Pictures of stable perturbations of codimension 1 singularities of maps from 2-space to 3-space

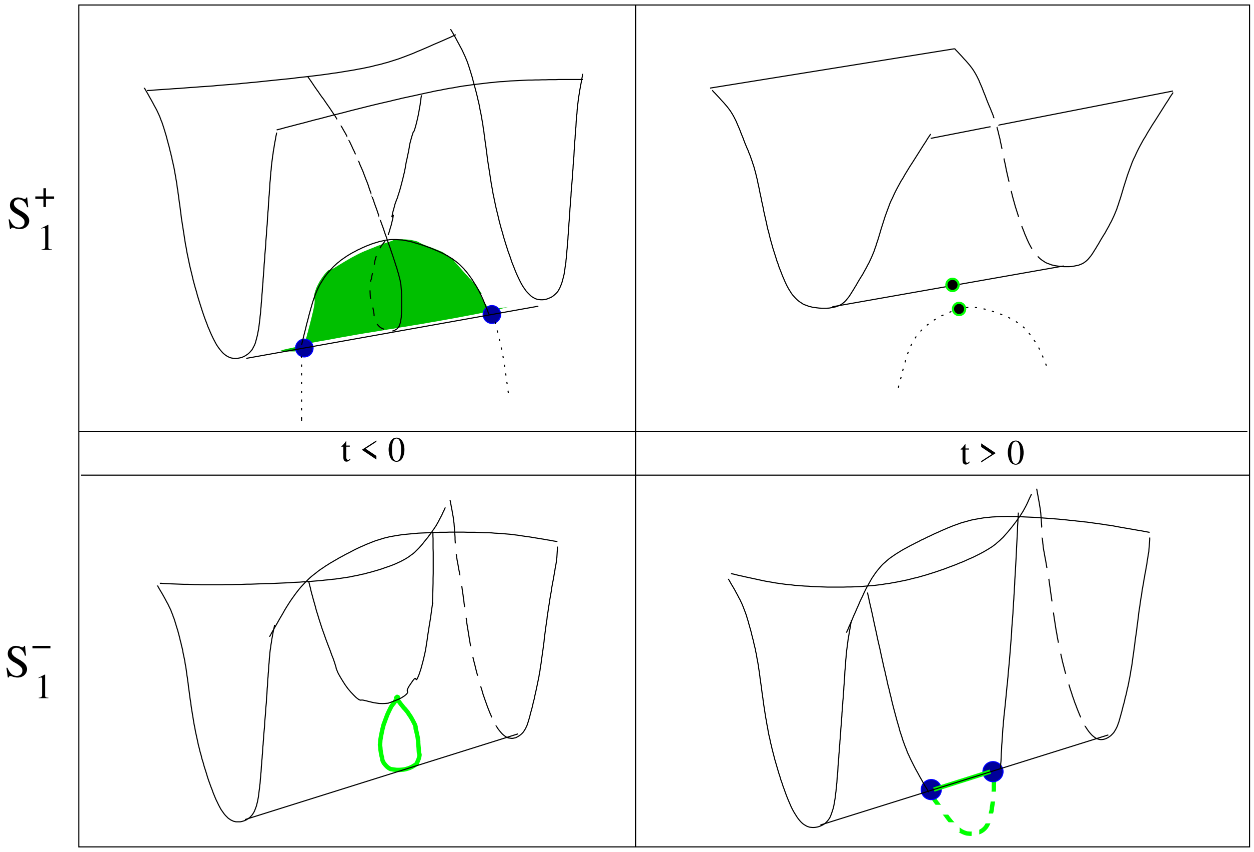
Theorem 0.1. *The algebraic closure of each real image is homotopy-equivalent to a k-sphere (the vanishing cycle) for some k between 0 and 2.*

Denote the value of k for $t < 0$ by k_-, and for $t > 0$ by k_+.

Theorem 0.2. *The complex homological monodromy is multiplication by $(-1)^{k_+ - k_-}$.*

Where $k = 2$, the the real perturbation is called *good*. In this case the inclusion of the algebraic closure of the real image in the complex image is a homotopy-equivalence. In this case it seems that the inclusion of the real image in its algebraic closure is a homotopy equivalence also. It is not a homotopy equivalence, in general, in the other cases.

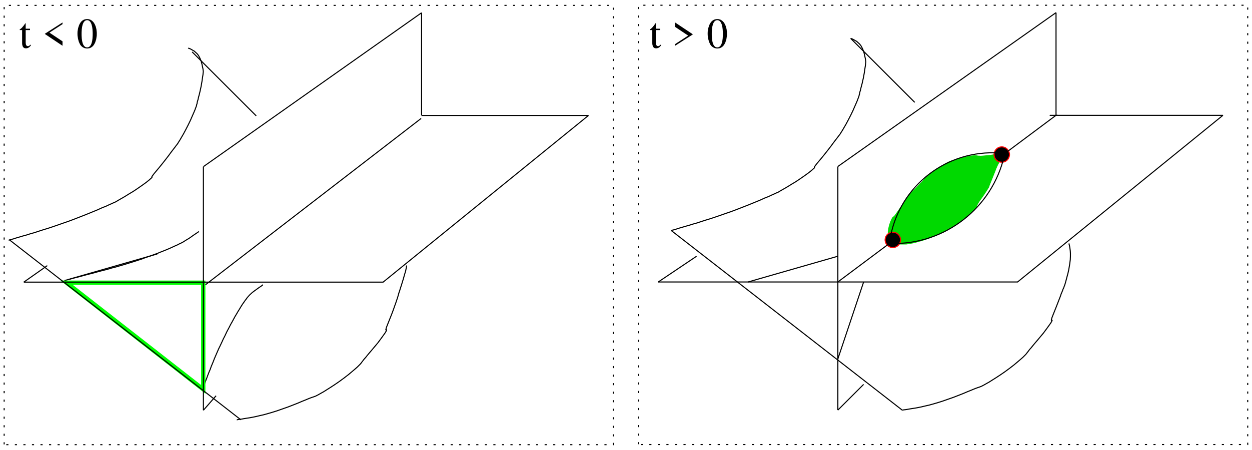
In each picture, the vanishing cycle is shown in green, and triple-points and Whitney umbrella points are shown in red and blue respectively. Dotted arcs indicate the complement of the real image in its algebraic closure. In stable perturbations of singularities of maps from surfaces to 3-space, these arcs always begin and end at Whitney umbrellas.



Good and bad real pictures of images of stable perturbations of $S_1^{\pm}$,

$$f_t(x, y) = (x, y^2, y^3 \pm x^2 y + ty).$$

The dimensions of the vanishing cycles are: 2 for S_1^+, $t < 0$; 0 for S_1^+, $t > 0$; 1 for S_1^-, $t < 0$; 1 for S_1^-, $t > 0$. For $S_1^{\pm}$, $t > 0$, the vanishing cycle necessarily leaves the real image and passes through the algebraic closure. On the picture for $S_1^+, t < 0$, an extra curve has been drawn in the yz plane, to indicate the shape.



Real pictures of birth of two triple points,

$$\begin{cases} (x_1, y_1) \mapsto (x_1, y_1, 0) \\ (x_2, y_2) \mapsto (0, x_2, y_2) \\ (x_3, y_3) \mapsto (x_3, y_3 - x_3^2 + t, -y_3 - x_3^2 + t). \end{cases}$$

The vanishing cycles are 1-dimensional for $t < 0$ and 2-dimensional for $t > 0$. The picture for $t > 0$ is good. Note two triple points for $t > 0$.

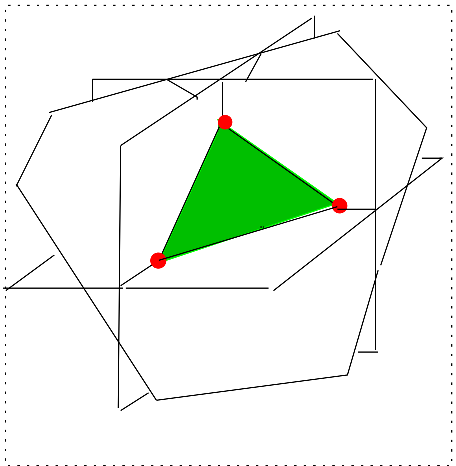
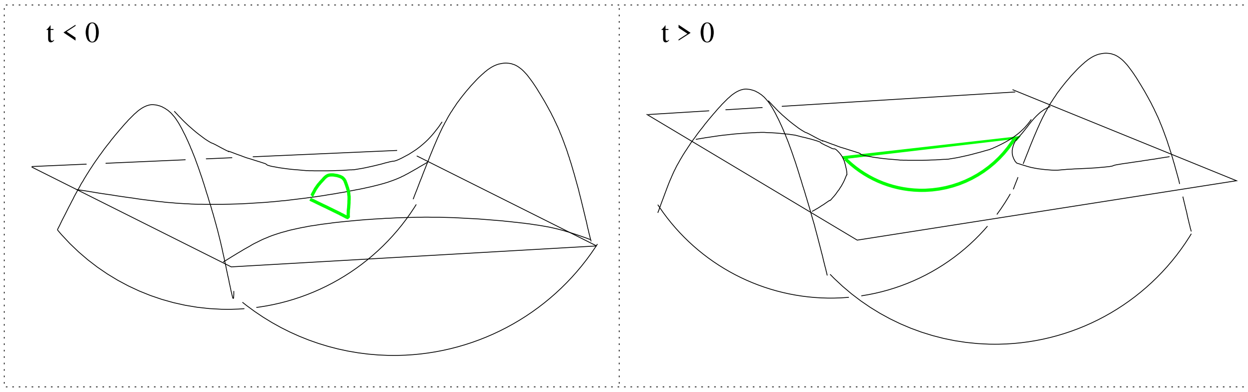


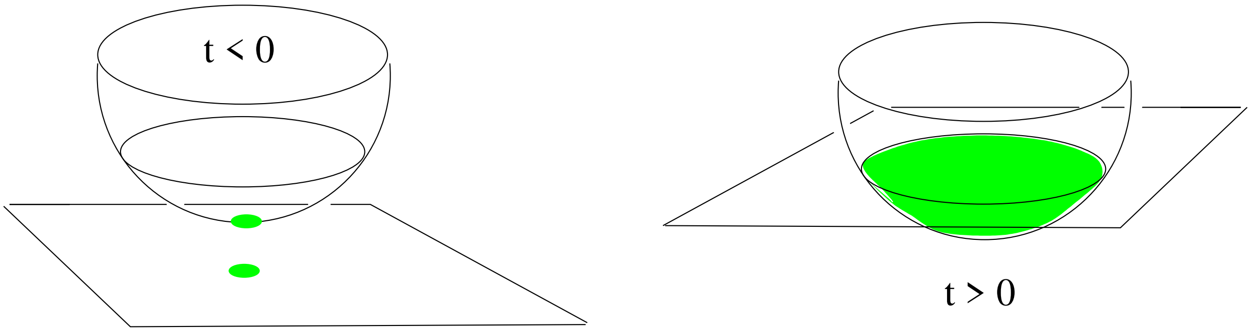
Image of stable perturbation of quadruple point.

Only one picture is required — the real images for $t < 0$ and $t > 0$ are diffeomorphic. There are 4 triple points, with 3 visible.



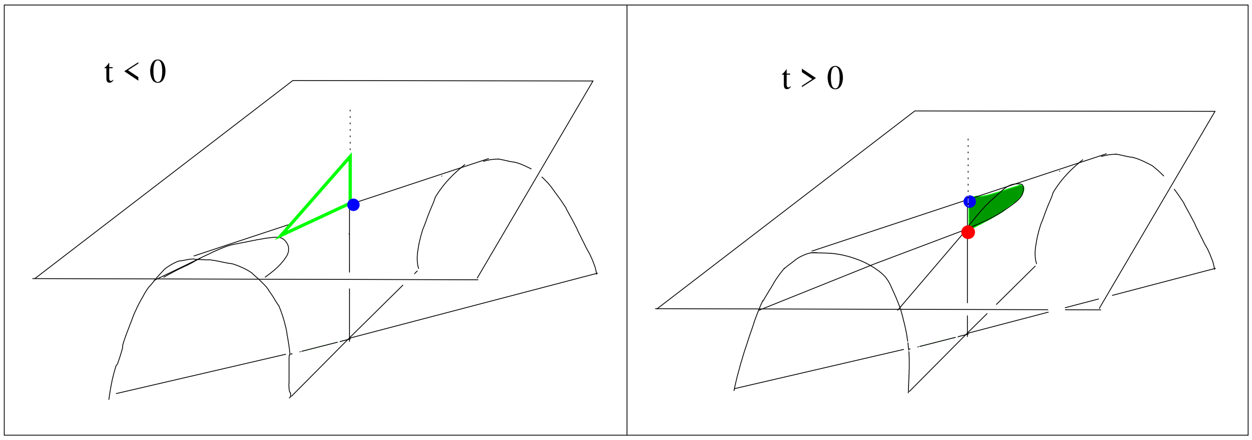
Images of stable perturbations of bi-germ consisting of two immersed sheets with tangential contact with Morse index 1,

$$\begin{cases}(x_1, y_1) \mapsto (x_1, y_1, t)\\(x_2, y_2) \mapsto (x_2, y_2, x_2^2 - y_2^2).\end{cases}$$



Images of stable perturbations of bi-germ consisting of two immersed sheets with tangential contact with Morse index 0,

$$\begin{cases}(x_1, y_1) \mapsto (x_1, y_1, t)\\(x_2, y_2) \mapsto (x_2, y_2, x_2^2 + y_2^2).\end{cases}$$



Images of stable perturbations of Whitney umbrella+immersed plane.

Note umbrella points (marked in blue), and triple point (marked in red) in the picture for $t > 0$, which is the good real picture. In the picture fo $t < 0$, the vanishing cycle is forced to meet the handle of the umbrella, which is not part of the real image.