

Interaction of Kelvin waves and nonlocality of energy transfer in superfluidsJason Laurie,¹ Victor S. L'vov,² Sergey Nazarenko,¹ and Oleksii Rudenko²¹*Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom*²*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*

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We argue that the physics of interacting Kelvin Waves (KWs) is highly nontrivial and cannot be understood on the basis of pure dimensional reasoning. A consistent theory of KW turbulence in superfluids should be based upon explicit knowledge of their interactions. To achieve this, we present a detailed calculation and comprehensive analysis of the interaction coefficients for KW turbulence, thereby, resolving previous mistakes stemming from unaccounted contributions. As a first application of this analysis, we derive a *local nonlinear* (partial differential) *equation*. This equation is much simpler for analysis and numerical simulations of KWs than the Biot-Savart equation, and in contrast to the completely integrable local induction approximation (in which the energy exchange between KWs is absent), describes the nonlinear dynamics of KWs. Second, we show that the previously suggested Kozik-Svistunov energy spectrum for KWs, which has often been used in the analysis of experimental and numerical data in superfluid turbulence, is irrelevant, because it is based upon an erroneous assumption of the locality of the energy transfer through scales. Moreover, we demonstrate the weak nonlocality of the inverse cascade spectrum with a constant particle-number flux and find resulting logarithmic corrections to this spectrum.

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I. PHYSICAL BACKGROUND, METHODOLOGY, AND OVERVIEW OF RESULTS**A. Kelvin waves (KWs) in superfluid turbulence**

The role of KWs in the dissipation of energy in zero-temperature quantum turbulence has long been discussed within the quantum-turbulence community. It is widely believed that KWs extend the transfer of a constant-energy flux from the fully three-dimensional Kolmogorov-type turbulence at large scales, through a crossover mechanism at scales comparable to the intervortex distance, to smaller scales via a local KW cascade on quantized vortices. Much theoretical work has been done recently, including the conjecture of a power-law scaling for the KW energy spectrum made by Kozik and Svistunov¹ in 2004 (hereafter the KS spectrum).

Nevertheless there remain important unanswered questions in quantum turbulence: (a) what are the relative roles of KWs and the other processes, e.g., vortex reconnections, in the transfer of energy to small scales? (b) What are the dominant physical mechanisms in the classical-quantum crossover range? Two alternative scenarios were put forward for this range: first, one relying on the idea that the polarization of vortex tangles suppress vortex reconnections, which lead to a bottleneck hump,^{2,3} and the second, implies that reconnections play an active role in removing the bottleneck.⁴ (c) If the KWs do play a key role at small scales, what kind of interaction processes are important for the transfer of energy toward smaller scales? Is it the resonant wave-wave interactions or a linear process of wave number evolution due to a large-scale curvature and/or slow time dependence of the underlying vortex line, or any other possibility? (d) If the KW energy transfer is dominated by the six-wave scattering, can one safely assume, as in Ref. 1 that this process is local, in the sense that the \mathbf{k} waves (with a given wave vector \mathbf{k}) are mainly affected by \mathbf{k}' waves (with a given wave vector \mathbf{k}' ,

where $k' = |\mathbf{k}'|$ is of the same order as $k = |\mathbf{k}|$ with contributions of \mathbf{k}' waves with $k' \ll k$ and $k' \gg k$ being vanishingly small?

In this paper we do not address all of these problems, particularly the ones about the role of reconnections and about the structure of the crossover range. We restrict our attention to the nonlinear interactions of weakly nonlinear KWs propagating on a single straight vortex line. This corresponds to the small-scale range of superfluid turbulence at near-zero temperature, where, because of the short wavelengths of KWs, one can ignore the influence of the neighboring vortex lines within the tangle, and assume that nonlinearity, being weak, is still strong enough for the nonlinear evolution to proceed faster than the large-scale (space and time) changes in the underlying vortex line.

Within this idealized setup, our immediate goal is to revise and advance the theory of weak-wave turbulence of KWs. In particular, we have clarified the structure of the nonlinear KW interactions, corrected the theory by including previously unaccounted leading-order contributions to the effective wave Hamiltonian, and explicitly calculated the interaction coefficients. Furthermore, we have used these results to: (a) first, derive a simple local nonlinear Eq. (8) for describing KW turbulence. (b) Second, to check locality of the KS spectrum assumed in the previous theory.

B. Hierarchy of the equations of motion for KWs

A commonly accepted model of superfluid turbulence comprises a randomly moving tangle of quantized vortex lines which can be characterized by the mean intervortex distance ℓ and the vortex core radius $a_0 \ll \ell$. There are two approaches in dealing with the vortex core. The first one is a “microscopic” model, in which the core is resolved: it is based on the Gross-Pitaevski equation

$$\frac{\partial \Psi}{\partial t} + \nabla^2 \Psi - \Psi |\Psi|^2 = 0, \quad (1)$$

where Ψ is the so-called condensate wave function.

This model was systematically derived for the Bose-Einstein condensate in supercold atoms and not for liquid Helium. Nevertheless, it is frequently used for describing superfluid flows in Helium because it contains several essential features of such superfluids, i.e., vortex quantization, acoustic waves (phonons) in the presence of a condensate, and the description of a gradual (nonsingular) vortex line reconnection.

However, the Gross-Pitaevski equation can be costly to study, and one often resorts to using the so-called Biot-Savart formulation of the Euler equations for ideal classical fluids, exploiting the fact that far away from the vortex cores the Gross-Pitaevski dynamics is isomorphic to the ideal classical flow via the Madelung transformation. In the Biot-Savart model, the vortices are postulated by a cutoff in the equations for the vortex line elements. Namely, the equations used are⁵⁻⁸

$$\dot{\mathbf{r}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3} \quad (2)$$

with a cutoff at the core radius a_0 , i.e., integration is over the range $|\mathbf{r} - \mathbf{s}| > a_0$. Here $\kappa \equiv 2\pi\hbar/m$ is the quantum of velocity circulation, m is the particle mass. In what follows, we will adopt the Biot-Savart equation (BSE) as a starting point for our derivation.

To consider the KW system, one has to start with an equilibrium state corresponding to an infinitely long straight vortex line and perturb it with small angle disturbances. This corresponds to a setup of weakly nonlinear KWs which are dispersive and that can be described by weak-wave turbulence theory. For this, one has to parametrize the transverse displacement vector (x, y) of the perturbed line by the distance z along the unperturbed line (the latter lies along the Cartesian z axis)

$$w(z, t) \equiv x(z, t) + iy(z, t). \quad (3)$$

In terms of w , the BSE can be written in a Hamiltonian form

$$i\kappa \frac{\partial w}{\partial t} = \frac{\delta H\{w, w^*\}}{\delta w^*}, \quad (4)$$

where $\delta/\delta w^*$ is the functional derivative, the asterisk stands for the complex conjugation and the Hamiltonian $H = H^{\text{BSE}}$ is defined as follows:⁹

$$H^{\text{BSE}} = \frac{\kappa}{4\pi} \int \int \frac{[1 + \text{Re}(w'^*(z_1)w'(z_2))] dz_1 dz_2}{\sqrt{(z_1 - z_2)^2 + |w(z_1) - w(z_2)|^2}}, \quad (5)$$

where we have used the notation $w'(z) = dw/dz$.

Then, one must expand in two small parameters: the perturbation inclination $w' \ll 1$ and $\Lambda_0^{-1} \ll 1$, where $\Lambda_0 = \ln(\ell/a_0)$. Such a simultaneous expansion is not easy. This is because the leading order in $1/\Lambda_0$ gives an integrable model called the local induction approximation (LIA) (Refs. 8 and 9) with a Hamiltonian

$$H^{\text{LIA}} = \frac{\kappa}{2\pi} \Lambda_0 \int dz \sqrt{1 + |w'(z)|^2}. \quad (6)$$

Because of the integrability, LIA conserves an infinite number of integrals of motion, and wave resonances are absent in all orders, which prevents energy exchange between KWs. Thus, LIA appears to be too simple, and to describe the KW energy transfer, one has to go to next order in $1/\Lambda_0$.

The second difficulty is that the lowest order process, the four-wave resonances, are absent for such one-dimensional systems with concaved dispersion relations. Thus, one must also go to the next order in small w' . The combination of these two facts makes finding the effective interaction Hamiltonian \mathcal{H}_{int} for KWs a hard task.

For numerical analysis purposes, the BSE model in Eq. (2) is also quite challenging because of the nonlocal (in the physical space) integral that has to be computed, especially when one has to resolve a wide range of turbulent scales and when the waves are weak so that the evolution times are long. Thus, there is a clear need for a simpler model for nonlinear KWs, which would be local in the physical space such as LIA (i.e., represent a nonlinear partial differential equation) but unlike LIA would be capable of describing the energy transfer over turbulent scales. Motivated by this need, an *ad hoc* model was introduced in Ref. 10 which has the simplest possible form with all the scaling properties and solutions of the original BSE model preserved. This model was called the truncated LIA (or TLIA); it has the following Hamiltonian:

$$H^{\text{TLIA}} = \frac{\kappa^2}{4\pi} \Lambda_0 \int \left[|w'|^2 - \frac{1}{4} |w'|^4 \right] dz. \quad (7)$$

The name TLIA arises from the fact that it can formally be obtained by expanding the LIA Hamiltonian in $|w'|^2$ and truncating at the fourth order. This truncation leads to the breaking of the LIA integrability while preserving all the important scalings.

The TLIA model turned out to be very efficient and useful for numerical simulations,¹⁰ even though it was suggested *ad hoc*, and motivated by the need for greater simplicity, rather than derived from first principles. In the present paper, we will obtain a very simple, local in z , nonlinear partial differential equation which is isomorphous to TLIA for weakly nonlinear KWs, and which is obtained rigorously from BSE by asymptotic expansions in $1/\Lambda_0$ and w' , and by a subsequent identification of the dominant type of interaction wave sextets.

C. New local-nonlinear equation for Kelvin waves

Our main goal in the paper is, being based on the BSE, is to systematically derive an effective motion equation for KWs, “as simple as possible but not more.” From one side, this new equation will include dynamical interaction (with energy exchange) of KWs (which are absent in LIA), and from the other, will contain the leading contribution obtained from local elements (in contrast to the TLIA model). More-

over, at the same time, being drastically simpler than the BSE itself. The resulting equation which we call the local nonlinear equation (LNE) reads

$$i\frac{\partial\tilde{w}}{\partial t} + \frac{\kappa}{4\pi}\frac{\partial}{\partial z}\left[\left(\Lambda - \frac{1}{4}\left|\frac{\partial\tilde{w}}{\partial z}\right|^4\right)\frac{\partial\tilde{w}}{\partial z}\right] = 0. \quad (8)$$

The variable $\tilde{w}(z,t)$ is related to $w(z,t)$ via a weakly nonlinear canonical transformation of type

$$\tilde{w}(z,t) = w(z,t) + O(w'^3). \quad (9)$$

The dimensionless parameter

$$\Lambda \equiv \Lambda_0 - \gamma - \frac{3}{2} \approx \Lambda_0 - 2, \quad (10)$$

where $\gamma=0.5772\dots$ is the Euler constant. The replacement $\Lambda_0 \rightarrow \Lambda$ is equivalent to a replacement of a_0 by an effective vortex core radius $a=a_0 \exp(\gamma+3/2) \approx 8a_0$, in the equation $\Lambda=\ln(\ell/a)$.

We entitled the result in Eq. (8) as the LNE to stress its three main features: *locality* (in physical space) of the KW interactions: according to Eq. (8), the evolution of the KW amplitude $\tilde{w}(z,t)$ depends only on the slope $\partial\tilde{w}(z,t)/\partial z$ of KWs and on its curvature [via $\partial^2\tilde{w}(z,t)/\partial z^2$] at the same point z . *Nonlinearity* plays a crucial role in the LNE, Eq. (8), as it is responsible for the energy transfer among the KW modes. Notice that LIA, being formally a nonlinear equation, allows (in the framework of the inverse scattering formulation) a linear formulation, in which the energy exchange between KWs is explicitly absent. The word *equation* (not an ‘‘approximation’’) stresses the fact that LNE, Eq. (8), is asymptotically exact in the triple limit

$$w' \ll 1, \quad \Lambda^{-1} \ll 1, \quad \text{and} \quad \lambda \ll \ell, \quad (11)$$

where λ is the characteristic wavelength of the the KWs.

It is instructive to represent LNE, Eq. (8), in a Hamiltonian form

$$i\kappa\frac{\partial\tilde{w}}{\partial t} = \frac{\delta}{\delta\tilde{w}^*} H^{\text{LNE}}\{\tilde{w}, \tilde{w}^*\}, \quad (12a)$$

where the LNE Hamiltonian is as follows:

$$H^{\text{LNE}} = \frac{\kappa^2}{4\pi} \int \left[\Lambda \left| \frac{\partial\tilde{w}}{\partial z} \right|^2 - \frac{1}{12} \left| \frac{\partial\tilde{w}}{\partial z} \right|^6 \right] dz, \quad (12b)$$

In the weakly nonlinear case, the LNE model is isomorphic to the TLIA model in Eq. (7), in which the variable w is related to \tilde{w} via the canonical transformation mentioned above and a proper rescaling. We repeat that the TLIA model was introduced in¹⁰ in an *ad hoc* way, requiring that it must have the same basic scaling properties as the original BSE, Eq. (2), but be simpler than the latter. In the present paper, the LNE, Eq. (8), is derived from the BSE *systematically* and, therefore, it is an asymptotically rigorous equation in the relevant KW turbulence limits. In spite of the remarkable simplicity of the LNE, Eq. (8), its asymptotical derivation is rather cumbersome, which possibly explains the fact that it has never been obtained before. To reach our goals we proceeded as follows.

D. Challenges of the KW turbulence: Nonlocality of the six-wave process

The first part of the paper is devoted to an explicit calculation of the effective six-wave interaction coefficient in the limit of small perturbation angles and small values of $1/\Lambda$ that culminate in the derivation of the LNE, Eq. (8). As we mentioned above, the respective expansion in two small parameters is not easy and rather cumbersome. To keep our presentation reasonably short and transparent, in Sec. II B we describe only the main steps of the calculation and move details to the Appendices. In this part, we fix a set of rather important technical errors made in Refs. 1 and 11, thereby preparing the mathematical problem for further analysis.

As an important application of the obtained result, we dedicate Sec. III to an analysis of the cascades of energy E and particle-number N caused by the $3 \leftrightarrow 3$ scattering of KWs in the context of the KS conjecture.¹ The KS theory predicts a power-law energy spectrum with constant E -flux, $E_{\text{KS}}(k) \propto k^{-7/5}$, and with constant N -flux, $E_{\text{N}}(k) \propto k^{-13}$. These spectra can only be valid if they are *local*. We show in Sec. III C that the KS spectrum is strongly nonlocal so that the KW dynamics are dominated by interactions of \mathbf{k}' waves with $k' \ll k$. Therefore, the KS spectrum is physically irrelevant, i.e., it cannot be realized in nature. In Sec. III C, we also demonstrate that the N -flux spectrum is weakly nonlocal in the sense that the dynamics of the \mathbf{k} waves are equally effected by all the \mathbf{k}' waves with $k' \leq k$. In Sec. III D, we ‘‘fix’’ locality of this spectrum by a logarithmic correction, $E_{\text{N}}(k) \propto k^{-3}[\ln(k\ell)]^{-1/5}$, Eq. (43).

Establishing nonlocality of the six-wave theory is a key step to the desired effective description. Indeed, it suggests that the dominant wave sextets must involve modes with widely separated wavelengths. We will show that in this case, the effective interaction coefficient looks remarkably simple: it is proportional to the product of the six wave numbers of the sextet modes, see Eq. (31). This immediately yields the LNE, Eq. (8). In conclusion, we discuss prospects of building a theory based upon the LNE, Eq. (8), which should lead to an alternative result to the invalid KS spectrum.

II. HAMILTONIAN DYNAMICS OF KWs

A. Introduction to the problem

To derive an effective KW Hamiltonian leading to the LNE, Eq. (8), we first briefly overview the Hamiltonian description of KWs initiated in Ref. 1 and further developed in Refs. 2 and 10. The main goal of Sec. II B is to start with the so-called ‘‘bare’’ Hamiltonian (5) for the Biot-Savart description of KWs, Eq. (2), and obtain expressions for the frequency Eqs. (21), four- and six-KWs interaction coefficients Eqs. (22) and (23) and their $1/\Lambda$ expansions, which will be used in further analysis. Equations (21)–(23) are starting points for further modification of the KW description, given in Sec. II C, in which we explore the consequences of the fact that *nontrivial* four-wave interactions of KWs are prohibited by the conservation laws of energy and momentum

$$\omega_1 + \omega_2 = \omega_3 + \omega_4, \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \quad (13)$$

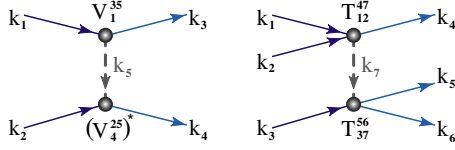


FIG. 1. (Color online) Examples of contribution of the triple vertices to the four-wave Hamiltonian when three-wave resonances are forbidden (left) and contributions of the quartet vertices to the six-wave Hamiltonian in resonances when four-wave resonances are forbidden (right). Intermediate virtual waves are shown by dash lines.

where $\omega_j \equiv \omega(k_j)$ is the frequency of the k_j wave. Only the *trivial* processes with $k_1 = k_3$, $k_2 = k_4$, or $k_1 = k_4$, $k_2 = k_3$ are allowed.

It is well known (see, e.g., Ref. 12) that in the case when nonlinear wave processes of the same kind (e.g., $1 \rightarrow 2$) are *forbidden* by conservation laws, the terms corresponding to this kind of processes can be eliminated from the interaction Hamiltonian by a weakly nonlinear canonical transformation. A famous example¹² of this procedure comes from a system of gravity waves on water surface in which three-wave resonances $\omega_1 = \omega_2 + \omega_3$ are forbidden. Then, by a canonical transformation to a new variable $b = a + O(a^2)$, the old Hamiltonian $H(a, a^*)$ is transformed to a new one, $\tilde{H}(b, b^*)$, where the three-wave (cubic) interaction coefficient $V_{1,2,3}^{2,3}$ is eliminated at the expense of appearance of an additional contribution to the next order term (i.e., four-wave interaction coefficient $T_{1,2}^{3,4}$) of the type

$$V_{1,2,3}^{3,5}(V_{4,5}^{2,5})^*[\omega_5 + \omega_3 - \omega_1]. \quad (14)$$

One can consider this contribution as a result of the second-order perturbation approach in the three-wave processes $k_1 \rightarrow k_3 + k_5$ and $k_5 + k_2 \rightarrow k_4$, see Fig. 1, left. The virtual wave k_5 oscillates with a forced frequency $\omega_1 - \omega_3$ which is *different* from its eigenfrequency ω_5 . The inequality $\omega(k_1) - \omega(k_3) \neq \omega(|k_1 - k_3|)$ is a consequence of the fact that the three-wave processes $\omega(k_1) - \omega(k_3) = \omega(|k_1 - k_3|)$ are forbidden. As the result the denominator in Eq. (14) is nonzero and the perturbation approach leading to Eq. (14) is applicable when the waves' amplitudes are small.

Strictly speaking our problem is different: as we mentioned above, not all $2 \leftrightarrow 2$ processes in Eq. (13) are forbidden but only the nontrivial ones that lead to energy exchange between KWs. Therefore, the use of a weakly nonlinear canonical transformation in Eq. (9) (as suggested in Ref. 1) should be done with extra caution. The transformation in Eq. (9) is supposed to eliminate the fourth order terms from the BSE-based interaction Hamiltonian by the price of appearance of extra contributions to the “full” six-wave interaction amplitude $\tilde{W}_{1,2,3}^{4,5,6}$, Eq. (25), of the following type (see Fig. 1, right):

$$\frac{T_{1,2,3,7}^{4,7} T_{3,7}^{5,6}}{\omega_7 + \omega_4 - \omega_1 - \omega_2}, \quad \omega_7 \equiv \omega(|k_1 + k_2 - k_4|). \quad (15)$$

Here all wave vectors are taken on the six-wave resonant manifold

$$\omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6,$$

$$k_1 + k_2 + k_3 = k_4 + k_5 + k_6. \quad (16)$$

The danger is seen from a particular example when $k_1 \rightarrow k_4$, $k_2 \rightarrow k_5$, and $k_3 \rightarrow k_6$, so $\omega_7 \rightarrow \omega_2$, and the denominator of Eq. (15) goes to zero, while the numerator remains finite. This means, that the perturbation contribution in Eq. (15) diverges and this approach becomes questionable.

However a detailed analysis of *all* contributions of the type in Eq. (15) performed *a posteriori* and presented in Sec. II C demonstrates cancelations of diverging terms with opposite signs such that the resulting full six-wave interaction coefficient remains finite and the perturbation approach in Eq. (15) appears to be eligible. The reason for this cancellation is hidden deep within the symmetry of the problem, which will not be discussed here.

Moreover, finding the full Hamiltonian is not enough for formulating the effective model. Such a model must include all contributions in the same $O(1/\Lambda)$, namely, the leading order allowing energy transfer in the k space. The full Hamiltonian still contains unexpanded in $(1/\Lambda)$ expressions for the KW frequencies. Leaving only the leading (LIA) contribution in the KW frequency ω , as it was done in,^{1,11} leads to a serious omission of an important leading order contribution. Indeed, the first subleading contribution to ω shifts the LIA resonant manifold, which upsets the integrability. As a result, the LIA part of $\tilde{W}_{1,2,3}^{4,5,6}$ yields a contribution to the effective model in the leading order. This (previously overlooked) contribution will be found and analyzed in Sec. II D.

B. “Bare” Hamiltonian dynamics of KWs

1. Canonical form of the bare KW Hamiltonian

Let us postulate that the motion of a tangle of quantized vortex lines can be described by the BSE model in Eq. (2) and assume that

$$\Lambda_0 \equiv \ln(\ell/a_0) \gg 1, \quad (17)$$

where a_0 is the vortex core radius. The BSE can be written in the Hamiltonian form (4) with Hamiltonian (5). Without the cutoff, the integral in H^{BSE} , Eq. (5), would be logarithmically divergent with the dominant contribution given by the leading order expansion of the integrand in small $z_1 - z_2$ that corresponds to H^{LIA} , Eq. (6).

As we have already mentioned, LIA represents a completely integrable system and it can be reduced to the one-dimensional nonlinear Schrödinger equation by the Hasimoto transformation.¹³ However, it is the complete integrability of LIA that makes it insufficient for describing the energy cascade and which makes it necessary to consider the next-order corrections within the BSE model.

For small-amplitude KWs when $w'(z) \ll 1$, we can expand the Hamiltonian (5) in powers of w'^2 , see Appendix A 1

$$H = H_2 + H_4 + H_6 + \dots \quad (18)$$

Here we omitted the constant term H_0 which does not contribute to the equation of motion Eq. (4). Assuming that the

boundary conditions are periodical on the length \mathcal{L} (the limit $k\mathcal{L} \gg 1$ to be taken later) we can use the Fourier representation

$$w(z,t) = \kappa^{-1/2} \sum_k a(\mathbf{k},t) \exp(i\mathbf{k}z). \quad (19)$$

The bold face notation of the one-dimensional wave vector is used for convenience only. Indeed, such a vector is just a real number, $\mathbf{k} \in \mathbb{R}$, in our case for KWs. For further convenience, we reserve the normal face notation for the length of the one-dimensional wave vector, i.e., $k = |\mathbf{k}| \in \mathbb{R}_+$. In Fourier space, the Hamiltonian equation also takes a canonical form

$$i \frac{\partial a(\mathbf{k},t)}{\partial t} = \frac{\delta \mathcal{H}\{a, a^*\}}{\delta a^*(\mathbf{k},t)}. \quad (20a)$$

The new Hamiltonian \mathcal{H} is the density of the old one

$$\mathcal{H}\{a, a^*\} = H\{w, w^*\} / \mathcal{L} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \dots \quad (20b)$$

The Hamiltonian

$$\mathcal{H}_2 = \sum_k \omega_k a_k a_k^*, \quad (20c)$$

describes the free propagation of linear KWs with the dispersion law $\omega_k \equiv \omega(k)$, given by Eq. (21), and the canonical amplitude $a_k \equiv a(\mathbf{k},t)$. The interaction Hamiltonians \mathcal{H}_4 and \mathcal{H}_6 describe the four-wave processes of $2 \leftrightarrow 2$ scattering and the six-wave processes of $3 \leftrightarrow 3$ scattering, respectively. Using a shorthand notation $a_j \equiv a(\mathbf{k}_j, t)$, they can be written as follows:

$$\mathcal{H}_4 = \frac{1}{4} \sum_{1+2=3+4} T_{1,2}^{3,4} a_1 a_2 a_3^* a_4^*, \quad (20d)$$

$$\mathcal{H}_6 = \frac{1}{36} \sum_{1+2+3=4+5+6} W_{1,2,3}^{4,5,6} a_1 a_2 a_3 a_4^* a_5^* a_6^*. \quad (20e)$$

Here $T_{1,2}^{3,4} \equiv T(\mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_3, \mathbf{k}_4)$ and $W_{1,2,3}^{4,5,6} \equiv W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 | \mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6)$ are bare four- and six-wave interaction coefficients, respectively. Summations over $\mathbf{k}_1, \dots, \mathbf{k}_4$ in \mathcal{H}_4 and over $\mathbf{k}_1, \dots, \mathbf{k}_6$ in \mathcal{H}_6 are constrained by $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$ and by $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6$, respectively.

2. Λ expansion of the bare Hamiltonian function

As will be seen below, the leading terms in the Hamiltonian functions ω_k , $T_{1,2}^{3,4}$, and $W_{1,2,3}^{4,5,6}$ are proportional to Λ , which correspond to the LIA, Eq. (6). They will be denoted further by a front superscript “ Λ ,” e.g. ${}^\Lambda \omega_k$, etc. Because of the complete integrability, there are no dynamics in the LIA. Therefore, the most important terms for us will be the ones of zeroth order in Λ , i.e. the ones proportional to $\Lambda^0 = \mathcal{O}(1)$. These will be denoted by a front superscript “ 1 ,” e.g., ${}^1 \omega_k$, etc.

Explicit calculations of the Hamiltonian coefficients must be done very carefully because even a minor mistake in the numerical prefactor can destroy various cancelations of large terms in the Hamiltonian coefficients. This could change the order of magnitude of the answers and the character of their

dependence on the wave vectors in the asymptotical regimes. Details of these calculations are presented in Appendix A 1 whereas the results are given below.

Together with Eq. (10), the Kelvin wave frequency is

$$\omega_k = {}^\Lambda \omega_k + {}^1 \omega_k + \mathcal{O}(\Lambda^{-1}), \quad (21a)$$

where

$${}^\Lambda \omega_k = \kappa \Lambda k^2 / 4\pi, \quad (21b)$$

$${}^1 \omega_k = -\kappa k^2 \ln(k\ell) / 4\pi. \quad (21c)$$

The bare four-wave interaction coefficient is

$$T_{1,2}^{3,4} = {}^\Lambda T_{1,2}^{3,4} + {}^1 T_{1,2}^{3,4} + \mathcal{O}(\Lambda^{-1}), \quad (22a)$$

$${}^\Lambda T_{1,2}^{3,4} = -\Lambda \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 / 4\pi, \quad (22b)$$

$${}^1 T_{1,2}^{3,4} = -(5\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 + \mathcal{F}_{1,2}^{3,4}) / 16\pi. \quad (22c)$$

The function $\mathcal{F}_{1,2}^{3,4}$ is symmetric with respect to $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$, $\mathbf{k}_3 \leftrightarrow \mathbf{k}_4$, and $\{\mathbf{k}_1, \mathbf{k}_2\} \leftrightarrow \{\mathbf{k}_3, \mathbf{k}_4\}$; its definition is given in Appendix A 2.

The bare six-wave interaction coefficient is

$$W_{1,2,3}^{4,5,6} = {}^\Lambda W_{1,2,3}^{4,5,6} + {}^1 W_{1,2,3}^{4,5,6} + \mathcal{O}(\Lambda^{-1}), \quad (23a)$$

$${}^\Lambda W_{1,2,3}^{4,5,6} = \frac{9\Lambda}{8\pi\kappa} \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 \mathbf{k}_5 \mathbf{k}_6, \quad (23b)$$

$${}^1 W_{1,2,3}^{4,5,6} = \frac{9}{32\pi\kappa} (7\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 \mathbf{k}_5 \mathbf{k}_6 - \mathcal{G}_{1,2,3}^{4,5,6}). \quad (23c)$$

The function $\mathcal{G}_{1,2,3}^{4,5,6}$ is symmetric with respect to $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2 \leftrightarrow \mathbf{k}_3$, $\mathbf{k}_4 \leftrightarrow \mathbf{k}_5 \leftrightarrow \mathbf{k}_6$, and $\{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\} \leftrightarrow \{\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6\}$; its definition is given in Appendix A 3.

Note that the full expressions for ω_k , $T_{1,2}^{3,4}$, and $W_{1,2,3}^{4,5,6}$ do not contain ℓ but rather $\ln(1/a_0)$. This is natural because in the respective expansions ℓ was introduced as an auxiliary parameter facilitating the calculations and it does not necessarily have to coincide with the intervortex distance. More precisely, we should have used some effective intermediate length scale, ℓ_{eff} , such that $\ell \ll \ell_{\text{eff}} \ll 2\pi/k$. However, since ℓ_{eff} is artificial and would have to drop from the full expressions anyway, we chose to simply write ℓ omitting subscript “eff.” Cancellation of ℓ is a useful check for verifying the derivations.

C. Full “six-KW” Hamiltonian dynamics

1. Full six-wave interaction Hamiltonian $\tilde{\mathcal{H}}_6$

Importantly, the four-wave dynamics in one-dimensional media with concaved dispersion laws $\omega(k)$ are absent because the conservation laws of energy and momentum allow only trivial processes with $\mathbf{k}_1 = \mathbf{k}_3$, $\mathbf{k}_2 = \mathbf{k}_4$, or $\mathbf{k}_1 = \mathbf{k}_4$, $\mathbf{k}_2 = \mathbf{k}_3$. Sec. II A explains that by a proper nonlinear canonical transformation $\{a, a^*\} \Rightarrow \{b, b^*\}$, \mathcal{H}_4 can be eliminated from the Hamiltonian description. This comes at a price of appearance of additional terms in the full interaction Hamiltonian $\tilde{\mathcal{H}}_6$

$$\mathcal{H}\{a, a^*\} \Rightarrow \tilde{\mathcal{H}}\{b, b^*\} = \tilde{\mathcal{H}}_2 + \tilde{\mathcal{H}}_4 + \tilde{\mathcal{H}}_6 + \dots, \quad (24a)$$

$$\tilde{\mathcal{H}}_2 = \sum_k \omega_k b_k b_k^*, \quad \tilde{\mathcal{H}}_4 \equiv 0, \quad (24b)$$

$$\tilde{\mathcal{H}}_6 = \frac{1}{36} \sum_{1+2+3=4+5+6} \tilde{W}_{1,2,3}^{4,5,6} b_1 b_2 b_3 b_4^* b_5^* b_6^*, \quad (24c)$$

$$\tilde{W}_{1,2,3}^{4,5,6} = W_{1,2,3}^{4,5,6} + Q_{1,2,3}^{4,5,6}, \quad (24d)$$

$$Q_{1,2,3}^{4,5,6} = \frac{1}{8} \sum_{i,j,m=1}^3 \sum_{p,q,r=4}^6 q_{i,j,m}^{p,q,r}, \quad (24e)$$

$$i \neq j \neq m \quad p \neq q \neq r$$

$$q_{i,j,m}^{p,q,r} \equiv \frac{T_{r,j+m-r}^{j,m} T_{i,p+q-i}^{q,p}}{\Omega_{j,m}^{r,j+m-r}} + \frac{T_{m,q+r-m}^{q,r} T_{p,i+j-p}^{i,j}}{\Omega_{m,q+r-m}^{m,q+r-m}},$$

$$\Omega_{1,2}^{3,4} \equiv \omega_1 + \omega_2 - \omega_3 - \omega_4 = \Lambda \Omega_{1,2}^{3,4} + {}^1\Omega_{1,2}^{3,4} + \mathcal{O}(\Lambda^{-1}). \quad (24f)$$

The Q terms in the *full six-wave interaction coefficient* $\tilde{W}_{1,2,3}^{4,5,6}$ can be understood as contributions of two four-wave scatterings into resulting six-wave process via a virtual KW with $\mathbf{k} = \mathbf{k}_j + \mathbf{k}_m - \mathbf{k}_r$ in the first term in Q and via a KW with $\mathbf{k} = \mathbf{k}_q + \mathbf{k}_r - \mathbf{k}_m$ in the second term; see Fig. 1, right.

2. $1/\Lambda$ expansion of the full interaction coefficient $\tilde{W}_{1,2,3}^{4,5,6}$

Similarly to Eq. (23a), we can present $\tilde{W}_{1,2,3}^{4,5,6}$ in the $1/\Lambda$ expanded form

$$\tilde{W}_{1,2,3}^{4,5,6} = \Lambda \tilde{W}_{1,2,3}^{4,5,6} + {}^1\tilde{W}_{1,2,3}^{4,5,6} + \mathcal{O}(\Lambda^{-1}). \quad (25)$$

Due to the complete integrability of the KW system in the LIA, even the six-wave dynamics must be absent in the interaction coefficient $\Lambda \tilde{W}_{1,2,3}^{4,5,6}$. This is true if function $\Lambda \tilde{W}_{1,2,3}^{4,5,6}$ vanishes on the LIA resonant manifold

$$\Lambda \tilde{W}_{1,2,3}^{4,5,6} \delta_{1,2,3}^{4,5,6} \delta(\Lambda \tilde{\Omega}_{1,2,3}^{4,5,6}) \equiv 0, \quad (26a)$$

where

$$\delta_{1,2,3}^{4,5,6} = \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6), \quad (26b)$$

$$\Lambda \tilde{\Omega}_{1,2,3}^{4,5,6} = \frac{\kappa \Lambda}{4\pi} [k_1^2 + k_2^2 + k_3^2 - k_4^2 - k_5^2 - k_6^2]. \quad (26c)$$

Explicit calculation of $\Lambda \tilde{W}_{1,2,3}^{4,5,6}$ in Appendix B 1 shows that this is indeed the case: the contributions $\Lambda W_{1,2,3}^{4,5,6}$ and $\Lambda Q_{1,2,3}^{4,5,6}$ in Eq. (24d) cancel each other (see Appendix B 1). (Such cancelation of complicated expressions was one of the tests of consistency and correctness of our calculations and our MATHEMATICA code). The same is true for all the higher interaction coefficients: they must be zero within LIA.

Thus we need to study the first-order correction to the LIA, which for the interaction coefficient can be schematically represented as follows:

$${}^1\tilde{W}_{1,2,3}^{4,5,6} = {}^1W_{1,2,3}^{4,5,6} + {}^1Q_{1,2,3}^{4,5,6} + {}^2Q_{1,2,3}^{4,5,6} + {}^3Q_{1,2,3}^{4,5,6}, \quad (27a)$$

$${}^1Q \sim \frac{\Lambda T \otimes {}^1T}{\Lambda \Omega}, \quad {}^2Q \sim \frac{{}^1T \otimes \Lambda T}{\Lambda \Omega}, \quad (27b)$$

$${}^3Q \sim - {}^1\Omega \frac{\Lambda T \otimes \Lambda T}{[\Lambda \Omega]^2}. \quad (27c)$$

Here 1W is the Λ^0 -order contribution in the bare vertex W , given by Eq. (23c), 1Q is the Λ^0 -order contribution in Q which consists of 1Q and 2Q originating from the part 1T in the four-wave interaction coefficient T , and 3Q originating from the ${}^1\Omega$ corrections to the frequencies Ω in Eqs. (24e) and (24f). Explicit Eqs. (B3) for ${}^1Q_{1,2,3}^{4,5,6}$, ${}^2Q_{1,2,3}^{4,5,6}$, and ${}^3Q_{1,2,3}^{4,5,6}$ are presented in Appendix B 2. They are very lengthy and were analyzed using MATHEMATICA, see Sec. II D 2.

D. Effective six-KW dynamics

1. Effective equation of motion

The LIA cancelation in Eq. (26a) on the full manifold in Eq. (16) is not exact

$$\Lambda \tilde{W}_{k,1,2}^{3,4,5} \delta_{k,1,2}^{3,4,5} \delta(\tilde{\Omega}_{k,1,2}^{3,4,5}) \neq 0,$$

$$\tilde{\Omega}_{k,1,2}^{3,4,5} \equiv \omega_k + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5 = \Lambda \tilde{\Omega}_{k,1,2}^{3,4,5} + {}^1\tilde{\Omega}_{k,1,2}^{3,4,5} + \mathcal{O}(\Lambda^{-1}).$$

The residual contribution due to ${}^1\tilde{\Omega}_{k,1,2}^{3,4,5}$ has to be accounted for—an important fact overlooked in the previous KW literature, including the formulation of the effective KW dynamics recently presented by KS in Ref. 11. Now we are prepared to take another crucial step on the way to the effective KW model by replacing the frequency ω_k by its leading order (LIA) part, Eq. (21b), and simultaneously compensating for the respective shift in the resonant manifold by correcting the effective vertex $\tilde{\mathcal{H}}$. This corresponds to the following Hamiltonian equation:

$$i \frac{\partial b_k}{\partial t} = \Lambda \omega_k b_k + \frac{1}{12} \sum_{k+1+2=3+4+5} \mathcal{W}_{k,1,2}^{3,4,5} b_1 b_2 b_3^* b_4^* b_5^*, \quad (28)$$

where the constraint $\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5$ holds. Here $\mathcal{W}_{k,1,2}^{3,4,5}$ is a corrected interaction coefficient, which is calculated in Appendix B 3

$$\mathcal{W}_{k,1,2}^{3,4,5} = {}^1\tilde{W}_{k,1,2}^{3,4,5} + {}^1\tilde{S}_{k,1,2}^{3,4,5}, \quad (29a)$$

$${}^1\tilde{S}_{1,2,3}^{4,5,6} = \frac{2\pi}{9\kappa} {}^1\tilde{\Omega}_{1,2,3}^{4,5,6} \sum_{\substack{i=\{1,2,3\} \\ j=\{4,5,6\}}} \frac{(\partial_j + \partial_i) \Lambda \tilde{W}_{1,2,3}^{4,5,6}}{(k_j - k_i) \Lambda}, \quad (29b)$$

where $\partial_j(\cdot) \equiv \partial(\cdot) / \partial k_j$.

Equation (28) represents a correct effective model and will serve as a basis for our future analysis of KW dynamics and kinetics. However, to make this equation useful we need to complete the calculation of the effective interaction coef-

efficient $\mathcal{W}_{k,1,2}^{3,4,5}$ and simplify it to a reasonably tractable form. The key for achieving this is in a remarkably simple asymptotical behavior of $\mathcal{W}_{k,1,2}^{3,4,5}$, which will be demonstrated in the next section. Such asymptotical expressions for $\mathcal{W}_{k,1,2}^{3,4,5}$ will allow us to establish nonlocality of the KS theory and thereby establish precisely that these asymptotical ranges with widely separated scales are the most dynamically active, which would lead us to the remarkably simple effective model expressed by the LNE, Eq. (8).

2. Analysis of the effective interaction coefficient $\mathcal{W}_{k,1,2}^{3,4,5}$

Now we will examine the asymptotical properties of the interaction coefficient which will be important for our study of locality of the KW spectra and formulation of the LNE, Eq. (8). The effective six-KW interaction coefficient $\mathcal{W}_{k,1,2}^{3,4,5}$ consists of five contributions given by Eqs. (27) and (29). The explicit form of $\mathcal{W}_{k,1,2}^{3,4,5}$ involves about 2×10^4 terms. However its asymptotic expansion in various regimes, analyzed by MATHEMATICA demonstrates very clear and physical transparent behavior, which we will study upon the LIA resonance manifold

$$\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5, \quad (30a)$$

$$k^2 + k_1^2 + k_2^2 = k_3^2 + k_4^2 + k_5^2. \quad (30b)$$

If the smallest wavevector (say \mathbf{k}_5) is much smaller than the largest wave vector (say \mathbf{k}) we have a remarkably simple expression

$$\mathcal{W}_{k,1,2}^{3,4,5} \rightarrow -\frac{3}{4\pi\kappa} \mathbf{k} \mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4 \mathbf{k}_5, \quad (31)$$

as

$$\frac{\min\{k, k_1, k_2, k_3, k_4, k_5\}}{\max\{k, k_1, k_2, k_3, k_4, k_5\}} \rightarrow 0.$$

We emphasize that in the expression (31), it is enough for the minimal wave vector to be much less than the maximum wave number and not all of the remaining five wave numbers in the sextet. This was established using MATHEMATICA and TAYLOR expanding $\mathcal{W}_{k,1,2}^{3,4,5}$ with respect to one, two, and four wave numbers.¹⁴ All of these expansions give the same leading term as in Eq. (31), see Appendices B 4 and B 5.

The form of expression (31) demonstrates a very simple physical fact: long KWs (with small k vectors) can contribute to the energy of a vortex line only when they produce curvature. The curvature, in turn, is proportional to wave amplitude b_k and, at a fixed amplitude, is inversely proportional to their wavelength, i.e., $\propto k$. Therefore, in the effective-motion equation each b_j has to be accompanied by k_j , if $k_j \ll k$. This statement is exactly reflected by formula (31).

Furthermore, a numerical evaluation of $\mathcal{W}_{k,1,2}^{3,4,5}$ on a set of 2^{10} randomly chosen wave numbers, different from each other at most by a factor of 2, indicate that in the majority of cases its values are close to the asymptotical expression (31) (within 40%). Therefore, for most purposes we can approximate the effective six-KW interaction coefficient by the simple expression (31). Finally, our analysis of locality seen

later in this paper, indicates that the most important wave sextets are those which include modes with widely separating wavelengths, i.e., precisely those described by the asymptotic formula (31).

This leads us to the conclusion that the effective model for KW turbulence should use the interaction coefficient in Eq. (31). Returning back to the physical space, we thereby obtain the desired LNE for KWs given by Eq. (8).

As we mentioned in the beginning of the present paper, LNA is very close (isomorphous for small amplitudes) to the TLIA model, Eq. (7), introduced and simulated in Ref. 10. It was argued in Ref. 10 that the TLIA model is a good alternative to the original Biot-Savart formulation due to it dramatically greater simplicity. In the present paper we have found a further support for this model, which is strengthened by the fact that now it follows from a detailed asymptotical analysis, rather than being introduced *ad hoc*.

3. Partial contributions to the six-wave effective interaction coefficient

It would be instructive to demonstrate the relative importance of different partial contributions, 1W , 1Q , 2Q , 3Q , and ${}^1\tilde{S}$ [see Eqs. (27) and (29)] to the full effective six-wave interaction coefficient. For this, we consider the simplest case, when four wave vectors are small, say $k_1, k_2, k_3, k_5 \rightarrow 0$. We have (see Appendix B 5)

$$\frac{{}^1W}{\mathcal{W}} \rightarrow -1 + \frac{3}{2} \ln(k\ell), \quad (32a)$$

$$\frac{{}^1Q}{\mathcal{W}} \rightarrow +\frac{1}{2} - \frac{3}{2} \ln(k\ell) - \frac{1}{6} \ln \frac{k_3}{k}, \quad (32b)$$

$$\frac{{}^2Q}{\mathcal{W}} \rightarrow +\frac{1}{2} - \frac{3}{2} \ln(k\ell) - \frac{1}{6} \ln \frac{k_3}{k}, \quad (32c)$$

$$\frac{{}^3Q}{\mathcal{W}} \rightarrow +1 + \frac{3}{2} \ln(k\ell) + \frac{1}{6} \ln \frac{k_3}{k}, \quad (32d)$$

$$\frac{{}^1\tilde{S}}{\mathcal{W}} \rightarrow \frac{1}{6} \ln \frac{k_3}{k}. \quad (32e)$$

One sees that Eqs. (32) for the partial contributions involve the artificial separation scale ℓ , which cancels out from ${}^1\tilde{W} = {}^1W + {}^1Q + {}^2Q + {}^3Q$. This is not surprising because the initial expressions Eqs. (23) do not contain ℓ but rather $\ln(1/a_0)$. This cancelation serves as one more independent check of consistency of the entire procedure.

Notice that in the KS paper,¹ contributions in Eqs. (32d) and (32e) were mistakenly not accounted for. Therefore the resulting KS expression for the six-wave effective-interaction coefficient depends on the artificial separation scale ℓ . This fact was missed in their numerical simulations.¹ In their recent paper,¹¹ the lack of contribution in Eq. (32d) in the previous work was acknowledged (also in Ref. 10) but the contribution in Eq. (32e) was still missing.

III. KINETIC DESCRIPTION OF KW TURBULENCE

A. Effective kinetic equation for KWs

The statistical description of weakly interacting waves can be reached¹² in terms of the kinetic equation (KE) shown below for the continuous limit $k\mathcal{L} \gg 1$

$$\partial n(\mathbf{k}, t) / \partial t = \text{St}(\mathbf{k}, t) \quad (33a)$$

for the spectra $n(\mathbf{k}, t)$ which are the simultaneous pair correlation functions, defined by

$$\langle b(\mathbf{k}, t) b^*(\mathbf{k}', t) \rangle = \left(\frac{2\pi}{\mathcal{L}} \right)^2 \delta(\mathbf{k} - \mathbf{k}') n(\mathbf{k}, t), \quad (33b)$$

where $\langle \dots \rangle$ stands for proper (ensemble, etc.) averaging. In the classical limit,¹⁵ when the occupation numbers of Bose particles $N(\mathbf{k}, t) \gg 1$ and $n(\mathbf{k}, t) = \hbar N(\mathbf{k}, t)$, the collision integral $\text{St}(\mathbf{k}, t)$ can be found in various ways,^{1,2,12} including the Golden Rule of quantum mechanics. For the $3 \leftrightarrow 3$ process of KW scattering, described by the motion Eq. (28)

$$\begin{aligned} \text{St}_{3 \leftrightarrow 3}(\mathbf{k}) &= \frac{\pi}{12} \int \int \int \int \int |\mathcal{W}_{k,1,2}^{3,4,5}|^2 \delta_{k,1,2}^{3,4,5} \delta(\Lambda \Omega_{k,1,2}^{3,4,5}) \times (n_k^{-1} \\ &+ n_1^{-1} + n_2^{-1} - n_3^{-1} - n_4^{-1} - n_5^{-1}) \\ &\times n_k n_1 n_2 n_3 n_4 n_5 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 d\mathbf{k}_5. \end{aligned} \quad (33c)$$

The KE, Eq. (33), conserves the total number of (quasi-) particles N and the total (bare) energy of the system ${}^\Lambda E$, defined respectively as follows:

$$N \equiv \int n_k d\mathbf{k}, \quad {}^\Lambda E \equiv \int {}^\Lambda \omega_k n_k d\mathbf{k}. \quad (34)$$

KE, Eq. (33), has a Rayleigh-Jeans solution

$$n_{\text{T}}(k) = \frac{T}{\hbar {}^\Lambda \omega_k + \mu}, \quad (35)$$

which corresponds to thermodynamic equilibrium of KWs with temperature T and chemical potential μ .

In various wave systems, including the KWs described by KE, Eq. (33a), there also exist flux-equilibrium solutions, $n_E(k)$ and $n_N(k)$, with constant k -space fluxes of energy and particles, respectively. The corresponding solution for $n_E(k)$ was suggested in the KS paper¹ under an (unverified) assumption of locality of the E flux. In Sec. III C, we will analyze this assumption in the framework of the derived KE, Eq. (33), and will prove that it is wrong. The N -flux solution $n_N(k)$ was discussed in Ref. 16. In Sec. III C, we will show that this spectrum is marginally nonlocal, which means that it can be “fixed” by a logarithmic correction.

B. Phenomenology of the E - and N -flux equilibrium solutions for KW turbulence

Conservation laws in Eq. (34) for E and N allow one to introduce the continuity equations for n_k and ${}^\Lambda E_k \equiv {}^\Lambda \omega_k n_k$ and their corresponding fluxes in the k space, μ_k and ε_k

$$\frac{\partial n_k}{\partial t} + \frac{\partial \mu_k}{\partial k} = 0, \quad \mu_k \equiv - \int_0^k \text{St}_{3 \leftrightarrow 3}(\mathbf{k}) d\mathbf{k}, \quad (36a)$$

$$\frac{\partial {}^\Lambda E_k}{\partial t} + \frac{\partial \varepsilon_k}{\partial k} = 0, \quad \varepsilon_k \equiv - \int_0^k {}^\Lambda \omega_k \text{St}_{3 \leftrightarrow 3}(\mathbf{k}) d\mathbf{k}. \quad (36b)$$

In scale-invariant systems, when the frequency and interaction coefficients are homogeneous functions of wave vectors, Eqs. (36) allow one to guess the scale-invariant flux equilibrium solutions of KE, Eq. (33) (Ref. 12)

$$n_E(k) = A_E k^{-x_E}, \quad n_N(k) = A_N k^{-x_N}. \quad (37)$$

Here A_E and A_N are some dimensional constants. Scaling exponents x_N and x_E can be found in the case of *locality* of the N and E fluxes, i.e., when the integrals over k_1, \dots, k_5 in Eqs. (33c) and (36) converge. In this case, the leading contribution to these integrals originate from regions where $k_1 \sim k_2 \sim k_3 \sim k_4 \sim k_5 \sim k$ and thus, the fluxes in Eq. (36) can be estimated as follows:

$$\mu_k \approx k^5 [\mathcal{W}(k, k, k | k, k, k)]^2 n_N^5(k) / \omega_k, \quad (38a)$$

$$\varepsilon_k \approx k^5 [\mathcal{W}(k, k, k | k, k, k)]^2 n_N^5(k). \quad (38b)$$

Stationarity of solutions of Eqs. (36) require constancy of their respective fluxes: i.e., μ_k and ε_k should be k independent. Together with Eqs. (38) this allows one to find the scaling exponents in Eq. (37).

Our formulation in Eq. (33) of KW kinetics belongs to the scale-invariant class¹⁷

$${}^\Lambda \omega_k \propto k^2, \quad \text{and for } \forall \eta,$$

$$\mathcal{W}(\eta \mathbf{k}, \eta \mathbf{k}_1, \eta \mathbf{k}_2 | \eta \mathbf{k}_3, \eta \mathbf{k}_4, \eta \mathbf{k}_5) = \eta^6 \mathcal{W}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2 | \mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_5).$$

Estimating $\mathcal{W}(k, k, k | k, k, k) \approx k^6 / \kappa$ and ${}^\Lambda \omega_k \approx \kappa \Lambda k^2$ in Eqs. (38), one gets for N -flux spectrum¹⁶

$$n_N(k) \approx (\mu \kappa / \Lambda)^{1/5} k^{-3}, \quad x_N = 3 \quad (39a)$$

and for E -flux KS spectrum¹

$$n_E(k) \approx (\varepsilon \kappa^2)^{1/5} k^{-17/5}, \quad x_E = 17/5. \quad (39b)$$

C. Nonlocality of the N and E fluxes by $3 \leftrightarrow 3$ scattering

Consider the $3 \leftrightarrow 3$ collision term in Eq. (33c) for KWs with the interaction amplitude $\mathcal{W}_{1,2,3}^{4,5,6}$. Note that in Eq. (33c) $\int d\mathbf{k}_j$ are one-dimensional integrals $\int_{-\infty}^{\infty} d\mathbf{k}_j$. Let us examine the “infrared” (IR) region ($k_5 \ll k, k_1, k_2, k_3, k_4$) in the integral in Eq. (33c), taking into account the asymptotics in Eq. (31), and observing that the expression

$$\begin{aligned} &\delta_{k,1,2}^{3,4,5} \delta(\Lambda \tilde{\Omega}_{k,1,2}^{3,4,5}) (n_k^{-1} + n_1^{-1} + n_2^{-1} - n_3^{-1} - n_4^{-1} - n_5^{-1}) \\ &\times n_k n_1 n_2 n_3 n_4 n_5 \rightarrow \delta_{k,1,2}^{3,4} \delta(\Lambda \Omega_{k,1,2}^{3,4}) (n_k^{-1} + n_1^{-1} + n_2^{-1} \\ &- n_3^{-1} - n_4^{-1}) \times n_k n_1 n_2 n_3 n_4 n_5 \sim n_5 \sim k_5^{-x}. \end{aligned}$$

Thus the integral over k_5 in the IR region can be factorized and written as follows:

$$2 \int_0^k k_5^2 n(k_5) d\mathbf{k}_5 \propto 2 \int_{1/\ell}^k k_5^{2-x} d\mathbf{k}_5. \quad (40)$$

The factor 2 here originates from the symmetry of the integration area and evenness of the integrand: $\int_{-\infty}^{\infty} = 2 \int_0^{\infty}$. The

lower limit 0 in this expression should be replaced by the smallest wave number where the assumed scaling behavior in Eq. (37) holds, and moreover, it depends on the particular way the wave system is forced. For example, this cut-off wave number could be $1/\ell$, where ℓ is the mean intervortex separation ℓ , at which one expects a cutoff of the wave spectrum. The crucial assumption of locality, under which both the E flux (KS) and the N -flux spectra were obtained, implies that the integral in Eq. (40) is independent of this cutoff in the limit $\ell \rightarrow 0$. Clearly, integral in Eq. (40) depends on the IR cutoff if $x \geq 3$, which is the case for both the E -flux (KS) and the N -flux spectra in Eq. (39). Note that all other integrals over $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and \mathbf{k}_4 in Eq. (33c) diverge exactly in the same manner as the integral over \mathbf{k}_5 , i.e., each of them leads to expression (40).

Even stronger IR divergence occurs when two wave numbers on the same side of the sextet (e.g., \mathbf{k}_1 and \mathbf{k}_2 , or \mathbf{k}_3 and \mathbf{k}_4 , etc.) are small. In this case, integrations over both of the small wave numbers will lead to the same contribution, namely, integral in Eq. (40), i.e., the result will be the integral in Eq. (40) squared. This appears to be the strongest IR singularity and the resulting behavior of the collision integral Eq. (33c) is

$$\text{St}^{\text{IR}} \sim \left(\int_{1/\ell} k_5^{2-x} dk_5 \right)^2. \quad (41)$$

When two wave numbers from the opposite sides of the sextet (e.g., \mathbf{k}_2 and \mathbf{k}_5) tend to zero simultaneously, we get an extra small factor in the integrand because in this case $(n_k^{-1} + n_1^{-1} - n_3^{-1} - n_4^{-1}) \rightarrow 0$. As a result we get IR convergence in this range. One can also show IR convergence when two wave numbers from one side and one on the other side of the sextet are small (the resulting integral is IR convergent for $x < 9/2$).

Divergence of integrals in Eq. (33c) means that both spectra in Eq. (39) with $x_N=3$ and $x_E=17/5 > 3$, obtained under *opposite* assumption of the convergence of these integrals in the limit $\ell \rightarrow \infty$ are not solutions of the $3 \leftrightarrow 3$ -KE, Eq. (33c), and thus cannot be realized in nature. One should find another self-consistent solution of this KE. Note that the proof of divergence at the IR limits is sufficient for discarding the spectra under the test whereas proving convergence would require considering all the singular limits including the ultraviolet (UV) ranges. However, we have examined these limits, too. At the UV end we have obtained convergence for the KS and for the inverse cascade spectra. Thus the most dangerous singularity appears to be in the IR range, when two wave numbers from the same side of the wave sextet are small simultaneously.

D. Logarithmic corrections for the N -flux spectrum in Eq. (39a)

Note that for the N -flux spectrum in Eq. (39a) $n_N(k) \propto k^{-3}$ that the integrals in Eqs. (40) and (41) diverge only logarithmically. The same situation happens, e.g., for the direct enstrophy cascade in two-dimensional turbulence: dimensional reasoning leads to the Kraichnan-1967 (Ref. 18) turbulent energy spectrum

$$E(k) \propto k^{-3} \quad (42a)$$

for which the integral for the enstrophy flux diverges logarithmically. Using a simple argument of constancy of the enstrophy flux, Kraichnan suggested¹⁹ a logarithmic correction to the spectra

$$E(k) \propto k^{-3} \ln^{-1/3}(k\ell) \quad (42b)$$

that permits the enstrophy flux to be k independent. Here ℓ is the enstrophy pumping scale.

Using the same arguments, we can substitute in Eq. (33c), a logarithmically corrected spectrum $n_N(k) \propto k^{-3} \ln^{-x}(k\ell)$ and find x by the requirement that the resulting N flux, μ_k , Eq. (36a) will be k independent. Having in mind that according to Eq. (38a) $\mu_k \propto n_N^5$, we can guess that $x=1/5$. Then, the divergent integral in Eq. (40) will be $\propto \ln^{4/5}(k\ell)$ while the remaining convergent integrals in Eq. (33c) will be $\propto \ln^{-4/5}(k\ell)$. Therefore, the resulting flux μ_k will be k independent as it should be.¹⁹ So, our prediction is that instead of a nonlocal spectrum in Eq. (39a) we have a slightly steeper log-corrected spectrum

$$n_N(k) \simeq \frac{(\mu\kappa)^{1/5}}{k^3 \ln^{1/5}(k\ell)}. \quad (43)$$

The difference is not large but the underlying physics must be correct; as one says on the Odessa market: “We can argue the price but the weight must be correct.”

IV. CONCLUSIONS

In this paper, we have derived an effective theory of KW turbulence based on asymptotic expansions of the Biot-Savart model in powers of small $1/\Lambda$ and small nonlinearity, by applying a canonical transformation eliminating nonresonant low-order (quadratic) interactions and by using the standard Wave Turbulence approach based on random phases.¹² In doing so, we have fixed errors arising from the previous derivations, particularly the latest one by KS,¹¹ by taking into account previously omitted and important contributions to the effective six-wave interaction coefficient. We have examined the resulting six-wave interaction coefficient in several asymptotic limits when one or several wave numbers are in the IR range. These limits are summarized in a remarkably simple expression (31). This allowed us to achieve three goals: (a) to derive a simple effective model for KW turbulence expressed in the local nonlinear Eq. (8). In addition to small $1/\Lambda$ and the weak nonlinearity, this model relies on the fact that our findings show, for dynamically relevant wave sextets, the interaction coefficient is a simple product of the six wave numbers, Eq. (31). For weak nonlinearities, the LNE is isomorphic to the previously suggested TLIA model.¹⁰ (b) To examine the locality of the E -flux (KS) and the N -flux spectra. We found that the KS spectrum is nonlocal and therefore cannot be realized in nature. (c) The N -flux spectrum is found to be marginally nonlocal and could be “rescued” by a logarithmic correction, which we constructed following a qualitative Kraichnan approach. However, it remains to be seen if such a spectrum can be realized in quantum turbulence, because, as it was shown in Ref. 20, the

vortex-line reconnections can generate only the forward cascade and not the inverse one (i.e., the reconnections produce an effectively large-scale wave forcing).

Finally we will discuss the numerical studies of KW turbulence. The earliest numerics by KS were reported in Ref. 1. They claimed that they observed the KS spectrum. At the same time they gave a value of the E -flux constant $\sim 10^{-5}$ which is unusually small. We have already mentioned that this work failed to take into account several important contributions to the effective interaction coefficient and thus these numerical results cannot be trusted. In particular, we showed that their interaction coefficient must have contained a spurious dependence on the scale ℓ which makes the numerical results arbitrary and dependent on the choice of such a cutoff. In addition, even if the interaction coefficient was correct, the Monte Carlo method used by KS is a rather dangerous tool when one deals with slowly divergent integrals (in this case $\int_0 x^{-7/5} dx$).

On the other hand, recent numerical simulations of the TLIA model also reported agreement with the KS scaling (as well as an agreement with the inverse cascade scaling).¹⁰ How can one explain this now when we showed analytically that the KS spectrum is nonlocal? It turns out that the correct KW spectrum, which takes into account the nonlocal interactions with long KW's, has an index which is close (but not equal) to the KS index, and it is also consistent with the data of Ref. 10. We will report these results in a separate publication.

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APPENDIX A: BARE INTERACTIONS

1. Actual calculation of the bare interaction coefficients

The geometrical constraint of a small-amplitude perturbation can be expressed in terms of a parameter

$$\epsilon(z_1, z_2) = |w(z_1) - w(z_2)| / |z_1 - z_2| \ll 1. \quad (\text{A1})$$

This allows one to expand Hamiltonian (5) in powers of ϵ and to rewrite it in terms consisting of the number of wave interactions, according to Eq. (18). Kozik and Svistunov (KS) found the exact expressions for H_2 , H_4 , and H_6 (Ref. 1)

$$H_2 = \frac{\kappa}{8\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \{2 \operatorname{Re}[w'^*(z_1)w'(z_2)] - \epsilon^2\},$$

$$H_4 = \frac{\kappa}{32\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \{3\epsilon^4 - 4\epsilon^2 \operatorname{Re}[w'^*(z_1)w'(z_2)]\},$$

$$H_6 = \frac{\kappa}{64\pi} \int \frac{dz_1 dz_2}{|z_1 - z_2|} \{6\epsilon^4 \operatorname{Re}[w'^*(z_1)w'(z_2)] - 5\epsilon^6\}.$$

(A2)

The explicit calculation of these integrals was analytically done in Ref. 10, by evaluating the terms in Eq. (A2) in Fourier space, and then expressing each integral as various cosine expressions.¹ Hamiltonian (A2) can be expressed in terms of a wave representation variable $a_k = a(\mathbf{k}, t)$ by applying a Fourier transform in Eq. (19) in the variables z_1 and z_2 , (for details see Refs. 1 and 10). The result is given by Eqs. (20), in which the cosine expressions for ω_k , T_{12}^{34} , and W_{123}^{456} were done in Ref. 1. In our notations they are

$$\omega_k = \frac{\kappa}{2\pi}[A - B], \quad T_{12}^{34} = \frac{1}{4\pi}[6D - E], \quad W_{123}^{456} = \frac{9}{4\pi\kappa}[3P - 5Q], \quad \text{where}$$

$$A = \int_{a_0}^{\infty} \frac{dz_-}{z_-} k^2 C^k, \quad B = \int_{a_0}^{\infty} \frac{dz_-}{z_-^3} [1 - C^k], \quad D = \int_{a_0}^{\infty} \frac{dz_-}{z_-^5} [1 - C_1 - C_2 - C^3 - C^4 + C_2^3 + C^{43} + C_2^4], \quad (\text{A3})$$

$$E = \int_{a_0}^{\infty} \frac{dz_-}{z_-^3} [k_1 k_4 (C^4 + C_1 - C^{43} - C_2^4) + k_1 k_3 (C^3 + C_1 - C^{43} - C_2^3) + k_3 k_2 (C^3 + C_2 - C^{43} - C_1^3) + k_4 k_2 (C^4 + C_2 - C^{43} - C_2^3)],$$

$$P = \int_{a_0}^{\infty} \frac{dz_-}{z_-^5} k_6 k_2 [C_2 - C_2^5 - C_{23} + C_{23}^5 - C_2^4 + C_2^{45} + C_{23}^4 - C_1^6 + C^6 - C^{56} - C_3^6 + C_3^{56} - C^{46} + C^{456} + C_3^{46} - C_{12}],$$

$$Q = \int_{a_0}^{\infty} \frac{dz_-}{z_-^7} [1 - C^4 - C_1 + C_1^4 - C^6 + C^{46} + C_1^6 - C_1^{46} - C^5 + C^{45} + C_1^5 - C_1^{45} + C^{65} - C^{456} - C_1^{56} + C_{23} - C_3 + C_3^4 + C_{13} - C_{13}^4 + C_3^6 - C_3^{46} - C_{13}^6 + C_2^5 + C_3^5 - C_3^{45} - C_{13}^5 + C_2^6 - C_3^{56} + C_{12} + C_2^4 - C_2]. \quad (\text{A4})$$

Here the variable, $z_- = |z_1 - z_2|$ and the expressions C , are cosine functions such that $C_1 = \cos(\mathbf{k}_1 z_-)$, $C_1^4 = \cos[(\mathbf{k}_4 - \mathbf{k}_1) z_-]$, $C_1^{45} = \cos[(\mathbf{k}_4 + \mathbf{k}_5 - \mathbf{k}_1) z_-]$, $C_{12}^{45} = \cos[(\mathbf{k}_4 + \mathbf{k}_5 - \mathbf{k}_1 - \mathbf{k}_2) z_-]$, and so on. The lower limit of integration a_0 is the induced cutoff of the vortex core radius $a_0 < |z_1 - z_2|$.

The trick used for explicit calculation of the analytical form of these integrals was suggested and used in Ref. 10. First one should integrate by parts all the cosine integrals so they can be expressed in the form of $\int_{a_0}^{\infty} \frac{\cos(z)}{z} dz$. Then, one can use a cosine identity for this integral³¹

$$\int_{a_0}^{\infty} \frac{\cos(z)}{z} dz = -\gamma - \ln(a_0) - \int_0^{a_0} \frac{\cos(z) - 1}{z} dz = -\gamma - \ln(a_0) - \sum_{k=1}^{\infty} \frac{(-a_0^2)^k}{2k(2k)!} = -\gamma - \ln(|a_0|) + \mathcal{O}(a_0^2), \quad (\text{A5})$$

where $\gamma=0.5772\dots$ is the Euler Constant. Therefore, in the limit of a small vortex core radius a_0 , we can neglect terms of order $\sim a_0^2$ and higher. For example, let's consider the following general cosine expression that can be found in Eqs. (A4): $\int_{a_0}^{\infty} z^{-3} \cos(\mathcal{K}z) dz$, where \mathcal{K} is an expression that involves a linear combination of wave numbers, i.e., $\mathcal{K}=k_1 - k_4$. Therefore, integration by parts will yield the following result for this integral:

$$\begin{aligned} \int_{a_0}^{\infty} \frac{\cos(\mathcal{K}z)}{z^3} dz &= \left[-\frac{\cos(\mathcal{K}z)}{2z^2} \right]_{a_0}^{\infty} + \left[\frac{\mathcal{K} \sin(\mathcal{K}z)}{2z} \right]_{a_0}^{\infty} \\ &= \frac{\mathcal{K}}{2} \int_{a_0}^{\infty} \frac{\cos(\mathcal{K}z)}{z} dz = \frac{\cos(\mathcal{K}a_0)}{2a_0^2} \\ &= \frac{\mathcal{K} \sin(\mathcal{K}a_0)}{2a_0} - \frac{\mathcal{K}^2}{2} \int_{\mathcal{K}a_0}^{\infty} \frac{\cos(y)}{y} dy. \end{aligned}$$

We then expand $\cos(\mathcal{K}a_0)$ and $\sin(\mathcal{K}a_0)$ in powers of a_0 , and apply the cosine formula (A5) for the last integral, where in the last step we have also changed integration variables: $y=\mathcal{K}z$. The final expression is then

$$\int_{a_0}^{\infty} \frac{\cos(\mathcal{K}z)}{z^3} dz = \frac{1}{2a_0^2} - \frac{3\mathcal{K}^2}{4} + \frac{\mathcal{K}^2}{2} [\gamma + \ln(|\mathcal{K}a_0|)] + \mathcal{O}(a_0^2).$$

By applying a similar procedure to the other cosine integrals, we find that all terms of negative powers of a_0 , (that will diverge in the limit $a_0 \rightarrow 0$) actually cancel in the final expression for each interaction coefficient. Applying this strategy to all interaction coefficients, we get the following analytical evaluation of the Hamiltonian functions:¹⁰

$$\Lambda_0 = \ln(\ell/a_0),$$

$$\omega_k = \frac{\kappa k^2}{4\pi} \left[\Lambda_0 - \gamma - \frac{3}{2} - \ln(k\ell) \right],$$

$$T_{12}^{34} = \frac{1}{16\pi} [k_1 k_2 k_3 k_4 (1 + 4\gamma - 4\Lambda_0) - \mathcal{F}_{1,2}^{3,4}],$$

$$W_{123}^{456} = \frac{9}{32\pi\kappa} [k_1 k_2 k_3 k_4 k_5 k_6 (1 - 4\gamma + 4\Lambda_0) - \mathcal{G}_{1,2,3}^{4,5,6}]. \quad (\text{A6})$$

Explicit equations for \mathcal{F}_{12}^{34} and \mathcal{G}_{123}^{456} are given below in Appendices A 2 and A 3. In the main text we introduced $\Lambda \equiv \Lambda_0 - \gamma - 3/2$. Writing $\Lambda = \ln(\ell/a)$, we see that $a = a_0 e^{\gamma+3/2} \simeq 8a_0$.

2. Bare four-wave interaction function $\mathcal{F}_{1,2}^{3,4}$

A rather cumbersome calculation, presented above, results in an explicit equation for the four-wave interaction function $\mathcal{F}_{1,2}^{3,4}$ in Eqs. (A6) and (22b). Function $\mathcal{F}_{1,2}^{3,4}$ is a symmetrical version of $F_{1,2}^{3,4}$: $\mathcal{F}_{1,2}^{3,4} \equiv \{F_{1,2}^{3,4}\}_S$ where the operator $\{\dots\}_S$ stands for the symmetrization $k_1 \leftrightarrow k_2$, $k_3 \leftrightarrow k_4$, and $\{k_1, k_2\} \leftrightarrow \{k_3, k_4\}$. In its turn $F_{1,2}^{3,4}$ is defined as following:

$$F_{1,2}^{3,4} \equiv \sum_{\mathcal{K} \in \mathcal{K}_1} \mathcal{K}^4 \ln(|\mathcal{K}\ell|) + 2 \sum_{i,j} \sum_{\mathcal{K} \in \mathcal{K}_{ij}} k_i k_j \mathcal{K}^2 \ln(|\mathcal{K}\ell|). \quad (\text{A7a})$$

The $\sum_{i,j}$ denotes sum of four terms with $(i,j) = \{(4,1), (3,1), (3,2), (4,2)\}$, \mathcal{K} is either a single wave vector or linear combination of wave vectors that belong to one of the following sets:

$$\mathcal{K}_1 = \{[-1], [-2], [-3], [-4], [+3], [+4], [+2], [+1]\},$$

$$\mathcal{K}_{41} = \{+[4], [+1], [-4], [-2]\},$$

$$\mathcal{K}_{31} = \{+[3], [+1], [-4], [-2]\},$$

$$\mathcal{K}_{32} = \{+[3], [+2], [-4], [-1]\},$$

$$\mathcal{K}_{42} = \{+[4], [+2], [-4], [-1]\}. \quad (\text{A7b})$$

Here we used the following shorthand notations with $\alpha, \beta, \gamma = 1, 2, 3, 4$: $[^\alpha] \equiv k_\alpha$, $[\beta] \equiv -k_\beta$, $[\alpha] \equiv k_\alpha - k_\beta$, $[^\alpha \gamma] \equiv k_\alpha + k_\gamma$, $[\beta \gamma] \equiv -k_\beta - k_\gamma$ and $+$ or $-$ signs before $[\dots]$ should be understood as prefactors $+1$ or -1 in the corresponding term in the sum. For example,

$$\mathcal{K}^4 \ln(|\mathcal{K}\ell|) \text{ for } \mathcal{K} \in \{[-1]\} \text{ is } -k_1^4 \ln(k_1\ell),$$

$$\mathcal{K}^4 \ln(|\mathcal{K}\ell|) \text{ for } \mathcal{K} \in \{+[2]\} \text{ is } +(\mathbf{k}_4 - \mathbf{k}_2)^4 \ln(|\mathbf{k}_4 - \mathbf{k}_2|\ell),$$

$$\mathbf{k}_i \mathbf{k}_j \mathcal{K}^2 \ln(|\mathcal{K}\ell|) \text{ for } i=4, j=1,$$

$$\mathcal{K} \in \{[-4]\} \text{ is } -\mathbf{k}_4 \mathbf{k}_1 (\mathbf{k}_4 + \mathbf{k}_3)^4 \ln(|\mathbf{k}_4 + \mathbf{k}_3|\ell).$$

3. Bare six-wave interaction function $\mathcal{G}_{1,2,3}^{4,5,6}$

Function $\mathcal{G}_{1,2,3}^{4,5,6} \equiv \{G_{1,2,3}^{4,5,6}\}_S$. The operator $\{\dots\}_S$ stands for the symmetrization $k_1 \leftrightarrow k_2 \leftrightarrow k_3$, $k_4 \leftrightarrow k_5 \leftrightarrow k_6$, and $\{k_1, k_2, k_3\} \leftrightarrow \{k_4, k_5, k_6\}$, and $G_{1,2,3}^{4,5,6}$ is defined as following:

$$G_{1,2,3}^{4,5,6} \equiv \sum_{\mathcal{K} \in \mathcal{K}_3} \mathbf{k}_6 \mathbf{k}_2 \mathcal{K}^4 \ln(|\mathcal{K}| \ell) + \frac{1}{18} \sum_{\mathcal{K} \in \mathcal{K}_4} \mathcal{K}^6 \ln(|\mathcal{K}| \ell), \quad (\text{A8a})$$

where

$$\mathcal{K}_3 = \{^+ [2], ^- [2], ^- [23], ^+ [23], ^- [2], ^+ [45], ^+ [4], ^- [23], ^- [6], ^+ [6], ^- [56], ^- [6], ^+ [56], ^- [46], ^+ [456], ^+ [46], ^- [12]\}, \quad (\text{A8b})$$

$$\begin{aligned} \mathcal{K}_4 = \{ & ^- [4], ^- [1], ^+ [4], ^- [6], ^+ [46], ^+ [6], ^- [46], ^- [5], ^+ [45], ^+ [5], ^- [45], ^+ [65], ^- [456], ^- [56], ^+ [23], ^- [3], ^+ [4], ^+ [13], ^- [4], \\ & ^+ [3], ^- [46], ^- [6], ^+ [5], ^+ [5], ^- [45], ^- [5], ^+ [6], ^- [65], ^+ [12], ^+ [4], ^- [2]\}. \end{aligned} \quad (\text{A8c})$$

APPENDIX B: EFFECTIVE SIX-KW INTERACTION COEFFICIENT

1. Absence of six-wave dynamics in LIA

According to Eqs. (24d) and (24e), the expression for ${}^\Lambda \tilde{W}_{1,2,3}^{4,5,6}$ is given by

$${}^\Lambda \tilde{W}_{1,2,3}^{4,5,6} = {}^\Lambda W_{1,2,3}^{4,5,6} + {}^\Lambda Q_{1,2,3}^{4,5,6}, \quad (\text{B1a})$$

$${}^\Lambda Q_{1,2,3}^{4,5,6} = \frac{1}{8} \sum_{i,j,m=1}^3 \sum_{p,q,r=4}^6 \Lambda q_{i,j,m}^{p,q,r}, \quad (\text{B1b})$$

$$\Lambda q_{i,j,m}^{p,q,r} \equiv \frac{\Lambda T_{r,j+m-r}^{j,m} \Lambda T_{i,p+q-i}^{q,p}}{\Lambda \Omega_{r,j,m}^{r,j+m-r}} + \frac{\Lambda T_{m,q+r-m}^{q,r} \Lambda T_{p,i+j-p}^{i,j}}{\Lambda \Omega_{m,q+r-m}^{m,q+r-r}}, \quad (\text{B1c})$$

where ${}^\Lambda \Omega_{1,2,3}^{3,4} \equiv {}^\Lambda \omega_1 + {}^\Lambda \omega_2 - {}^\Lambda \omega_3 - {}^\Lambda \omega_4$. We want to compute this equation on the LIA manifold in Eq. (30). To do this we express two wave vectors in terms of the other four²² using the LIA manifold constraint in Eq. (30)

$$\mathbf{k}_1 = \frac{(\mathbf{k}_3 - \mathbf{k})(\mathbf{k}_2 - \mathbf{k}_3)}{\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_5} + \mathbf{k}_5, \quad (\text{B2a})$$

$$\mathbf{k}_4 = \frac{(\mathbf{k}_3 - \mathbf{k})(\mathbf{k}_2 - \mathbf{k}_3)}{\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_5} + \mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3. \quad (\text{B2b})$$

Then ${}^\Lambda \tilde{W}_{1,2,3}^{4,5,6}$ is easily simplified to zero with the help of MATHEMATICA. This gives an independent verification of the validity of our initial Eqs. (24) for full interaction coefficient ${}^\Lambda \tilde{W}_{1,2,3}^{4,5,6}$ which is needed for the calculations of the $\mathcal{O}(1)$ contribution ${}^1 \tilde{W}_{1,2,3}^{4,5,6}$. Another way to see the cancelation is to use the Zakharov-Schulman variables²³ that parameterise the LIA manifold in Eq. (30).

2. Exact expression for ${}^1 \tilde{W}$

We get expressions for ${}^1 Q$, ${}^2 Q$, and ${}^3 Q$, introduced by Eqs. (27), from Eqs. (24d) and (24e). Namely,

$${}^1 Q_{1,2,3}^{4,5,6} = \frac{1}{8} \sum_{i,j,m=1}^3 \sum_{p,q,r=4}^6 \left[\frac{\Lambda T_{r,j+m-r}^{j,m} \Lambda T_{i,p+q-i}^{q,p}}{\Lambda \Omega_{r,j,m}^{r,j+m-r}} + \frac{\Lambda T_{m,q+r-m}^{q,r} \Lambda T_{p,i+j-p}^{i,j}}{\Lambda \Omega_{m,q+r-m}^{m,q+r-r}} \right], \quad (\text{B3a})$$

$${}^2 Q_{1,2,3}^{4,5,6} = \frac{1}{8} \sum_{i,j,m=1}^3 \sum_{p,q,r=4}^6 \left[\frac{{}^1 T_{r,j+k-r}^{j,k} \Lambda T_{i,p+q-i}^{q,p}}{\Lambda \Omega_{r,j,m}^{r,j+m-r}} + \frac{{}^1 T_{m,q+r-m}^{q,r} \Lambda T_{p,i+j-p}^{i,j}}{\Lambda \Omega_{m,q+r-m}^{m,q+r-r}} \right], \quad (\text{B3b})$$

$${}^3 Q_{1,2,3}^{4,5,6} = \frac{1}{8} \sum_{i,j,m=1}^3 \sum_{p,q,r=4}^6 \left[\frac{\Lambda T_{r,j+m-r}^{j,m} \Lambda T_{i,p+q-i}^{q,p}}{(\Lambda \Omega_{r,j,m}^{r,j+m-r})^2} \cdot {}^1 \Omega_{r,j+m-r} + \frac{\Lambda T_{m,q+r-m}^{q,r} \Lambda T_{p,i+j-p}^{i,j}}{(\Lambda \Omega_{m,q+r-m}^{m,q+r-r})^2} \cdot {}^1 \Omega_{m,q+r-m} \right]. \quad (\text{B3c})$$

Again, using MATHEMATICA we substitute Eqs. (B2) into Eqs. (B3a)–(B3c). Clearly, the resulting equations are too cumbersome to be presented here. But we will analyze them in various limiting cases, see below.

3. Derivation of Eq. (29b) for ${}^1 \tilde{S}_{k,1,2}^{3,4,5}$

First of all, let us find a parametrization for the full resonant manifold, by calculation of the correction to the LIA parametrization in Eq. (B2), namely,

$$\mathbf{k}_1 = {}^\Lambda \mathbf{k}_1 + {}^1 \mathbf{k}_1, \quad \mathbf{k}_4 = {}^\Lambda \mathbf{k}_4 + {}^1 \mathbf{k}_4, \quad (\text{B4})$$

where ${}^\Lambda \mathbf{k}_1$ and ${}^\Lambda \mathbf{k}_4$ are given by the right-hand sides of Eqs. (B2), respectively. Corrections ${}^1 \mathbf{k}_1$ and ${}^1 \mathbf{k}_4$ are found so that the resonances in \mathbf{k} , Eq. (30a), and (full) ω are satisfied. The resonances in \mathbf{k} fixes ${}^1 \mathbf{k}_1 = {}^1 \mathbf{k}_4$. Then the ω resonance in the leading order in $1/\Lambda$ gives

$$\tilde{\Omega}_{1,2,3}^{4,5,6} = {}^1 \mathbf{k}_1 \frac{\partial {}^\Lambda \omega_1}{\partial \mathbf{k}_1} - {}^1 \mathbf{k}_4 \frac{\partial {}^\Lambda \omega_4}{\partial \mathbf{k}_4} + {}^1 \tilde{\Omega}_{1,2,3}^{4,5,6} + \mathcal{O}(\Lambda^{-1}) = 0. \quad (\text{B5})$$

Thus

$${}^1\mathbf{k}_1 = {}^1\mathbf{k}_4 \approx \frac{2\pi}{\Lambda\kappa} \frac{{}^1\tilde{\Omega}_{1,2,3}^{4,5,6}}{(\mathbf{k}_4 - \mathbf{k}_1)}. \quad (\text{B6})$$

This allows us to write down the contribution of ${}^\Lambda\tilde{W}$ from the deviation of the LIA resonant surface

$$\begin{aligned} {}^1\tilde{S}_{1,2,3}^{4,5,6} &= {}^1\mathbf{k}_1 \frac{\partial {}^\Lambda\tilde{W}_{1,2,3}^{4,5,6}}{\partial \mathbf{k}_1} + {}^1\mathbf{k}_4 \frac{\partial {}^\Lambda\tilde{W}_{1,2,3}^{4,5,6}}{\partial \mathbf{k}_1} + \mathcal{O}(\Lambda^{-1}) \\ &\approx \frac{2\pi}{\Lambda\kappa} \frac{{}^1\tilde{\Omega}_{1,2,3}^{4,5,6} (\partial_4 + \partial_1) {}^\Lambda\tilde{W}_{1,2,3}^{4,5,6}}{(\mathbf{k}_4 - \mathbf{k}_1)} \end{aligned} \quad (\text{B7})$$

with $\partial_j(\cdot) = \partial_j(\cdot) / \partial \mathbf{k}_j$. It is obvious that instead of \mathbf{k}_1 and \mathbf{k}_4 we could use parametrizations in terms of other pairs \mathbf{k}_i and \mathbf{k}_j with $i=1, 2$, or 3 and $j=4, 5$, or 6 . This enables us to write a fully symmetric expression for 1S

$${}^1\tilde{S}_{1,2,3}^{4,5,6} = \frac{2\pi}{9\Lambda\kappa} {}^1\tilde{\Omega}_{1,2,3}^{4,5,6} \sum_{\substack{i=\{1,2,3\} \\ j=\{4,5,6\}}} \frac{(\partial_j + \partial_i) {}^\Lambda\tilde{W}_{1,2,3}^{4,5,6}}{(\mathbf{k}_j - \mathbf{k}_i)}. \quad (\text{B8})$$

This is the required expression Eq. (29b).

4. Analytical expression for \mathcal{W} on the LIA manifold when two wave numbers are small

Let us put together the coefficients to the interaction coefficient $\mathcal{W}_{k,1,2}^{3,4,5}$ given in Eqs. (27a)–(27c), (23c), (29a), and (29b), and use in these expressions the formulas obtained in the previous appendices and the parametrization of the LIA surface in Eq. (B2). Using MATHEMATICA, and TAYLOR expanding $\mathcal{W}_{k,1,2}^{3,4,5}$ with respect to one wave number, e.g., \mathbf{k}_5 , we obtain a remarkably simple result—expression (31).

Now we will consider the asymptotical limit when two of the wave numbers, say \mathbf{k}_2 and \mathbf{k}_5 (let them be on the opposite sides of the resonance conditions), are much less than the other wave numbers in the sextet. Using MATHEMATICA and TAYLOR expanding $\mathcal{W}_{k,1,2}^{3,4,5}$ with respect to two wave numbers \mathbf{k}_2 and \mathbf{k}_5 , we have

$$\lim_{\substack{k_2 \rightarrow 0 \\ k_5 \rightarrow 0}} \mathcal{W}_{k,1,2}^{3,4,5} = -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5. \quad (\text{B9})$$

Simultaneously, see Eq. (B2)

$$\lim_{\substack{k_2 \rightarrow 0 \\ k_5 \rightarrow 0}} \mathbf{k}_1 \rightarrow \mathbf{k}_3, \quad \lim_{\substack{k_2 \rightarrow 0 \\ k_5 \rightarrow 0}} \mathbf{k}_4 \rightarrow \mathbf{k}. \quad (\text{B10})$$

Therefore, Eq. (B9) coincides with Eq. (31). Note that this was not obvious *a priori* because formally Eq. (31) was obtained when \mathbf{k}_5 is much less than the rest of the wave numbers, including \mathbf{k}_2 .

For reference, we provide expressions for the different contributions to the interaction coefficient $\mathcal{W}_{k,1,2}^{3,4,5}$ given in Eqs. (27) and (29). For $\mathbf{k}_2, \mathbf{k}_5 \rightarrow 0$

$$\begin{aligned} {}^1W \rightarrow & -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left\{ +\frac{3}{2} \ln(k\ell) - \frac{1}{24} \left[49 \right. \right. \\ & - \frac{(1-x)^2(7+10x+7x^2)}{x^2} \ln|1-x| + 2x(12+7x) \ln|x| \\ & \left. \left. - 7 \frac{(1+x)^4}{x^2} \ln|1+x| \right] \right\}, \end{aligned} \quad (\text{B11a})$$

$$\begin{aligned} {}^1Q = {}^1Q \rightarrow & -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left\{ -\frac{3}{2} \ln(k\ell) + \frac{1}{48} \left[59 \right. \right. \\ & - \frac{(1-x)^2(9+10x+9x^2)}{x^2} \ln|1-x| + 2 \left(9x^2 + 14x - 6 \right. \\ & \left. \left. + \frac{2}{1-x} \right) \ln|x| - 9 \frac{(1+x)^4}{x^2} \ln|1+x| \right] \right\}, \end{aligned} \quad (\text{B11b})$$

$$\begin{aligned} {}^1Q \rightarrow & -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left\{ +\frac{3}{2} \ln(k\ell) + \frac{1}{48} \left[7 \right. \right. \\ & + \frac{(1-x)^2(1+x^2)}{x^2} \ln|1-x| + 2 \frac{1-5x+x^3}{1-x} \ln|x| \\ & \left. \left. + \frac{(1+x)^4}{x^2} \ln|1+x| \right] \right\}, \end{aligned} \quad (\text{B11c})$$

$${}^1S \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[\frac{1}{6} \frac{1+x}{1-x} \ln|x| \right], \quad (\text{B11d})$$

$${}^1\tilde{W} \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[1 - \frac{1}{6} \frac{1+x}{1-x} \ln|x| \right], \quad x \equiv k_3/k. \quad (\text{B11e})$$

Another possibility is for two small wave numbers to be on the same side of the sextet. We have checked that on the resonant manifold, this also leads to Eq. (31).

5. Analytical expression for \mathcal{W} on the LIA manifold when four wave numbers are small

Now let us, using MATHEMATICA, calculate the asymptotic behavior of \mathcal{W} when four wave vectors are smaller than the other two; on the LIA manifold this automatically simplifies to $k_1, k_2, k_3, k_5 \ll k, k_4$ [remember that on the LIA manifold \mathbf{k}_1 and \mathbf{k}_4 are expressed in terms of the other wave numbers using Eq. (B2) thus from Eq. (B10)] we have

$$\lim_{k_{1,2,3,5} \rightarrow 0} \mathcal{W}_{k,1,2}^{3,4,5} = -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5. \quad (\text{B12})$$

Again, we have got an expression which coincides with Eq. (31). We emphasize that this was not obvious *a priori* because formally Eq. (31) was obtained when k_5 is much less than the rest of the wave numbers, including k_1, k_2, k_3 .

Therefore we conclude that the expression (31) is valid when k_5 is much less than just one other wave number in the sextet, say k , and not only when it is much less than all of the remaining wave numbers. For a reference, we give the term

by term results for the limit $k_1, k_2, k_3, k_5 \ll k, k_4$

$${}^1W \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[-1 + \frac{3}{2} \ln(k\ell) + 0 \right],$$

$${}^1Q \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[+\frac{1}{2} - \frac{3}{2} \ln(k\ell) - \frac{1}{6} \ln \frac{k_3}{k} \right],$$

$${}^1Q \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[+\frac{1}{2} - \frac{3}{2} \ln(k\ell) - \frac{1}{6} \ln \frac{k_3}{k} \right],$$

$${}^1Q \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[+1 + \frac{3}{2} \ln(k\ell) + \frac{1}{6} \ln \frac{k_3}{k} \right],$$

$${}^1S \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 \left[0 + 0 + \frac{1}{6} \ln \frac{k_3}{k} \right].$$

The sum of this contributions is very simple

$${}^1\tilde{W} \rightarrow -\frac{3}{4\pi\kappa} k^2 k_2 k_3^2 k_5 [+1 + 0 + 0].$$

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- ¹E. Kozik and B. Svistunov, Phys. Rev. Lett. **92**, 035301 (2004).
²V. S. L'vov, S. V. Nazarenko, and O. Rudenko, Phys. Rev. B **76**, 024520 (2007).
³V. S. L'vov, S. V. Nazarenko, and O. Rudenko, J. Low Temp. Phys. **153**, 140 (2008).
⁴E. V. Kozik, B. V. Svistunov, Phys. Rev. B **77**, 060502(R) (2008).
⁵R. J. Donnelly, *Quantized Vortices in He II* (Cambridge University Press, Cambridge, 1991).
⁶*Quantized Vortex Dynamics and Superfluid Turbulence*, Lecture Notes in Physics Vol. 571, edited by C. F. Barenghi, R. J. Donnelly, and E. F. Vinen, (Springer-Verlag, Berlin, 2001).
⁷K. W. Schwarz, Phys. Rev. B **31**, 5782 (1985); Phys. Rev. B **38**, 2398 (1988).
⁸R. J. Arms and F. R. Hama, Phys. Fluids **8**, 553 (1965).
⁹B. V. Svistunov, Phys. Rev. B **52**, 3647 (1995).
¹⁰G. Boffetta, A. Celani, D. Dezzani, J. Laurie, and S. Nazarenko, J. Low Temp. Phys. **156**, 193 (2009).
¹¹E. Kozik and B. Svistunov, J. Low Temp. Phys. **156**, 215 (2009).
¹²V. E. Zakharov, V. S. L'vov, and G. E. Falkovich, *Kolmogorov Spectra of Turbulence* (Springer-Verlag, Berlin, 1992).
¹³H. Hasimoto, J. Fluid Mech. **51**, 477 (1972).
¹⁴The limit of three small wave numbers is not allowed by the resonance conditions. Indeed, putting three wave numbers to zero, we get a $1 \leftrightarrow 2$ process which is not allowed in one dimension for $\omega \sim k^2$.
¹⁵Here, we evoke a quantum-mechanical analogy as an elegant shortcut, allowing us to introduce KE and the respective solutions easily. However, the reader should not get confused with this analogy and understand that our KW system is purely classical. In particular, the Planck's constant \hbar is irrelevant outside of this analogy and should be simply replaced by 1.
¹⁶S. Nazarenko, JETP Lett. **83**, 198 (2006).
¹⁷It is evident for the approximation, Eq. (31). For the full expression, Eq. (29a), it was confirmed by symbolic computation with the help of MATHEMATICA.
¹⁸R. Kraichnan, Phys. Fluids **10**, 1417 (1967).
¹⁹R. Kraichnan, J. Fluid Mech. **47**, 525 (1971).
²⁰S. V. Nazarenko, JETP Lett. **84**, 585 (2007).
²¹I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980).
²²It is appropriate to remind the reader that we use the bold face notation of the one-dimensional wave vector for convenience only. Indeed, such a vector is just a real number, $k \in \mathbb{R}$.
²³V. E. Zakharov and E. I. Schulman, Physica D **4**, 270 (1982).