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Quantum turbulence cascades in the Gross-Pitaevskii model

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We present a numerical study of quantum turbulence within the three-dimensional Gross-Pitaevskii equation, concentrating on the direct energy cascade in the case of a forced-dissipated system. We show that the behavior of the system is very sensitive to the properties of the model at the scales greater than the forcing scale, and we identify three different regimes: (1) a nonstationary regime with condensation and transition from a four-wave to a three-wave interaction process when the largest scales are not dissipated, (2) a steady weak wave turbulence regime when largest scales are dissipated with a friction-type dissipation, and (3) a state with a scale-by-scale balance of the linear and the nonlinear time scales when the large-scale dissipation is a hypoviscosity.

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Quantum turbulence (QT) is an important subject in the field of low temperature physics and has attracted the attention of many researchers in the recent years. Systems where could be observed are for example Bose-Einstein condensates [1–3] of cold atomic vapors and liquid helium II below the transition temperature. In this framework the Gross-Pitaevskii equation (GPE), which describes a Bose gas at very low temperature [4], has been widely used as a model for QT. Written in adimensional form it results in

$$i\frac{\partial \psi}{\partial t} + \nabla^2\psi - |\psi|^2\psi = \mathcal{F} + \mathcal{D}, \tag{1}$$

where ψ is the order parameter indicating the condensate wave function and $\mathcal{F}$ and $\mathcal{D}$ represent possible external forcings and dissipation mechanisms. In general, when $\mathcal{F}=0$ and $\mathcal{D}=0$, GPE conserves total energy and particles

$$H = \int \frac{1}{2}|\nabla\psi|^2 d\mathbf{x} + \int \frac{1}{4}|\psi|^4 d\mathbf{x} = H_{LIN} + H_{NL}, \tag{2a}$$

$$N = \int |\psi|^2 d\mathbf{x}. \tag{2b}$$

This model has been used to study the formation of a condensate in [5–7]. Moreover GPE can be mapped, using the Madelung transformation, to the Euler equation for ideal fluid flows with the extra quantum pressure term. This is why many concepts arising from the fluid dynamics have been discussed and studied with GPE, for example vortices and their reconnection [8]. It was also suggested that this model allows statistical motions similar to classical fluid turbulence and a number of papers [9–13] have been devoted to finding the Kolmogorov spectrum in a such turbulent system. On the other hand, if the system has small nonlinearity, dispersive waves that are solution of the linearized Eq. (1) are involved in nonlinear interactions and an approach known as weak wave turbulence (WWT) can be developed for GPE. Generally, WWT describes statistics on large ensembles of weakly nonlinear waves in different applications, i.e., water waves or waves in plasmas [14]. Such waves interact with each other in a resonant way, e.g., in triads or quartets, thereby transferring energy (or/and any other invariants) through the scale space forming turbulent cascades similar to the classical Kolmogorov cascade in hydrodynamic turbulence. One remarkable property of WWT is that, in contrast to hydrodynamic turbulence, power-law spectra corresponding to such cascades, known as Kolmogorov-Zakharov (KZ) spectra, have been found as exact stationary solutions of the corresponding wave kinetic equation [14].

WWT for GPE turbulence was developed in [5,15]; the following wave kinetic equation was derived

$$\frac{\partial n(\mathbf{k}_1,t)}{\partial t} = \gamma(\mathbf{k}_1,t) + 4\pi\int n_1n_2n_3n_4\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right) \times \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,d\mathbf{k}_{234}, \tag{3}$$

where $n_i=(L/2\pi)^d\langle\hat{\psi}(\mathbf{k}_i,t)\hat{\psi}^*(\mathbf{k}_i,t)\rangle$ is the *wave-action* spectrum averaged over many realizations (here L is the size of the bounding box and d is the dimension of the space), γ includes general forcing $\hat{\mathcal{F}}$ and damping $\hat{\mathcal{D}}$ terms in Fourier space, $\omega_i=k_i^2$ is the wave frequency, and $k_i=|\mathbf{k}_i|$.

If $\gamma=0$ Eq. (3) conserves the total wave-action $N=\int n_1 d\mathbf{k}_1$ and the total energy $E=\int\omega_1 n_1 d\mathbf{k}_1$ which correspond to the GPE invariants (2a) and (2b), respectively. In this case Eq. (3) has a stationary solution which is the well-known Rayleigh-Jeans distribution $n_{th}(k)=T/(\mu+k^2)$, where T is the temperature and μ is the chemical potential. Besides this thermodynamic solution, in the presence of forcing and damping terms, the kinetic equation has two nonequilibrium stationary isotropic solutions of the form of $n(k)\sim k^{-\alpha}$ corresponding to constant fluxes of energy or wave-action (KZ spectra). These predictions are formally obtained considering a source $\hat{\mathcal{F}}$ at $|\mathbf{k}|\to 0$ and a sink $\hat{\mathcal{D}}$ at $|\mathbf{k}|\to+\infty$ for the direct energy cascade and viceversa for the inverse wave-action cascade. Usually, even if all experimental measures and numerical simulations consider a finite range of k values, WWT predictions are expected to be valid if the nonlinearity is small and $\hat{\mathcal{D}}$ and $\hat{\mathcal{F}}$ act at different scales in order to have in between a wide inertial range where a cascade can develop.

In three-dimensional (3D) GPE the *direct energy cascade*

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spectrum has slope $\alpha=3$ and the *inverse wave-action cascade* has $\alpha=7/3$. We will present our results in terms of the one-dimensional (1D) wave-action spectrum $n^{1D}(k)=4\pi k^2 n(k)$, i.e., after integration over the solid angle. For such a spectrum the WWT prediction for the direct cascade is $n^{1D}(k)\sim k^{-1}$. Note that in hydrodynamic turbulence the results are usually discussed in terms of the 1D energy spectrum $E^{1D}(k)$ [e.g., Kolmogorov $E^{1D}(k)\sim k^{-5/3}$]. For WWT we have the relation $E^{1D}(k)=\omega(k)n^{1D}(k)$, which for GPE means $E^{1D}(k)\sim k^{-\alpha+4}$. Further, it was predicted in [5] that in presence of a condensate (due to the inverse wave-action cascade) the four-wave resonant interaction will eventually be replaced by a three-wave process with an acoustic-type KZ spectrum.

Previously, there has been a number of numerical simulations of turbulence in two-dimensional (2D) GPE case and comparisons with the WWT predictions [5,7]. For the 3D case, we are aware of a number of simulations of GPE in freely decaying case [12] or for the unforced undamped simulation where the initial condition relaxes to the thermodynamic solution [6]. As far a we know no steady state (in the sense of cascade) has ever been reached in any simulation, and no direct comparison with the WWT predictions has ever been attempted. The purpose of the present work is to revisit the problem of 3D GPE turbulence in the direct energy cascade range using the numerical simulations and compare the results with some theoretical predictions from WWT. Our goal consist in finding the spectrum and verifying if (and when) it agrees with the theoretical predictions from the WWT. We stress that here our aim is to study some fundamental properties of turbulent cascades in GPE in simplified settings: triple-periodic cube volume, forcing and dissipation well separated by nondissipative inertial range of scales. The situation is very similar to the classical turbulence, where theory and numerics most often deal with similar idealized setups before more realistic for experiments geometries can be studied. Our numerical domain is a cube with uniform mesh of 256^3 points and periodic boundary conditions. We integrate Eq. (1) by a standard split step method. In order to observe the cascade, energy and wave-action are injected directly in Fourier space at wave numbers $\in[9\Delta k, 10\Delta k]$ by a forcing term $\hat{\mathcal{F}}=-if_0 e^{i\varphi(k)}$ with random phases φ which are independent for different ks and short correlated in time and f_0 being a constant forcing coefficient. To absorb energy at high wave numbers and prevent accumulation or thermalization, a dissipative hyperviscous term $\mathcal{D}=i\nu_h(-\nabla^2)^n\psi$ is included in Eq. (1), with $\nu_h=2\times10^{-6}$ and $n=8$. From an experimental point of view, forcing could be implemented, e.g., by optical "spoons" (laser beams) whereas dissipation can occur naturally via "evaporative cooling," i.e., via loosing the high-momentum (short scale) component of BEC over the potential barrier of the bounding trap. Of course, before implementing such a setup experimentally, it would be desirable to compute BEC turbulence in a more realistic volume or trap with more realistic forcing and dissipation mechanisms. This task is numerically challenging but not impossible, and it could be undertaken at the next step, once the fundamental GPE turbulence properties are clarified.

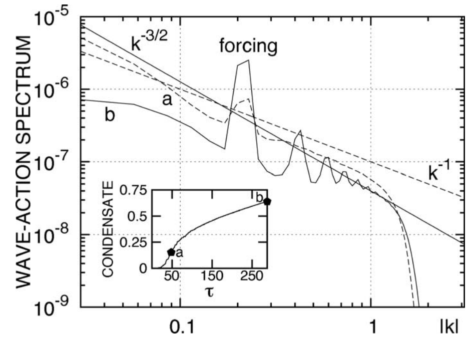

FIG. 1. Spectrum $n^{1D}(k,t)$ at different stages of the evolution. In this simulation the dissipation is only hyperviscosity. Slopes k^{-1} and $k^{-3/2}$ are also plotted. Inset: $c_0=|\hat{\psi}(0,\tau)|$ as a function of nondimensional time $\tau=t/T_f$, where T_f is the linear characteristic period of the forced scale.

The initial condition is $\psi(\mathbf{k},t=0)=0$; thanks to the random forcing term, mass and energy start growing. If GPE is integrated with a forcing at low wave numbers and a dissipation at only high wave numbers, it will never reach a steady solution. This is because it admits also the inverse cascade of wave-action which will start feeding wave number $\mathbf{k}=0$ and its close vicinity, building a strong condensate $c_0=|\hat{\psi}(0,t)|$ that changes the type of interactions from four to three-wave resonances [5]. This become clear by looking at Fig. 1 where we show the spectrum at two stages. In the first one, (a) the spectrum exhibits at wave numbers larger than forcing a power low close to k^{-1}, consistently with the WWT prediction. As the simulation evolves, the condensate grows and the spectrum starts to deviate from the pure -1 scaling: a set of well-defined peaks appears in the spectrum, curve (b). These peaks are probably the result of three-wave interactions similar to ones previously observed for a three-wave system in [16] and could be viewed, at first order, as resonances of the forcing term with the condensate fraction. In the inset we present the condensate as a function of time: the dots (a) and (b) correspond to the instant of times at which the two spectra are computed.

Regime where the condensate is prevalent was theoretically considered in [5,17]. In this case, the wave field in Eq. (1) can be decomposed as $\psi(\mathbf{x},t)=c(t)+\phi(\mathbf{x},t)$, where ϕ represents small fluctuations, i.e., $\phi\ll c$. The condensate part evolves as $c(t)=c_0e^{i\rho_0 t}$, with $\rho_0=|\hat{\psi}(0,t)|^2$. Linearizing the system the dispersion relation $\omega(k)=\rho_0\pm k\sqrt{k^2+2\rho_0}$ can be found, known in the literature as Bogoliubov dispersion. In Fig. 2 we present the numerical evaluation of the dispersion relation taken at the final stage of simulation, case (b); results show excellent agreement with the theoretical Bogoliubov curve. For very strong condensate, $\rho_0\gg k^2$, the Bogoliubov waves become acoustic and $\omega(k)=k\sqrt{2\rho_0}$ (in a reference frame rotating with the condensate speed ρ_0). Such acoustic WWT was considered in [18] and the respective KZ spectrum is $E^{1D}(k)\sim k^{-3/2}$ (very close to Kolmogorov $-5/3$). To make comparison with our results we have to take into account that, in this regime, $E^{1D}\sim\rho_0 n^{1D}$ [17]. By looking at the late time spectrum (b) in Fig. 1 we see that our results are not contradicting the Zakharov-Sagdeev prediction $k^{-3/2}$ for

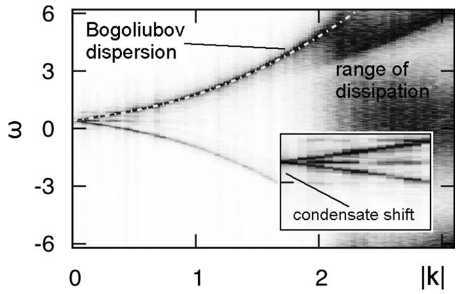

FIG. 2. Dispersion relation in the simulation with only hyperviscosity evaluated at final time, see Fig. 1 case (b). The Bogoliubov dispersion curve (only positive branch) is superposed with white dashed line. Inset: zoom in the zone of low ks to observe the condensate (horizontal branch).

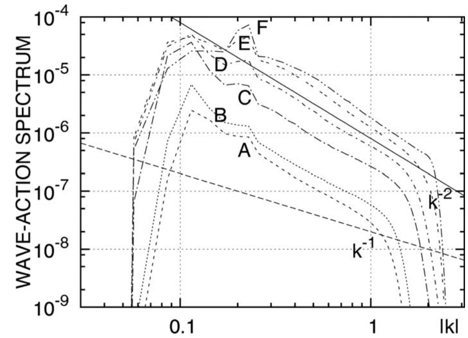

FIG. 4. Wave-action $n^{1D}(k,t)$ spectrum at final stage of simulation with hypoviscosity for different forcing coefficient: (a) f_0=0.05, (b) f_0=0.1, (c) f_0=0.5, (d) f_0=1.0, (e) f_0=2, and (f) f_0=3. A k^{-1} and k^{-2} slopes are also plotted.

the three-wave acoustic turbulence, although the spectrum reveals a long transient range with peaks, and the resulting scaling range is too tiny to make any decisive conclusion about the slope. This is very similar to the picture observed in [16] who computed weak acoustic turbulence. Long transient could be explained by low dispersion of the acoustic-like waves and to make a more definite conclusion about the scaling one has to increase the inertial range.

Note that the condensate keeps growing in the above simulation. In order to avoid such growth, a dissipation can be included at wave numbers lower that the ones corresponding to forcing. Different options are available. First, we will use a friction-type dissipation which takes the form, in Fourier space, $\hat{\mathcal{D}}=i\mu\theta(k^*-k)\hat{\psi}$, where θ is the Heaviside step function, $k^*=9\Delta k$ corresponds to lowest wave number forced, and μ is a friction coefficient which has been set to $\mu=1\times10^{-4}$. We present our stationary state solution in Fig. 3. The resulting spectral slope is consistent with the prediction of the WWT theory. The growth of the condensate is now stopped by friction, as shown in the inset, and transition from the four-wave to a three-wave regime is prevented.

Another common way of damping the low wave numbers consists, in analogy to what is done at high wave numbers, in including a *hypoviscosity* term $\mathcal{D}=i\nu_l(\nabla^{-2})^m\psi$ in Eq. (1) and suppressing the condensate in Fourier space (mode $\mathbf{k}=0$). In our simulations, we have chosen $\nu_l=1\times10^{-18}$ and $m=8$. In Fig. 4 we show the stationary states achieved with this new damping term for different forcing coefficient f_0. The observed spectrum is clearly much steeper than the WWT prediction, and it is reasonably fitted by a power law k^{-2} for forcing f_0 in a wide range (two orders of magnitude). It seems that the direct energy cascade is strongly influenced by the accumulation of wave-action at wave numbers below the forcing. In other words, a sharp dissipative term at low wave numbers can cause an *infrared bottleneck effect*. Similar behavior (steeper spectrum) has been observed recently in numerical simulations for water waves [19].

To understand these results we try to catch the level of nonlinearity in the system by considering the ratio $\eta=H_{NL}/H_{LIN}$. Note that integral quantities are not always relevant because we are interested at η in the inertial range and both energies may be strongly influenced by what happens, for example, in the forcing or in the low wave number region. In the case where WWT prediction are confirmed (Fig. 3), $\eta\approx1.06$; apparently WWT condition is not valid in this case but probably most of the nonlinear energy, in Fourier space, is stacked at low wave numbers and so, in the inertial range, the nonlinearity remains weak. It is instructive to look now at η in simulations with hypoviscosity that give the k^{-2} slope. As we can see in Fig. 5, even by increasing the forcing coefficient f_0 by two order of magnitude, the ratio η remains of order one. In those cases it is reasonable to think that the infrared bottleneck accumulation lead to the growth

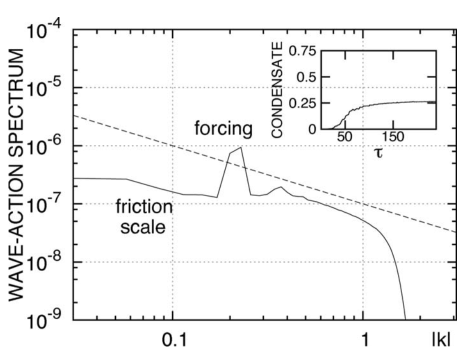

FIG. 3. Spectrum $n^{1D}(k,t)$ at final stage of simulation in the presence of the friction term. The k^{-1} prediction of WWT is also shown. Inset: $c_0=|\hat{\psi}(0,\tau)|$ as a function of non-dimensional time $\tau=t/T_f$, where T_f is the linear characteristic period of the forced scale.

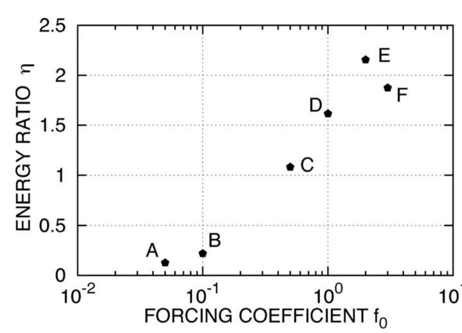

FIG. 5. Energy ratio $\eta=H_{NL}/H_{LIN}$ evaluated at steady nonequilibrium state with the hypoviscous dissipation for different forcing coefficient f_0 (see Fig. 4 and its label).

of the nonlinear terms until they become comparable to the linear ones in the inertial range.

This observations lead to a "critical balance" (CB) conjecture that the systems saturates in a state where the linear and the nonlinear time scales are balanced on a *scale-by-scale* basis. The name CB is borrowed from MHD turbulence were it was originally proposed in [20]. Even though not called by this name, CB-like ideas were put forward in the past for several other physical wave systems, notably the water surface gravity waves where the CB condition leads to the famous Phillips spectrum. Indeed, in the Phillips spectrum the wave steepness is saturated by the wave breaking process when a fluid particle cannot stay attached to the water surface because its downward acceleration becomes equal to the gravity constant, which occurs when the nonlinear time scale becomes of order of the linear one.

Let us estimate the CB spectrum in the GPE model. Equating the linear and the nonlinear terms for Eq. (1) written in Fourier space at each k, we have

$$k^2|\hat{\psi}_k| \sim |\hat{\psi}_k|^3 k^6, \tag{4}$$

where we replaced each of $\mathbf{k}$ integration in the nonlinear term by multiplication by k^3 thereby assuming that the nonlinear interaction is local in the k space and that the wave numbers of similar sizes are correlated. Then Eq. (4) gives for the 1D wave-action spectrum

$$n^{1\mathrm{D}}(k) = 4\pi k^2 n(k) = 4\pi k^2 (L/2\pi)^3 |\hat{\psi}_k|^2 \sim k^{-2}. \tag{5}$$

The k^{-2} prediction in Eq. (5) is consistent with our numerical simulations of GPE turbulence with hypoviscous dissipation shown in Fig. 4.

Why CB state forms in GPE turbulence with hypoviscous dissipation (and possibly for another kind of low-$\mathbf{k}$ dissipation which is sharp enough to lead to the infrared bottleneck)? If the low-$\mathbf{k}$ range is overdissipated by strong hypoviscosity, the inverse cascade tendency tends to accumulate the spectrum at low $\mathbf{k}$s until the critical balance is reached and the spectrum is saturated. When the size of the nonlinear term, locally in Fourier space, becomes of the same order as the linear, which is the CB condition, the inverse cascade is arrested and the further (infrared) bottleneck accumulation is halted. This is because the Fjørtoft's argument about the double-cascade behavior applies only when both the cascading invariants are quadratic in the wave amplitude, which is only true for GPE if $H_{LIN} \gg H_{NL}$. One could also qualitatively view this as a set of nonlinear coherent structures, in this case solitons or/and vortices, whose amplitude is limited by the linear dispersion: stronger solitons would break into the weaker ones and incoherent waves.

Concluding, we have performed numerical simulations of the 3D GPE with forcing and dissipation. The direct energy cascade range is strongly influenced by the second conserved quantity, the wave-action N, which has an inverse cascade tendency; results depend on how the low ks are damped. We have observed three different types of universal behavior roughly corresponding to situations where the largest scales are either nondissipative or damped by an efficient (e.g., friction-type) dissipation or damped by an inefficient (e.g., hypoviscosity) dissipation. In the first case turbulence is not steady: initial direct energy cascade, with a spectrum in good agreement with predictions of the WWT theory, is followed by condensation at the largest scales and a transition from a four to a three-wave interactions with a clearly Bogoliubov dispersion relation characteristic to this regime. In the second case, the wave-action cascade is effectively absorbed so that there is no condensation, and we observe a steady-state spectrum which is in good agreement with the WWT theory. In the third case, the dissipation is not so efficient and an infrared bottleneck forms in the spectrum. In this regime we observe a robust steady-state spectrum which could be explained by a phenomenological "critical balance" proposition where the linear and nonlinear time scales are balanced on the scale-by-scale basis.

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