is manifested in a two dimensionalisation of the turbulence spectrum in a plane transverse to the locally dominant magnetic field and in inhibiting spectral energy transfer along the direction parallel to the field (Montgomery and Turner, 1981; Montgomery and Matthaeus, 1995; Matthaeus et al., 1996; Kinney and McWilliams, 1998). Replacing the 3D isotropy assumption by a 2D one, and retaining the rest of the IK dimensional analysis, gives a $k_\perp^{-2}$ spectrum ($\mathbf{B_0} = B_0\, \hat{\mathbf{e}}_{||}$, the applied magnetic field, $k_{||} = \mathbf{k} \cdot \hat{\mathbf{e}}_{||}$, $\mathbf{k}_\perp = \mathbf{k} - k_{||}\, \hat{\mathbf{e}}_{||}$, $k_\perp = |\mathbf{k}_\perp|$).

MHD turbulence in presence of a strong uniform magnetic field has been recently investigated by several authors. Ng and Bhattacharjee (1996, 1997) developed a theory of weakly interacting Alfvén wave packets which takes into account anisotropy. This leads to certain predictions for the turbulence spectra based on some additional phenomenological assumptions. More recently Galtier et al. (2000) developed a rigorous theory of weak Alfvén turbulence in incompressible MHD. The derivation of such a theory, at the level of three–wave interactions, via a systematic asymptotic expansion in powers of small nonlinearity leads to some exact results which will be discussed in detail in the next section. One of the main results obtained is that the $k_\perp^{-2}$ spectrum is an exact finite flux Kolmogorov solution of the wave turbulence equation and that it corresponds to local turbulence: the collision integral converges on this spectrum. This property is a necessary and sufficient condition to apply the Zakharov transformation which reveals the $k_\perp^{-2}$ solution. Convergence means that the interactions between triads of waves with large wavevector separations decay sufficiently fast for the collision integral to exist and that therefore the main contribution comes from triads with similar wavevectors. On the other hand, according to the IK picture, the existence of the (short) Alfvén waves themselves in MHD turbulence is due to the action of a strong magnetic field carried by the large scales. This is obviously another type of non-locality which is assumed here and which is not going to be examined.

Observations of turbulence in the solar wind and the interstellar medium reveal the presence of density variations. Therefore any theory based on incompressible MHD has to be seen as a first attempt to describe turbulent plasmas. However it is a (much) more difficult task to include compressibility effects in a rigorous theory like weak turbulence. In section 3 we present the basis on which a weakly compressible turbulence theory can be derived. In addition to Alfvén waves, fast and slow magneto–acoustic waves have now to be taken into account. The resonant manifolds are much richer than in the incompressible case and they allow tranfer along the strong uniform magnetic field. The way by which this transfer is possible is like the one for acoustic waves, namely along rays in all directions, but also through tilted planes and spheres. The two last mechanisms of transfer are due to the interaction between magnetic and acoustic waves. We also show that the presence of double resonances may lead to different asymptotics according to the dimension of the problem.

2 Weak alfvénic turbulence

There is an extensive literature devoted to weak turbulence (see *e.g.* Vedenov (1967); Benney and Newell (1969); Sagdeev and Galeev (1969); Zakharov (1974); Akhiezer et al. (1975); McIvor (1977); Zakharov, Lvov and Falkovich (1992)). Basically the essence of weak turbulence is the statistical study of large ensembles of weakly interacting waves via a systematic asymptotic expansion in powers of small nonlinearity. This approach leads finally to the derivation of “kinetic equations” for quantities like the wave energy. Previously it was believed that this asymptotic expansion was consistent only for dispersive or semi–dispersive (*e.g.* acoustic) waves. The study by Galtier et al. (2000) shows that non-dispersive Alfvén waves are in fact a unique exception from this rule which gives properties of weak alfvénic turbulence very different to those observed for dispersive waves. The physical reason is that for non–dispersive waves all wavepackets propagate with the same group velocity even if their wavenumbers are different. Thus, no matter how weak the nonlinearity is, the energy exchanged between the wavepackets will be accumulated over a long time and it may not be considered small, as it would be required in the weak turbulence theory. Alfvén waves are an exception because the nonlinear interaction coefficient for co-propagating waves is null, whereas the counter-propagating wavepackets pass through each other in a finite time and exchange only small amounts of energy over the waveperiod timescale which makes the weak turbulence approach applicable in this case.

2.1 Kinetic equation for the energy spectrum

In this section we would like to review briefly the kinetic equation for the energy spectrum. Details of the derivation are given in Galtier et al. (2000) but for the coherence of this paper, some important steps of the derivation will be outlined.

We start with the 3D incompressible MHD equations for the velocity $\mathbf{v}$ and the Alfvén velocity $\mathbf{b}$

$$(\partial_t + \mathbf{v} \cdot \nabla)\, \mathbf{v} = -\nabla P_* + \mathbf{b} \cdot \nabla\, \mathbf{b} + \nu \nabla^2 \mathbf{v}\,, \tag{1}$$

$$(\partial_t + \mathbf{v} \cdot \nabla)\, \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{v} + \eta \nabla^2 \mathbf{b}\,, \tag{2}$$

$$\nabla \cdot \mathbf{v} = 0\,, \tag{3}$$

$$\nabla \cdot \mathbf{b} = 0\,, \tag{4}$$

where P_* is the total pressure, ν the viscosity, η the magnetic diffusivity, $\mathbf{b} = \mathbf{B}/\sqrt{\mu_0 \rho_0}$, with ρ_0 the uniform density and μ_0 the magnetic permeability. We follow the eulerian wave turbulence approach which can be found for instance in Benney and Newell (1969).

We assume the presence of a strong uniform magnetic field B_0 and that the wave amplitudes are small which allow us to introduce a small parameter $\epsilon \ll 1$. Introducing potentials ψ^s and ϕ^s which are related to the Elsässer fields (whose amplitude are proportional to ϵ) $\mathbf{z^s} = \mathbf{v} + s\mathbf{b}$ $(s = \pm 1)$ via

$$\mathbf{z^s} = \nabla \times (\psi^s \hat{\mathbf{e}}_{\|} + \nabla \times \phi^s \hat{\mathbf{e}}_{\|}) \,, \tag{5}$$

we derive (Galtier et al., 2000) a closed system of kinetic equations for the second correlations of the Fourier transforms $\hat{\psi}^s(\mathbf{k})$ and $\hat{\phi}^s(\mathbf{k})$. The physically important combinations of these correlators are the total energy $e^s(\mathbf{k})$, poloidal and toroidal energies, $\Phi^s(\mathbf{k})$ and $\Psi^s(\mathbf{k})$, pseudo magnetic $I^s(\mathbf{k})$ and magnetic $R^s(\mathbf{k})$ helicities whose definitions are

$$\langle \hat{\psi}^s(\mathbf{k}) \hat{\psi}^s(\mathbf{k}') \rangle = \delta(\mathbf{k} + \mathbf{k}') \Psi^s(\mathbf{k}') \,, \tag{6}$$

$$\langle \hat{\phi}^s(\mathbf{k}) \hat{\phi}^s(\mathbf{k}') \rangle = \delta(\mathbf{k} + \mathbf{k}') \Phi^s(\mathbf{k}') \,, \tag{7}$$

$$\langle \hat{\psi}^s(\mathbf{k}) \hat{\phi}^s(\mathbf{k}') \rangle + \langle \hat{\phi}^s(\mathbf{k}) \hat{\psi}^s(\mathbf{k}') \rangle = \delta(\mathbf{k} + \mathbf{k}') R^s(\mathbf{k}) \,, \tag{8}$$

$$\langle \hat{\psi}^s(\mathbf{k}) \hat{\phi}^s(\mathbf{k}') \rangle - \langle \hat{\phi}^s(\mathbf{k}) \hat{\psi}^s(\mathbf{k}') \rangle = \delta(\mathbf{k} + \mathbf{k}') I^s(\mathbf{k}) \,, \tag{9}$$

and

$$e^s(\mathbf{k}) = \mathbf{k}_\perp^2 (\Psi^s(\mathbf{k}) + k^2 \Phi^s(\mathbf{k})) \,, \tag{10}$$

where $\langle\rangle$ denotes average.

The resulting kinetic equations are lengthy (Galtier et al., 2000) and for what follows it suffices to write only the equation for the energy spectrum $e^s(\mathbf{k})$,

$$\frac{\partial e^s(\mathbf{k})}{\partial t} = \frac{\pi \varepsilon^2}{b_0} \int \Bigg[\left(L_\perp^2 - \frac{X^2}{k^2} \right) \Psi^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L^2} \right) \Psi^s(\mathbf{k}) + \left(L_\perp^2 L^2 - \frac{k_\|^2 W^2}{k^2} \right) \Phi^s(\mathbf{L}) \tag{11}$$
$$- \left(k_\perp^2 k^2 - \frac{k_\|^2 Y^2}{L^2} \right) \Phi^s(\mathbf{k}) + \left(\frac{k_\| X Y}{L^2} \right) I^s(\mathbf{k}) - \left(\frac{k_\| X W}{k^2} \right) I^s(\mathbf{L}) \Bigg] Q_k^{-s}(\boldsymbol{\kappa}) \delta(\kappa_\|) \, \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} d\boldsymbol{\kappa} \, d\mathbf{L} \,,$$

with

$$\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}} = \delta(\mathbf{L} + \boldsymbol{\kappa} - \mathbf{k}) \,,$$

$$Q_k^{-s}(\boldsymbol{\kappa}) = X^2 \Psi^{-s}(\boldsymbol{\kappa}) + X(k_\| \kappa_\perp^2 - \kappa_\| Y) I^{-s}(\boldsymbol{\kappa}) + (\kappa_\| Y - k_\| \kappa_\perp^2)^2 \phi^{-s}(\boldsymbol{\kappa}) \,, \tag{12}$$

$$X = k_\perp \kappa_\perp \sin\theta \,,$$

$$Y = k_\perp \kappa_\perp \cos\theta \,,$$

$$W = k_\perp^2 - L_\perp^2 - k_\perp \kappa_\perp \cos\theta \,,$$

where θ is the angle between $\mathbf{k}_\perp$ and $\boldsymbol{\kappa}_\perp$, and with

$$d\boldsymbol{\kappa}_\perp = L_\perp d\kappa_\perp dL_\perp / k_\perp \sin\theta \,,$$

$$\cos\theta = (\kappa_\perp^2 + k_\perp^2 - L_\perp^2)/2\kappa_\perp k_\perp \,.$$

The integral on the RHS of (11) (called the collision integral or CI) is generally non-zero. It is an equation for the slow time change of the leading term $e^s(\mathbf{k})$ in an asymptotic expansion for the energy density and is chosen so as to remove all secular terms $(\epsilon^2 t)^N$, $N = 1, 2, 3, ...$, which appear due to the three wave processes. Higher order corrections to the kinetic equations can arise from higher order processes, such as four wave interactions, but they do not affect the result on the time scale $t_{tr} = \sqrt{\mu_0 \rho_0}/\epsilon^2 B_0 k_\perp$. Even at longer time scales, these corrections are only significant if they cause a redistribution of energy in $\mathbf{k}$ space not mediated by three wave processes. The main point, however, is that on the time scale t_{tr}, three wave interactions dominate. This subtle point is at the origin of a recent controversy (Sridhar and Goldreich, 1994; Goldreich and Sridhar, 1995). The original error of Goldreich and Shridhar (GS) was in failing to recognise that the power (the Fourier transform of the two-point correlation) contained in a zero mean signal need not be zero even at zero wavenumber (frequency).

2.2 General properties

The delta function in the CI arises because of the three wave frequency resonance condition. Thus, in any resonantly interacting wave triad ($\mathbf{k}$, $\boldsymbol{\kappa}$, $\mathbf{L}$), there is always one wave that corresponds to a purely 2D motion – having no dependence on the direction parallel to the uniform magnetic field – whereas the other two waves have equal parallel components of their corresponding wavenumbers, *viz.* $L_\parallel = k_\parallel$. Such a property was recently discussed in Kinney and McWilliams (1998). This means that the parallel components of the wavenumber enter in the kinetic equation of the total energy $e^s(\mathbf{k})$ as an external parameter and that the dynamics is decoupled at each level of $k_\parallel$. In other words, there is no energy transfer associated with the three wave resonant interaction along the $k_{||}$-direction in $\mathbf{k}$-space for the total energy : the resonant manifolds of the energy redistribution process foliate $\mathbf{k}$-space. The absence of the parallel transfer was also discussed in Kinney and McWilliams (1998) who also presented a numerical confirmation of this fact. Further, for every triad of Alfvén waves $\mathbf{k}, \mathbf{L}$ and $\boldsymbol{\kappa}$ (such that $\kappa_{||} = 0$) the energy is conserved within two co-propagating waves having wavevectors $\mathbf{k}$ and $\mathbf{L}$ because of the symmetry of CI with respect to changing $\mathbf{k} \leftrightarrow \mathbf{L}$ and $\boldsymbol{\kappa} \leftrightarrow -\boldsymbol{\kappa}$.

The first consequence of the fact that there is no transfer of the total energy in the $k_{||}$ direction in $\mathbf{k}$-space is an asymptotic two-dimensionalisation of the energy spectrum $e^s(\mathbf{k})$. Namely, the 3D initial spectrum spreads over the transverse wavenumbers, $\mathbf{k}_\perp$, but remains of the same size in the $k_{||}$ direction, and the support of the spectrum becomes very flat (pancake-like) for large time. The two-dimensionalisation of weak MHD turbulence has been observed in laboratory experiments (Robinson and Rusbridge, 1971), and in the the solar wind data (Belcher and Davis, 1971; Bavassano et al., 1982; Horbury et al., 1995; Bieber, Wanner and Matthaeus, 1996), as well as in many direct numerical simulations of the three–dimensional MHD equations (Oughton, Priest and Matthaeus, 1994; Kinney and McWilliams, 1998). From the mathematical point of view, the two-dimensionalisation of the total energy means that, for large time, the energy spectrum $e^s(\mathbf{k})$ is supported on a volume of wavenumbers such that for most of them $k_\perp \gg k_{||}$. This implies that $\Psi^s(\mathbf{k})$ and $\Phi^s(\mathbf{k})$ are also supported on the same anisotropic region of wavenumbers because both of them are non-negative. This fact allows one to expand the integrands in the kinetic equations in powers of small $k_{||}/k_\perp$. At leading order (and taking $\Phi^s(\mathbf{k}) = 0$), one obtains

$$\frac{\partial\left[k_\perp^2 \Psi^s(\mathbf{k})\right]}{\partial t} = \qquad (13)$$

$$\frac{\pi\varepsilon^2}{b_0} \int \left[\left(L_\perp^2 - \frac{X^2}{k_\perp^2}\right) \Psi^s(\mathbf{L}) - \left(k_\perp^2 - \frac{X^2}{L_\perp^2}\right) \Psi^s(\mathbf{k}) \right] X^2 \Psi^{-s}(\boldsymbol{\kappa})\, \delta(\kappa_{||})\, \delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d\boldsymbol{\kappa}\, d\mathbf{L}\,.$$

An important consequence of the dynamical decoupling at different $k_{||}$'s within the kinetic equation formalism is that the set of purely 2D modes (corresponding to $k_{||} = 0$) evolve independently of the 3D part of the spectrum (with $k_{||} \neq 0$) and can be studied separately. This property was also observed in direct numerical simulations (Kinney and

McWilliams, 1998).

It is important to note that the kinetic equations themselves are applicable to a description of $k_{||} = 0$ modes only if the correlations of the dynamical fields decay in all directions, so that their spectra are sufficiently smooth for all wavenumbers including the ones with $k_{||} = 0$. To be precise, the characteristic scale $k_{||}$ over which the spectra can experience significant changes must be greater than ϵ^2 because the kinetic equation arises from a limit process $B_0 k_{||} t \to \infty$, $\epsilon^2 B_0 k_\perp t$ finite, where $k_{||}$ and $k_\perp$ are dominant parallel and perpendicular wavenumbers. The parallel wavenumber $k_{||}$ enters equation (13) only as an external parameter. In other words, the wavenumber space is foliated into the dynamically decoupled planes $k_{||} =$ constant. Thus, the large-time asymptotic solution can be found in the following form,

$$\Psi^s(\mathbf{k}_\perp, \mathbf{k}_{||}) = f(k_{||}) \Psi^s(\mathbf{k}_\perp, 0) \,, \tag{14}$$

where f is some arbitrary function of $k_{||}$ satisfying the conditions $f(0) = 1$. Substituting this formula into (13), one can readily see that the solution of the 3D equations is reduced to solving a 2D problem for $\Psi^s(\mathbf{k}_\perp, 0)$.

Using the Zakharov transform, we solved this 2D problem and found the exact stationary solutions in power laws which are

$$E^s_\perp(k_\perp, 0) = k_\perp^3 \Psi^s(k_\perp, 0) \sim k_\perp^{n_s} \,, \tag{15}$$

where $n_+ + n_- = -4$. In the particular case of a zero correlation between the velocity and the magnetic field one has $n_+ = n_- = -2$. The Kolmogorov constants appearing in front of the spectra are computed exactly and found to depend on the amount of correlation between the velocity and the magnetic field. We also prove that the collision integral converges for all $-3 < n_s < -1$, which gives an a posteriori justification for the application of the Zakharov transform. Numerical simulations show the $k_\perp^{-2}$ solution to be attractive and that the small scales are reached in a finite time *i.e.* in a catastrophic way. In particular, and very much to our surprise, it was observed (Galtier et al., 2000) that the spectral energy density propagates to large wavenumbers following a stationary $k_\perp^{-7/3}$– spectrum and not a $k_\perp^{-2}$– spectrum, the latter obtaining during the self–similar decay phase. Then, when the dissipative scale is reached, the $k_\perp^{-7/3}$ solution suddenly turns into the finite energy flux spectrum $k_\perp^{-2}$ with a change of the slope propagating from small scales to large scales. This picture is very different from the scenario proposed by Falkovich and Shafarenko (1991) for finite capacity spectra and has many of the hallmarks of a dramatic new type of phase transition. Note that a similar behaviour is also observed in the formation of Bose condensates (Svistunov, 1991; Lacaze et al, 1999).

3 Compressible MHD

We would like now to emphasise the important effect of weak compressibility (see *e.g.* Kaburaki and Uchida (1971); Zank and Matthaeus (1993); Bhattacharjee et al. (1998)) in weak MHD turbulence. Our aim here is not to derive the complete theory (the detailed analysis is being carried out by us in a separate work) but rather to show, by considering the resonant manifolds in the limit of small compressibility, that the physical picture of spectral densities redistribution may differ significantly from the incompressible case. The main difference is that transfer along the strong uniform magnetic field is a priori no longer forbidden. The way by which this transfer is allowed is like for acoustic waves, namely along rays in all directions (Newell and Aucoin, 1971; L'vov et al, 1997), but in addition we also have transfer through tilted planes and spheres. The two last mechanisms find their origin in the interaction between magnetic and acoustic waves.

3.1 MHD waves

We consider now the 3D compressible inviscid MHD equations

$$\partial_t \rho + \nabla(\rho \mathbf{v}) = 0\,, \tag{16}$$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho} + \frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B}\,, \tag{17}$$

$$\partial_t \mathbf{B} = -\nabla \times (\mathbf{B} \times \mathbf{v})\,, \tag{18}$$

$$P = P_0 (\frac{\rho}{\rho_0})^\gamma\,, \tag{19}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{20}$$

where P is the pressure and γ the polytropic index. We consider waves with small amplitude in the presence of a strong uniform magnetic field B_0 and we introduce the following notations

$$\mathbf{B} = B_0\,\hat{\mathbf{e}}_{||} + \epsilon B_0 \mathbf{b_1}\,, \tag{21}$$

$$\mathbf{v} = \epsilon\, C_A \mathbf{v_1}\,, \tag{22}$$

$$\rho = \rho_0 + \epsilon\rho_0\rho_1\,, \tag{23}$$

where $\epsilon \ll 1$, $C_A \equiv B_0/\sqrt{\mu_0\rho_0}$ is the Alfvén speed and $\hat{\mathbf{e}}_{||} = (0,0,1)$. Let us use a 2.5D geometry to derive the well-known dispersion relations. MHD equations linearised and expressed in terms of the adimensionalised variables (21)–(23) give in the Fourier space (the symbol $\hat{}$ stands for the Fourier transform)

$$\partial_t \hat{\mathbf{W}} = -iC_A k \begin{pmatrix} 0 & M_A\sin\theta & 0 & M_A\cos\theta & 0 & 0 \\ M_A\sin\theta & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\cos\theta \\ M_A\cos\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\cos\theta & 0 & 0 & 0 \end{pmatrix} \hat{\mathbf{W}}\,, \tag{24}$$

where $\hat{\mathbf{W}} \equiv (M_A\hat{\rho}_1, \hat{v}_{1x}, \hat{v}_{1y}, \hat{v}_{1z}, \hat{b}_{1x}/\cos\theta, \hat{b}_{1y})$, $M_A \equiv C_S/C_A$, $C_s \equiv \sqrt{\gamma P_0/\rho_0}$ is the sound speed, $\mathbf{k} = k(\sin\theta, 0, \cos\theta)$, $\theta = (\widehat{\mathbf{k}, \mathbf{B_0}})$ and $k = |\mathbf{k}|$. Note that in (24) the divergence free condition for the magnetic field is used to reduce the initial system of seven variables to six variables. With this choice of variables the expression for total energy of the system per unit volume E_T (the sum of kinetic, magnetic and internal energies) is then simply written as

$$\langle E_T \rangle = \frac{B_0^2}{2\mu_0} + \frac{P_0}{\gamma - 1} + \frac{1}{2}\,\epsilon^2\rho_0\, C_A^2 \sum_{\ell=1}^{6} \langle W_\ell^2 \rangle\,, \tag{25}$$

where W_ℓ is the ℓ-th component of $\mathbf{W}$. The diagonalisation of the matrix leads finally to the well-known dispersion relations (see *e.g.* Jeffrey and Taniuti (1964))

$$\partial_t A_j^s(\mathbf{k}) + i\omega_j^s A_j^s(\mathbf{k}) = 0\,, \tag{26}$$

$$\omega_j^s = s\omega_j(\mathbf{k}) = sk\, C_A\, M_j \qquad (j=1,2,3)\,, \tag{27}$$

$$M_1 \equiv M_F = \sqrt{\frac{1+M_A^2+\sqrt{(1+M_A^2)^2-4M_A^2\cos^2\theta}}{2}}\,, \tag{28}$$

$$M_2 \equiv M_S = \sqrt{\frac{1+M_A^2-\sqrt{(1+M_A^2)^2-4M_A^2\cos^2\theta}}{2}}\,, \tag{29}$$

$$M_3 = \cos\theta\,, \tag{30}$$

where $M_{F/S}$ and M_A are respectively the phase velocity of fast/slow magneto–acoustic wave and Alfvén wave; $s=\pm$ refers to the direction of propagation of waves; the normal vector is

$$\mathbf{A}^s(\mathbf{k}) = \begin{pmatrix} \frac{-1}{2(M_F^2-M_S^2)}\{M_A\sin\theta(\hat{W}_1+sM_S\hat{W}_4)+(M_S^2-1)(\hat{W}_5-sM_F\hat{W}_2)\} \\ \frac{1}{2(M_F^2-M_S^2)}\{M_A\sin\theta(\hat{W}_1+sM_F\hat{W}_4)+(M_F^2-1)(\hat{W}_5-sM_S\hat{W}_2)\} \\ \frac{1}{2}\{\hat{W}_6-s\hat{W}_3\} \end{pmatrix}. \tag{31}$$

Then, after having inversed the previous system to express $\hat{\mathbf{W}}$ in terms of the normal vector, we can write the expressions for the invariants of compressible MHD whose density spectra are, for total energy (at order ϵ^2)

$$E_T(\mathbf{k}) = \rho_0\, C_A^2\{2(1-M_F^2)q_{11}^{+-}(\mathbf{k})+2(1-M_S^2)q_{22}^{+-}(\mathbf{k})+q_{33}^{++}(\mathbf{k})+q_{33}^{--}(\mathbf{k})\}\,, \tag{32}$$

and for the cross-correlation (whose definition is $E_C=\frac{1}{2}\langle\mathbf{v}\cdot\mathbf{B}\rangle$ in the physical space)

$$E_C(\mathbf{k}) = \frac{1}{2}(q_{33}^{--}(\mathbf{k})-q_{33}^{++}(\mathbf{k}))\,, \tag{33}$$

where the energy tensor is defined as

$$q_{j_1j_2}^{s_1s_2}(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}') \equiv \langle A_{j_1}^{s_1}(\mathbf{k})A_{j_2}^{s_2}(\mathbf{k}')\rangle\, e^{-i(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})t} = \langle a_{j_1}^{s_1}(\mathbf{k})a_{j_2}^{s_2}(\mathbf{k}')\rangle\,. \tag{34}$$

The last invariant, the magnetic helicity H_M, is null in the weak turbulence limit since it involves terms of type $q_{3\,j}^{s_1s_2}$ with $j=1,2$.

The normal vector is the starting point to derive in the interacting representation the fundamental equation for the wave amplitude where the slow variation in time is due to weak nonlinearities. Formally this equation may be written as

$$\partial_t a_j^s(\mathbf{k},t) = i\epsilon\int C_{m,n,s_1,s_2}^{j,s}(\mathbf{k},\boldsymbol{\kappa},\mathbf{L})\, a_m^{s_1}(\boldsymbol{\kappa})\, a_n^{s_2}(\mathbf{L})\, e^{i(s\omega_j(\mathbf{k})-s_1\omega_m(\boldsymbol{\kappa})-s_2\omega_n(\mathbf{L}))t}\delta_{\mathbf{k},\boldsymbol{\kappa}\mathbf{L}}\, d_{\boldsymbol{\kappa}\mathbf{L}}\,, \tag{35}$$

where the rule of summation over repeated indices is assumed (*i.e.* for m, n, s_1 and s_2). The interacting coefficient $C^{j,s}_{m,n,s_1,s_2}(\mathbf{k},\boldsymbol{\kappa},\mathbf{L})$ which contains information on nonlinearities is generally much more complicated than in the incompressible case. Zero interacting coefficients are observed for

$$C^{j,s}_{m,n,s_1,s_2} = 0 \qquad \text{for } \; j = 1,2 \text{ and } \begin{cases} m = 3;\; n = 1,2 \\ \text{or} \\ m = 1,2;\; n = 3 \end{cases}, \tag{36}$$

$$C^{j,s}_{m,n,s_1,s_2} = 0 \qquad \text{for } \; j = 3 \text{ and } \begin{cases} m = n \\ \text{or} \\ n,m = 1,2 \end{cases}. \tag{37}$$

In particular it means that for this choice of geometry if Alfvén waves are initially absent they cannot be produced by nonlinearities (Nakariakov et al, 1997). The exponentially oscillating term in (35) plays a fundamental role in the closure procedure: its exponent should not vanish when $\mathbf{k} = \boldsymbol{\kappa} + \mathbf{L}$. We shall look now for the resonant manifolds solution of the system

$$\begin{cases} s\omega_j(\mathbf{k}) = s_1\omega_m(\boldsymbol{\kappa}) + s_2\omega_n(\mathbf{L})\,, \\ \mathbf{k} = \boldsymbol{\kappa} + \mathbf{L}\,. \end{cases} \tag{38}$$

But in the general case we have to deal with a set of 40 systems of 2 equations which are generally not obvious to solve. Therefore in the next section we will restrict our attention to the weakly compressible limit.

3.2 Weakly compressible limit in 3D

We want to determine the resonant manifolds for three wave interactions in the 3D case in the weakly compressible limit. As it was explained before in incompressible MHD the resonant manifolds foliate wavevector space which leads to an absence of energy transfer along the main magnetic field. We will see that this transfer is no longer forbidden when magneto–acoustic waves are taken into account.

The starting point of our study is the function

$$h(\boldsymbol{\kappa};\mathbf{k}) = s_1\omega_m(\boldsymbol{\kappa}) + s_2\omega_n(\mathbf{k}-\boldsymbol{\kappa}) - s\omega_j(\mathbf{k})\,. \tag{39}$$

The resonant manifolds M is then defined as the ensemble of $\boldsymbol{\kappa}$ (s, s_1, s_2 and $\mathbf{k}$ given) for which $h(\boldsymbol{\kappa};\mathbf{k}) = 0$. As it will be explained later we are also interested to know if these resonances are double, *i.e.* if in the Taylor expansion of h in the neighbourhood of M

$$h(\boldsymbol{\kappa};\mathbf{k}) = h(\boldsymbol{\kappa}^{(0)};\mathbf{k}) + (\boldsymbol{\kappa} - \boldsymbol{\kappa}^{(0)}) \cdot \nabla_{\boldsymbol{\kappa}} h(\boldsymbol{\kappa}^{(0)};\mathbf{k}) + ... \qquad (\boldsymbol{\kappa}^{(0)} \in M) \tag{40}$$

the gradient is zero or not. Indeed the presence of double resonances may change the asymptotic according to the dimension of the problem. This is true, for example, for acoustic waves (Newell and Aucoin, 1971; L'vov et al, 1997).

In the weakly compressible limit ($M_A \to +\infty$, C_A finite) we have

$$\omega_F(\mathbf{k}) = k\, C_A(M_A + \frac{\sin^2\theta}{2}\frac{1}{M_A}) + O(\frac{1}{M_A^3}) \,, \tag{41}$$

$$\omega_S(\mathbf{k}) = k_{||}\, C_A + O(\frac{1}{M_A^2}) \,, \tag{42}$$

$$\omega_A(\mathbf{k}) = k_{||}\, C_A \,, \tag{43}$$

which means that slow magneto–acoustic waves degenerate with the Alfvén waves. At the main order the gradient of h reads (only the projection on $\mathbf{n}$, vector normal to the resonant manifold M, is considered since the tangential part is automatically null)

$$\nabla_{\boldsymbol{\kappa}} h \cdot \mathbf{n} = -[\mathcal{F}_{F/S}(\boldsymbol{\kappa}) + \mathcal{F}_{F/S}(\mathbf{k}-\boldsymbol{\kappa})]\boldsymbol{\kappa}^2 + \mathcal{F}_{F/S}(\mathbf{k}-\kappa)\mathbf{k}\cdot\kappa \\ -s_2\, \omega_{F/S}(\mathbf{k}-\boldsymbol{\kappa}) \frac{(k_{||}\kappa_\perp - k_\perp \kappa_{||})}{(\mathbf{k}-\boldsymbol{\kappa})^2} \,, \tag{44}$$

with

$$\mathcal{F}_{F/S}(\mathbf{k}) = \pm s \frac{C_A^2}{\omega_{F/S}(\mathbf{k})} \frac{k_\perp k_{||}}{k^2} \,, \tag{45}$$

where a positive sign must be taken for F and a negative sign for S. In particular we note the useful property

$$\mathcal{F}_{F/S}(a\mathbf{k}) = \mathcal{F}_{F/S}(\mathbf{k})/|a| \,. \tag{46}$$

For Alfvén waves we note that

$$\nabla_{\boldsymbol{\kappa}}\, \omega_A(\boldsymbol{\kappa}) = -\nabla_{\boldsymbol{\kappa}}\, \omega_A(\mathbf{k}-\boldsymbol{\kappa}) = C_A\, \hat{\mathbf{e}}_{||} \,. \tag{47}$$

We are now going to consider the different possibilities for three wave interactions.

3.2.1 FFF interaction

For interaction between three fast magneto–acoustic waves (FFF), we have to solve at the main order

$$h(\boldsymbol{\kappa};\mathbf{k}) = C_S\left(s_1\kappa + s_2|\mathbf{k}-\boldsymbol{\kappa}| - sk\right) = 0\,. \tag{48}$$

If we do not consider the trivial case $s = -s_1 = -s_2$ for which the solution is made by three null vectors, the resonances always occur between wave vectors purely collinear. More precisely if we define $\boldsymbol{\kappa} = \alpha\mathbf{k}$ we have

$$\alpha \leq 0 \quad \text{if: } s = -s_1 = s_2\,, \tag{49}$$

$$0 \leq \alpha \leq 1 \quad \text{if: } s = s_1 = s_2\,, \tag{50}$$

$$1 \leq \alpha \quad \text{if: } s = s_1 = -s_2\,. \tag{51}$$

It is the same situation as for pure acoustic waves. From (45) and (46) we see that we have in fact double resonances.

3.2.2 SSS/SSA/SAA/AAA interaction

Interaction between three slow magneto–acoustic waves (SSS) gives

$$h(\boldsymbol{\kappa};\mathbf{k}) = C_A\left((s_1 - s_2)\kappa_{\|} - (s - s_2)k_{\|}\right) = 0\,. \tag{52}$$

For case $s = s_1 = s_2$ the relation is always satisfied which means that the closure procedure does not work. However this conclusion depends of course on the value of the interacting coefficient. This point will be discussed in section 3.3. For $s = s_1 = -s_2$ the resonant manifold is the plane $\kappa_{\|} = k_{\|}$; for $s = -s_1 = s_2$ it is the plane $\kappa_{\|} = 0$; and for $-s = s_1 = s_2$ it is the plane $k_{\|} = 0$ (so for $\kappa_{\|} = 0$). Double resonances are observed for cases $s_1 = s_2$. For interactions of type SSA, SAA and AAA the resonance conditions are the same and so are the results. Note that we already know for incompressible MHD (AAA) that interacting coefficients are non-zero only for $s_1 = -s_2$.

3.2.3 FFS/FFA interaction

Interaction between two fast and one slow magneto–acoustic waves (FFS) leads to (if the S–wave is supported on the wavevector $\mathbf{k}$)

$$h(\boldsymbol{\kappa};\mathbf{k}) = C_S\,(s_1\kappa + s_2|k-\kappa|) = 0\,. \tag{53}$$

When $s_1 = s_2$ the trivial solution is $\boldsymbol{\kappa} = 0$, otherwise the resonant manifolds is defined by $\kappa = |k-\kappa|$ which defines a (tilted) plane intersecting $\mathbf{k}$ at the middle of its length. If the S–wave is supported on the wavevector $\mathbf{k}-\boldsymbol{\kappa}$ the non–trivial solution for the resonant manifolds is $\kappa = k$ which defines a sphere of radius k. Finally if the S–wave is supported on the wavevector $\boldsymbol{\kappa}$ then the resonant manifolds is just $\boldsymbol{\kappa} = 0$. The resonances are generally simple except if vectors are collinear. Interaction between two fast magneto–acoustic waves and one Alfvén wave (FFA) leads to the same type of results.

3.2.4 FSS/FSA/FAA interaction

For interaction between one fast and two slow magneto–acoustic waves (FSS) the resonant manifolds is readily obtained: in the weakly compressible limit the wavevector which supports the F–wave has to be zero. Then the resonant manifolds is defined by $\boldsymbol{\kappa} = 0$, or $\boldsymbol{\kappa} = \mathbf{k}$ (if we do not consider the irrelevant case $\mathbf{k} = 0$). We only have simple resonances. Interaction of type FSA and FAA gives the same kind of results.

3.3 General discussion

The resonant manifold in weakly compressible MHD is richer than for the incompressible case. Fig. 1 shows the Fourier-space representation of the resonant manifolds. We see now how can the spectral densities redistribute along the mean magnetic field $\mathbf{B_0}$: first along rays, *i.e.* by a resonance between collinear wave vectors, secondly by a spheric resonance, and thirdly by tilted planes. The first mechanism of redistribution is similar to what we know for acoustic waves (Newell and Aucoin, 1971; L'vov et al, 1997) since in the weakly compressible limit F–waves tend to pure acoustic waves. However the two last mechanisms involving interaction between F-waves and S– or A–waves are completely new.

It is likely that these different mechanisms of redistribution do not have the same efficiency at large M_A. (Note that the fast speed of waves does not automatically mean that the nonlinear processes are fast!) To find the relative strength of the different three-wave processes one has to take into account the competition between the asymptotic decrease of interaction coefficients and the asymptotic amplification because some waves become less dispersive for large M_A (the frequency Jacobian becomes large in the collisional integral). The exact answer depends on the asymptotic behaviour in the weakly compressible limit of the collisional integral which is yet to be derived. At the level of this study we can only speculate about it and outline different physical possibilities. For example the most efficient mechanism could be a transfer along spheres. This mechanism does not degenerate Fourier space by foliating it into noninteracting manifolds and thus energy can be redistributed throughout $\mathbf{k}$–space. Consequently the study of other sub-redistributions which could appear later is unnecessary. However if the most efficient transfer mechanism degenerates the Fourier space, like for a transfer along planes or rays, then it is relevant to know the second most efficient mechanism of transfer because it will tell us how does the redistribution take place at different time scale. For instance we can imagine a transfer dominated by alfvénic turbulence, *i.e.* along planes perpendicular to $\mathbf{B_0}$. For this scenario we see that our knowledge on alfvénic turbulence (Galtier et al., 2000) could be useful to describe the dynamics at least for small times. On the other hand if the dominant transfer occurs like for pur acoustic waves, *i.e.* along rays, then the acoustic turbulence developed in Newell and Aucoin (1971) and in L'vov et al (1997) provides the theory for small times. We see that in any case the competition between the different mechanisms gives a richness that we do not have for afvénic turbulence.

An important other question is about the presence (or absence) of double resonances. According to the previous study relevant double resonances are observed for FFF, SSS, SSA and SAA interactions. The presence of double resonances can affect significantly the asymptotic behaviour of certain singular integrals of the form

$$\int_{-\infty}^{+\infty} g(\boldsymbol{\kappa}, \mathbf{k}) \Delta(h) \, d\boldsymbol{\kappa} \,, \tag{54}$$

with

$$\Delta(h) = \frac{e^{iht} - 1}{ih} \,, \tag{55}$$

whose evaluation is required in obtaining the closure equations (this point is studied in detail in Newell and Aucoin (1971)). Briefly the reason is that the major contribution to (54) is due to values $\boldsymbol{\kappa}$ near a zero of $h(\boldsymbol{\kappa};\mathbf{k})$. If double resonances are present, *i.e.* if in the Taylor expansion (40) the gradient is zero, then the behaviour of (54) is essentially governed by the next term in the Taylor expansion which depends on the dimension of the problem. But as it was said before, such a conclusion depends of course on the asymptotic behaviour of interaction coefficients and of the Jacobian in the collisional integral.

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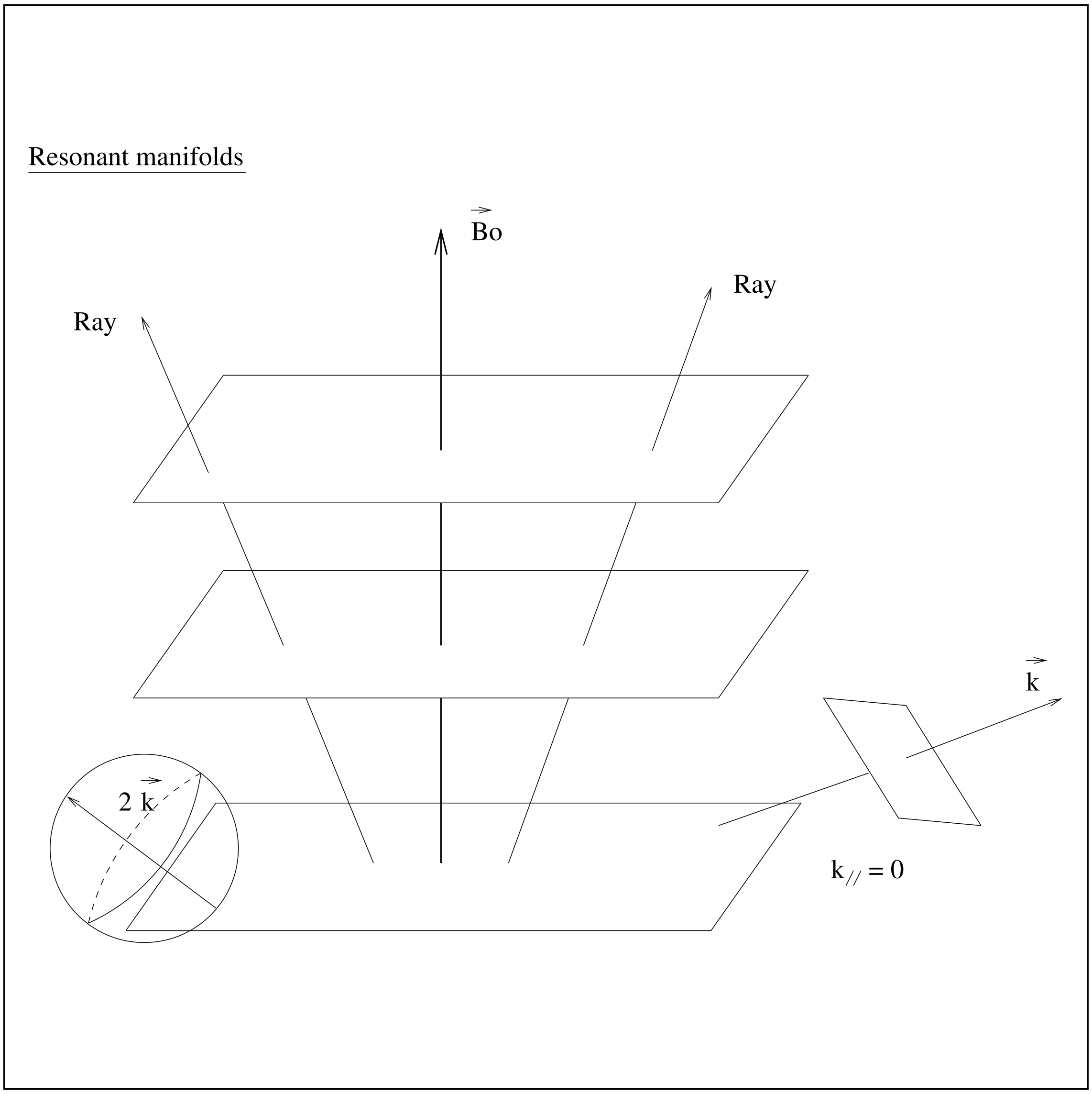
Figure Captions

Fig. 1. Fourier-space representation of the resonant manifolds in the weakly compressible limit. Three different types of spectral densities redistribution mechanisms are visible: along rays, spheres and planes.

Figures



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