Introduction to Topology   Exercises I   October 2012

There are more exercises here than you will most likely want to do at once. This means there should be a good number remaining for revision next spring. As usual, the exercises in Section A are reasonably straightforward, those in Section B a bit less so, and those in Section C a bit harder still. Some of the exercises ask for proofs of results which are proved in the lectures/lecture notes. This is not necessarily a mistake. It is the opportunity for you to reconstruct a proof from the seed of the idea you have retained from having heard it before.

Section A

1. Find homeomorphisms

(a) $(0,1) \to \mathbb{R}$.

(b) $(0,1) \to (0,\infty)$

(c) $\mathbb{R} \times S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$

(d) $[0,1] \times S^1 \to \{(x,y) \in \mathbb{R} : 1 \leq \|(x,y)\| \leq 2\}$.

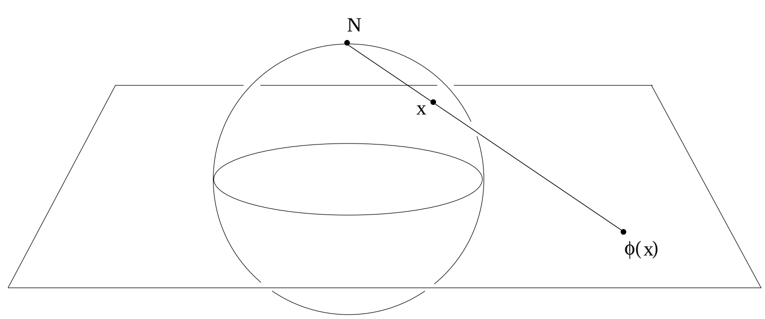
2. Show that the following pairs of spaces are *not* homeomorphic:

(a) S^1 and $[0,1]$ (Hint: consider what happens if you remove a single point from each.)

(b) The letter "X" (four line-segments sharing a common end-point) and the letter "Y" (three line segments sharing a common end-point).

(c) S^2 and the torus.

3. Find an explicit formula for the stereographic projection $\phi : S^2 \setminus \{0\} \to \mathbb{R}^2 \times \{0\}$, and for its inverse.



Conclude that stereographic projection is a homeomorphism, and that S^2 is a 2-dimensional manifold.

4. A topological space X is *path-connected* if for every pair of points $x_0, x_1 \in X$, there is a path in X which joins them – that is, there is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Show that if $U \subset \mathbb{R}^2$ is open and path-connected, and if $p \in U$, then $U \setminus \{p\}$ is still path-connected. Hint: there is a very simple pictorial argument.

5. Recall the definition of *connectedness*: a topological space X is connected if it is not possible to express X as the union of two disjoint non-empty open subsets. Remind yourself of the proof that every interval in $\mathbb{R}$ is connected (Metric Spaces Lecture Notes, Lemma 4.7).

6. Prove the following statement about quotient topologies: Suppose Q is the quotient of a topological space X by an equivalence relation $\sim$, with the quotient topology. Suppose that $f : X \to Y$ has the property that whenever $x_1 \sim x_2$ then $f(x_1) = f(x_2)$. The the map $\bar{f} : Q \to Y$ which is (well-) defined by setting $\bar{f}([x]) = f(x)$, is continuous.

This is quite straightforward – just put together the definition of continuity with the definition of the quotient topology.

7. Show that if X is a compact topological space and $\sim$ is an equivalence relation, then $X/\sim$ is also compact.

8. Suppose that $f : X \to Y$ is a map. Define a relation $\sim$ on X by setting $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Show that $\sim$ is an equivalence relation – it is the "equivalence relation induced by f", and that if f is surjective then there is a natural bijection $X/\sim \; \to Y$. It is useful in many cases to identify $X/\sim$ and Y by means of this bijection.

Section B

9. Show that if X is path-connected then it is connected. Hint: Suppose that X is not connected: by definition, this means there exist disjoint, non-empty, open subsets of X, X_0 and X_1, such that $X = X_0 \cup X_1$. Pick $x_0 \in X_0$ and $x_1 \in X_1$, and a path $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$, $\gamma(1) = x_1$. Let $I_0 = \gamma^{-1}(X_0)$ and $I_1 = \gamma^{-1}(X_1)$.

10. Suppose that $U \subset \mathbb{R}^n$ is open. Show that for each $x_0 \in U$, the set of points in U to which x_0 can be connected by a path in U is open in U. Hint: recall the definition of open set and draw a picture.

11. Recall that the *n-dimensional real projective space* $\mathbb{R}\mathbb{P}^n$ is the quotient of S^n by the equivalence relation identifying antipodal points. This exercise introduces another way of obtaining $\mathbb{R}\mathbb{P}^n$, as the space of lines through 0 in $\mathbb{R}^{n+1}$.

We denote by q the quotient map $S^n \to \mathbb{R}\mathbb{P}^n$ sending $x \in S^n$ to its equivalence class.

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by $x \sim y$ if there exists $\lambda \neq 0$ in $\mathbb{R}$ such that $\lambda x = y$.

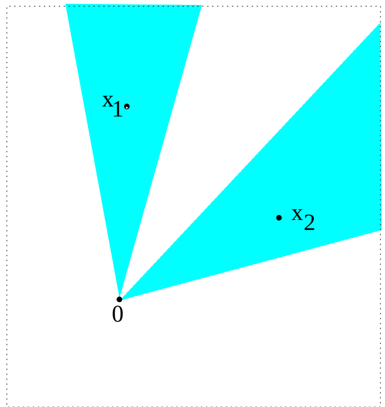
(a) What are the equivalence classes?

(b) Let Q be the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$, and let $p : \mathbb{R}^{n+1} \setminus \{0\} \to Q$ be the quotient map. Consider the diagram

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[d, "q"'] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "p"] \\ \mathbb{R}\mathbb{P}^n \arrow[r, dashed, "\bar{i}"'] & Q \end{tikzcd}$$

where i is inclusion. Prove that there *is* a continuous map $\bar{i}$ as indicated, making the diagram commutative (i.e. such that $p \circ i = \bar{i} \circ q$). This uses Exercise 6, of course, with $p \circ i$ in place of f.

(c) Show that Q is Hausdorff. Suggestion: this diagram of $\mathbb{R}^{n+1} \setminus \{0\}$ shows two points with different images in Q.



(d) Prove that $\bar{i}$ is a homeomorphism. Hint: Use the following
Lemma If A and B are topological spaces, with A compact and B Hausdorff, and if $f : A \to B$ is continuous and bijective, then f is a homeomorphism.

12. Prove the Lemma in the last exercise.

13. Suppose, in Exercise 8, that X and Y are topological spaces, and let $X/\sim$ be the quotient of X by the equivalence relation induced by f. Show that if X is compact, Y is Hausdorff and f is surjective then f passes to the quotient to give a homeomorphism $X/\sim\ \to Y$. Which theorem of group theory does this resemble?

14. Give an example of a continuous bijection which is not a homeomorphism.

Section C

15. Suppose that $U \subset \mathbb{R}^n$ is open and connected. Show that it is path connected. Hint: by the previous exercise, for any $x_0 \in U$ the set of points to which x_0 can be connected by a path in U is open. Show that its complement is also open.

16. Show that $\mathbb{R}\mathbb{P}^n$ is an n-dimensional manifold.

17. Suppose that X is the lower hemisphere $\{(x, y, z) \in S^2 : z \leq 0\}$ and Y the upper hemisphere $\{(x, y, z) \in S^2 : z \geq 0\}$. If we glue the two together along their common boundary, by declaring that $(x_1, x_2, 0) \in X$ is equivalent to $(y_1, y_2, 0) \in Y$ if $x_1 = y_1$ and $x_2 = y_2$ and then use the quotient topology, then we get a space, Z, that ought to be homeomorphic to S^2, and indeed it is. Prove this. Suggestion: find a continuous bijection from the glued space Z to S^2, and use the previous exercise and the following statement.

18. Denote by $G_{k,n}$ the set of k-dimensional vector subspaces (“k-planes”) of $\mathbb{R}^n$ (the “G” stands for “Grassmannian”). There is nice way to give it a topology as a quotient of the group $\mathrm{Gl}_n(\mathbb{R})$ of invertible $n \times n$ matrices.

Pick any k-plane L_0 in $\mathbb{R}^n$. If $A \in \mathrm{Gl}_n(\mathbb{R})$ then A sends L_0 to some other k-plane, which we will denote by $A \cdot L_0$. Consider the map $\mathrm{Gl}(n, \mathbb{R}) \to G_{k,n}$ defined by $f_0(A) = A \cdot L$; clearly f_0 is surjective. We give $G_{k,n}$ the quotient topology defined by the equivalence relation induced by f_0.

To do:

(a) Show that if we start instead with another k-plane L_1 and define $f_1 : \mathrm{Gl}_n(\mathbb{R}) \to G_{k,n}$ by $f_1(A) = A \cdot L_1$, then topology induced on $G_{k,n}$ by f_1 is the same as the topology induced by f_0.

(b) Show that if $k = 1$ then the space obtained is homeomorphic to $\mathbb{R}\,\mathbb{P}^{n-1}$.

(c) Show that with this quotient topology, $G_{k,n}$ is compact. Hint: find a compact subgroup H of $\mathrm{Gl}_n(\mathbb{R})$ such that the composite $H \hookrightarrow \mathrm{Gl}_n(\mathbb{R}) \xrightarrow{f_0} G_{k,n}$ is surjective.