Stable mappings and logarithmic relative symplectic forms

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Abstract

Let D be the image of a stable, weighted homogenenous map $f : \mathbb{C}^n \to \mathbb{C}^{n+1}$, with $\dim_{\mathbb{C}}\mathrm{Ker}\ (df_0) = 1$, and which is not a trivial deformation of a lower-dimensional map. By proving a variant of the Buchsbaum- Eisenbud structure theorem for grade 3 Gorenstein quotients, we show the existence of a form $\omega \in \Omega^2(\log D)$ which restricts to a non-degenerate holomorphic 2-form on the Milnor fibres of D; experiments with the computer algebra programme *Macaulay* suggest this restriction is closed, and is thus a holomorphic symplectic form.

1 Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}^{n+1}$ be a weighted homogeneous stable mapping such that the kernel of df_0 has rank 1, and with the additional property of minimality: f is not left-right equivalent to a mapping of the form $g \times \mathrm{id}_{\mathbb{C}} : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$. This forces n to be even and indeed, up to right-left equivalence, there is one such mapping $\mathbb{C}^{2k} \to \mathbb{C}^{2k+1}$, for each positive integer k. For example, when $k = 1$, this map parametrises the pinch-point (cross-cap, or Whitney umbrella). Let D be the image of f and let $\Omega^p(\log D)$ be the sheaf of germs of differential forms with logarithmic poles along D (see [20]). Let $h \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ be a local defining equation for D and let $\mathrm{Der}(\log h)_0$ be the $\mathcal{O}_{\mathbb{C}^{n+1},0}$-module of germs at 0 of holomorphic vector fields on $\mathbb{C}^{n+1}$ annihilating h, and thus tangent to all of the level sets of h. We prove:

Theorem 1.1 *There exists an algebraic 2-form $\omega_r \in \Omega^2(\log D)_0$ inducing a perfect pairing $\mathrm{Der}(\log h)_0 \times \mathrm{Der}(\log h)_0 \to \mathcal{O}_{\mathbb{C}^{n+1},0}$.*

If h is weighted homogeneous (as f is weighted homogenenous such an equation exists), then ω_r can be defined globally, and induces a global perfect pairing of $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules, $\mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \to \mathcal{O}_{\mathbb{C}^{n+1}}$.

Let $X_t = h^{-1}(t)$. For any $t \neq 0$ and any $x \in X_t$, the tangent space $T_x X_t$ is equal to the set of values at x of germs $\xi \in \mathrm{Der}(\log h)_x$, and moreover $\mathrm{Der}(\log h)$ is free at x. It therefore follows from Theorem 1.1 that $\omega_r|_{X_t}$ is a non-degenerate 2-form.

Calculations with the computer algebra programme Macaulay ([3]) strongly support the following conjecture:

Conjecture 1.2 *Let h be a weighted homogenenous defining equation for D. Then there is an algebraic 2-form $\omega_r \in \Omega^2(\log D)$, inducing the perfect pairing of Theorem 1.1, whose restriction to each level set $X_t = h^{-1}(t), t \neq 0$ is closed. That is, $\omega_r|_{X_t}$ is an algebraic symplectic form.*

Closedness is obvious when $k = 1$, and can be proved by an argument involving the dimension of certain graded components of $\Omega^3(\log D)$ in case $k = 2$. Using Macaulay we have checked the conjecture for $k = 3$. In the latter case, closedness amounts to the equality of two 7×7 skew-symmetric matrices each of whose entries is a polynomial in 7 variables. Each of these polynomials contains approximately 20 monomials.

A Macaulay script which finds the 2-form ω_r inducing the perfect pairing on $\mathrm{Der}(\log h)$ in the case $k = 3$ can be found at the Web site `http://www.shef.ac.uk/~ms/staff/holland/`.

There are other symplectic forms in some way associated with these maps:

1. If one defines R_f to be the ring $\{g \in \mathcal{O}_{\mathbb{C}^n} : dg \in \mathcal{O}_{\mathbb{C}^n} \cdot f^{-1}\Omega^1_{\mathbb{C}^{n+1}}\}$ (i.e.the kernel of the natural derivation $\mathcal{O}_{\mathbb{C}^n} \to \Omega_{\mathbb{C}^n/\mathbb{C}^{n+1}}$) then $Y = \mathrm{Spec}\, R_f$ is a singular Lagrangian subvariety (the so-called open Whitney umbrella) of a certain complex cotangent bundle T^*M ([8],[10],[11]); since $f^{-1}\mathcal{O}_{\mathbb{C}^{n+1}} \subseteq R_f$, it follows that f factors through the parametrisation $\mathbb{C}^n \to Y$, and moreover that vector fields tangent to $D = f(\mathbb{C}^n)$ lift to vector fields tangent to Y. Thus there is a (non-unique) $\mathcal{O}_{\mathbb{C}^{n+1}}$-linear pairing $\mathrm{Der}(\log D) \times \mathrm{Der}(\log D) \to I_Y$, where I_Y is the ideal defining Y in T^*M.

2. If (in the notation of Section 2) one forgets the last coordinate of the map f, then one obtains a versal unfolding of the hypersurface singularity of type A_k. Adding new variables to the domain, and their squares to the nth component of this deleted map, one obtains a versal unfolding of a positive-dimensional singularity of type A_k. The base of this unfolding inherits a

closed 2-form from the intersection form on the Minor fibre, pulled-back by the period mapping; this form is non-degenerate in case k is even ([7], AGV III.15).

We have not succeeded in explaining the behaviour described in this paper by means of these associated symplectic structures.

Our results show that the Milnor fibration of the image of a stable map of corank 1 carries a non-degenerate relative 2-form; conjecturally, for some choice of defining equation of the image (apparently the weighted homogenenous equation), this form is relatively closed. Our argument also shows that this is not the case for stable maps of corank greater than 1 (see Proposition 3.4 below): here there exists no non-degenerate relative 2-form. The special behaviour in the corank 1 case is explained by a simple fact: it is just in this case that the ramification algebra $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ (where $\mathcal{R}_f$ is the ideal generated by the maximal minors of the jacobian matrix of f) is Gorenstein. In fact the perfect pairing of Theorem 1.1 is obtained by constructing a resolution of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ over $\mathcal{O}_{\mathbb{C}^{n+1},0}$, and the skew-symmetry of the pairing follows from the skew-symmetry of the matrix in the middle of this resolution (*cf* Proposition 3.6); this itself is due to the fact that $\mathcal{O}_{\mathbb{C}^n,0}/R_f$ is Gorenstein of grade 3 over $\mathcal{O}_{\mathbb{C}^{n+1},0}$. Were $\mathcal{O}_{\mathbb{C}^n,0}/R_f$ a *quotient* of $\mathcal{O}_{\mathbb{C}^{n+1},0}$, skew symmetry would follow from the Buchsbaum-Eisenbud structure theorem ([5]). As it is, our argument amounts to a generalisation of this theorem [1]. To summarise: the existence of a (conjecturally) symplectic structure on the Milnor fibration of the image in the corank 1 case and not in the case of higher corank, reflects the different structures of the ramification algebra in these two cases.

Given a variety D with smooth normalisation, it is in general a difficult problem to determine the differential properties of the normalisation map directly from the geometry of D, that is, without computing the normalisation. Theorem 1.1 goes some way towards doing this; a proof of the conjecture 1.2 would of course do so in a much more satisfactory way.

2 Stable maps of corank 1

Here we recall some basic facts we shall need in the sequel, and establish our notation.

[1] David Eisenbud has informed me that he and Bernd Ulrich have recently proved precisely such a generalisation, in a less restrictive context than the one considered here.

The map-germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is *stable* if every 1-parameter level-preserving deformation $F : \mathbb{C}^n \times \mathbb{C}, (0,0) \to \mathbb{C}^p \times \mathbb{C}, (0,0)$ of f is trivial, in the sense that there exist level-preserving germs of analytic automorphisms Φ of $\mathbb{C}^n \times \mathbb{C}, 0$ and Ψ of $\mathbb{C}^p \times \mathbb{C}, 0$ such that $F = \Psi \circ (f \times \mathrm{id}_{\mathbb{C}}) \circ \Phi$. If F is a 1-parameter deformation of f, its derivative with respect to the parameter t, evaluated when $t = 0$, is a germ of vector field along f, that is, a germ of map $\hat{f} : \mathbb{C}^n, 0 \to T\mathbb{C}^p$ such that $\pi \circ \hat{f} = f$ (where $\pi : T\mathbb{C}^p \to \mathbb{C}^p$ is the bundle projection). The set of all such infinitesimal deformations of f is denoted $\theta(f)$. There are natural maps $tf : \theta_{\mathbb{C}^n} \to \theta(f)$, obtained by composing on the left with the derivative of f, and $\omega f : \theta_{\mathbb{C}^p} \to \theta(f)$, obtained by composing on the right with f.

The sum $tf(\theta_{\mathbb{C}^n}) + \omega f(\theta_{\mathbb{C}^p})$ is denoted by $T\mathcal{A}_e f$ (it is the "$\mathcal{A}_e$ tangent space of f"), and the dimension of the quotient $\theta(f)/T\mathcal{A}_e f$ as $\mathbb{C}$-vector space is called the $\mathcal{A}_e$-codimension of f. When $T\mathcal{A}_e f = \theta(f)$ then f is said to be infinitesimally stable. It was shown by John Mather in [13] that infinitesimal stability implies stability (the converse implication being obvious).

It is an easy consequence of Mather's work in [14], of which a good account may be found in [12, Chapters XVI and XVII], that if $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ is a minimal stable map of corank 1 then $n = 2k$, for some k, and that in appropriate coordinates f may be written in the form

$$f(u_1, \ldots, u_{k-1}, v_1, \ldots, v_k, x) = (u, v, x^{k+1} + \sum_{i=1}^{k-1} u_i x^{k-i}, \sum_{i=1}^{k} v_i x^{k-i+1}).$$

We denote the coordinates in the codomain by $(U_1, \ldots, U_{k-1}, V_1, \ldots, V_k, Z, T)$.

In particular, there is a globally defined mapping $\mathbb{C}^n \to \mathbb{C}^{n+1}$, also denoted by f, which is equivariant for the 2-torus (i.e. $\mathbb{C}^* \times \mathbb{C}^*$) actions on domain and codomain, defined as follows:

$$\begin{aligned}
\lambda \cdot (u, v, x) &= (\lambda^2 u_1, \ldots, \lambda^k u_{k-1}, \lambda v_1, \ldots, \lambda^k v_k, \lambda x) \\
\lambda \cdot (U, V, Z, T) &= (\lambda^2 U_1, \ldots, \lambda^k U_{k-1}, \lambda V_1, \ldots, \lambda^k V_k, \lambda^{k+1} Z, \lambda^{k+1} T) \\
\mu \cdot (u, v, x) &= (\mu^2 u_1, \ldots, \mu^k u_{k-1}, \mu^{-k} v_1, \ldots, \mu^{-1} v_k, \mu x) \\
\mu \cdot (U, V, Z, T) &= (\mu^2 U_1, \ldots, \mu^k U_{k-1}, \mu^{-k} V_1, \ldots, \mu^{-1} V_k, \mu^{k+1} Z, T).
\end{aligned}$$

We denote the infinitesimal generator of the first $\mathbb{C}^*$-action by χ_e—it will be clear from the context whether this refers to the vector field on $\mathbb{C}^n$ or on $\mathbb{C}^{n+1}$—and the infinitesimal generator of the second by χ_r. We will later consider various induced $\mathbb{C}^*$-actions, for example on $\mathrm{Der}(\log h)$. Whenever we speak of weights unadorned, the reader may safely assume that these are with respect to χ_e. Up to scalar multiples, there is a unique reduced equation h for the image D of f which is weighted homogeneous with respect to both systems of weights; it has weight $(k+1)^2$ with respect to χ_e (that is, $\chi_e \cdot h = (k+1)^2 h$) and weight 0 with respect to χ_r (see [17, Algorithm 2.2 and Proposition 3.1]).

In the introduction we discussed a globally defined weighted homogeneous map $\mathbb{C}^n \to \mathbb{C}^{n+1}$ which was stable. On the other hand, in this section, we began with a stable map-germ and observed that it arose from a global weighted homogeneous map. We can now reconcile these two points of view by showing that the map of this section is indeed everywhere stable.

Remark 2.1 *The globally defined mapping $\mathbb{C}^n \to \mathbb{C}^{n+1}$ given by f, above, is stable at all points.*

Proof Note that, by Mather's criterion for stability in terms of infinitesimal stability, this map is stable at the points outside the support of a certain coherent sheaf on $\mathbb{C}^{n+1}$. Since the germ of f at the origin is stable it follows that f is stable in an (analytic) neighbourhood of the origin. On the other hand, it is easy to see that if $x \in \mathbb{C}^{n+1}$ then $(T\mathcal{A}_e f)_x \cong (T\mathcal{A}_e f)_{\lambda \cdot x}$. Since the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ is good (that is, all the weights are positive) it follows that f is stable at every point. $\square$

3 Construction of the perfect pairing

Let $\mathrm{Der}(\log D)$ be the sheaf of germs at the origin of holomorphic vector fields on $\mathbb{C}^{n+1}$ which are tangent to D; in other words, $\mathrm{Der}(\log D) = \{\xi \in \theta_{\mathbb{C}^{n+1}} : \xi \cdot h \in (h)\}$ where h is a defining equation for D. Recall that $\mathrm{Der}(\log h)$ is the subsheaf of vector fields which annihilate h. It follows from the weighted homogeneity of f that $\mathrm{Der}(\log h)$ is a direct summand of $\mathrm{Der}(\log D)$. The argument is this: if h_0 is a weighted homogeneous defining equation for D and χ_e is the Euler vector field, then $\chi_e \cdot h_0 = h_0$. If $h = uh_0$ is any other defining equation of D then u is a unit, and so $u + \chi_e \cdot u$ is a unit and $u(u + \chi_e \cdot u)^{-1} \chi_e \cdot h = h$. Thus, for each defining equation h, there is a vector field χ_0 such that $\chi_0 \cdot h = h$. Now if $\xi \in \mathrm{Der}(\log D)$, $\xi = (\xi \cdot h)\chi_0 + (\xi - (\xi \cdot h)\chi_0)$; the second summand here evidently annihilates h, and so

$$\mathrm{Der}(\log D) = \mathcal{O}_{\mathbb{C}^{n+1}} \cdot \chi_0 \oplus \mathrm{Der}(\log h).$$

For almost all of the remainder of this section our focus will be exclusively local. Thus, we will work with a stable map-germ $\mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ of corank one. We will only return to the global perspective at the very end of the section when we finally prove Theorem 1.1. However, the reader will notice that although we have chosen to work in the category of $\mathcal{O}_{\mathbb{C}^{n+1},0}$-modules, we could equally well carry out the analysis in the category of graded $\mathcal{O}_{\text{alg}}(\mathbb{C}^{n+1})$-modules. In that scenario, one considers the graded module of globally defined algebraic vector fields in $\text{Der}(\log h)$. In fact, the proof of Theorem 1.1, at the end of the section, precisely relies upon uniting these two points of view. Of course, our computations with Macaulay in Section 5, also fall into the graded framework.

In order to simplify notation as much as possible we will generally use global notation for local objects; that is, we write $\mathcal{O}_{\mathbb{C}^{n+1}}$ in place of $\mathcal{O}_{\mathbb{C}^{n+1},0}$, $\text{Der}(\log h)$ in place of $\text{Der}(\log h)_0$ etc.

Lemma 3.1 *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ be a stable map-germ which is not an immersion. Then the $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules $\text{Der}(\log D)$ and $\text{Der}(\log h)$ have projective dimension 1.*

Proof Any vector field tangent to D can be lifted to a vector field on $\mathbb{C}^n$, and vice versa, so there is a short exact sequence

$$0 \longrightarrow \text{Der}(\log D) \longrightarrow \theta_{\mathbb{C}^{n+1}} \longrightarrow \theta(f)/tf(\theta_{\mathbb{C}^n}) \longrightarrow 0. \tag{1}$$

On the other hand, it is clear that $tf : \theta_{\mathbb{C}^n} \to \theta(f)$ is injective. Further, this map is not split, since f is not an immersion. Thus, $\text{pd}_{\mathcal{O}_{\mathbb{C}^n}} \theta(f)/tf(\theta_{\mathbb{C}^n}) = 1$. Since $\theta(f)/tf(\theta_{\mathbb{C}^n})$ has the same depth, considered as a $\mathcal{O}_{\mathbb{C}^{n+1}}$-module, or as an $\mathcal{O}_{\mathbb{C}^n}$-module, it follows that $\text{pd}_{\mathcal{O}_{\mathbb{C}^{n+1}}} \theta(f)/tf(\theta_{\mathbb{C}^n}) = 2$. Using (1) we obtain that $\text{pd}_{\mathcal{O}_{\mathbb{C}^{n+1}}} \text{Der}(\log D) = \text{pd}_{\mathcal{O}_{\mathbb{C}^{n+1}}} \theta(f)/tf(\theta_{\mathbb{C}^n}) - 1 = 1$, proving the statement for $\text{Der}(\log D)$. The statement for $\text{Der}(\log h)$ follows because $\text{Der}(\log h)$ is a direct summand of $\text{Der}(\log D)$ with free complementary summand. $\square$

Corollary 3.2 *For any stable map-germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ which is not an immersion we have*

1. $\text{pd}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^{n+1}}/J_h) = 3$ *and*
2. $\text{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\text{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \text{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^{n+1}}/J_h, \mathcal{O}_{\mathbb{C}^{n+1}})$

Proof The first statement follows immediately from Lemma 3.1 and the exact sequence $0 \longrightarrow \mathrm{Der}(\log h) \longrightarrow \theta_{\mathbb{C}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}/J_h \longrightarrow 0$. The second statement follows by dimension shifting. $\square$

Let $\mathcal{R}_f$ be the *ramification ideal* of f; that is, $\mathcal{R}_f$ is the ideal of $\mathcal{O}_{\mathbb{C}^n}$ generated by the $n \times n$ minors of the matrix of df.

Lemma 3.3 *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ be a stable map-germ which is not an immersion, and suppose that $D = Image(f)$ has a defining equation h_0 such that $h_0 \in m_{\mathbb{C}^{n+1},0} J_{h_0}$ (e.g. f is weighted homogenenous). Then for any defining equation h of D,*

1.
$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f};$$

2.
$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h)^*, \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f,$$

   (where $\mathrm{Der}(\log h)^* = \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}})$*);*

3. *if f has corank 1 then*
$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f.$$

Proof

Consider the short exact sequence

$$0 \longrightarrow \mathcal{F}_1/J_h \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}/J_h \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}/\mathcal{F}_1 \longrightarrow 0. \qquad (2)$$

Here, $\mathcal{F}_1 = \mathcal{F}_1^{\mathcal{O}_{\mathbb{C}^{n+1}}}(f_*(\mathcal{O}_{\mathbb{C}^n}))$ is the first Fitting ideal of $\mathcal{O}_{\mathbb{C}^n}$ as $\mathcal{O}_{\mathbb{C}^{n+1}}$-module. It is known (see, for example [17, Theorem 3.4]) that $\mathcal{F}_1$ is a determinantal ideal of height 2, and it follows from the long exact sequence of Ext arising from the short exact sequence (2) that the inclusion $\mathcal{F}_1/J_h \to \mathcal{O}_{\mathbb{C}^{n+1}}/J_h$ induces an ismorphism $\mathrm{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^{n+1}}/J_h, \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \mathrm{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{F}_1/J_h, \mathcal{O}_{\mathbb{C}^{n+1}})$.

The argument at the start of this section shows that $h \in J_h$; hence $\mathcal{F}_1/J_h \simeq \mathcal{F}_1\mathcal{O}_D/J_h\mathcal{O}_D$. Moreover, $\mathcal{F}_1\mathcal{O}_D = \mathcal{C}$, where $\mathcal{C}$ is the conductor ideal—the annihilator over $\mathcal{O}_D$ of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{O}_D$; see [17, Proposition 3.5]. Note that $\mathcal{C}$ is also an ideal in $\mathcal{O}_{\mathbb{C}^n}$, where it is principal (since $\mathcal{O}_{\mathbb{C}^n}$ is Gorenstein). Now in [16, Proposition 2.1], it is shown that for any map-germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ of finite $\mathcal{A}_e$-codimension, $J_h\mathcal{O}_{\mathbb{C}^n}/J_h\mathcal{O}_D \simeq \theta(f)/T\mathcal{A}_e f$; since here f is stable, it follows that $J_h\mathcal{O}_D = J_h\mathcal{O}_{\mathbb{C}^n}$. Thus, $\mathcal{F}_1/J_h \simeq \mathcal{C}/J_h\mathcal{O}_{\mathbb{C}^n}$.

If c is an $\mathcal{O}_{\mathbb{C}^n}$-generator of $\mathcal{C}$, then we have $c\mathcal{R}_f = J_h\mathcal{O}_{\mathbb{C}^n}$ (see [19, Theorem 1, Example 1]); thus multiplication by c defines an $\mathcal{O}_{\mathbb{C}^{n+1}}$-isomorphism $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \simeq \mathcal{C}/J_h\mathcal{O}_{\mathbb{C}^n}$. Putting this together with what we have deduced so far, we get

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\operatorname{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \operatorname{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f, \mathcal{O}_{\mathbb{C}^{n+1}}).$$

Let $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^{n+1})$ be the space of complex $n \times (n+1)$-matrices, and let Σ be the subvariety of $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^{n+1})$ formed by all matrices of rank less than n. Then Σ has codimension 2, and is smooth at each matrix of rank $n-1$. As f is stable, $df : \mathbb{C}^n \to \mathcal{L}(\mathbb{C}^n, \mathbb{C}^{n+1})$ is transverse to the natural stratification of Σ, and so $df^{-1}(\Sigma) = V(\mathcal{R}_f)$ has codimension 2 in $\mathbb{C}^n$. It follows from the Hilbert-Burch theorem (see, for example [4, Theorem 1.4.16]) that $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ is Cohen-Macaulay. Thus, by [4, Theorem 3.3.7],

$$\operatorname{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f, \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f}.$$

This proves the first statement of the lemma.

For the second statement, we complete the sequence

$$0 \leftarrow \omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f} \leftarrow \mathcal{O}^a \xleftarrow{r} \mathcal{O}^b$$

(where $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1}}$ and r is the matrix of a presentation of $\operatorname{Der}(\log h)$) to a resolution

$$0 \leftarrow \omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f} \leftarrow \mathcal{O}^a \xleftarrow{r} \mathcal{O}^b \xleftarrow{\alpha} \mathcal{O}^c \xleftarrow{\beta} \mathcal{O}^d \leftarrow 0.$$

Thus β is the matrix of a presentation of $\operatorname{Der}(\log h)^* = \ker(r^t)$. Dualising the resolution over $\mathcal{O}_{\mathbb{C}^{n+1}}$ and using the fact that the double dual of a Cohen-Macaulay module M is isomorphic to M itself, we deduce that

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\operatorname{Der}(\log h)^*, \mathcal{O}_{\mathbb{C}^{n+1}}) = \operatorname{Coker}(\beta^t) = \operatorname{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f}, \mathcal{O}_{\mathbb{C}^{n+1}})$$

$$\simeq \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f.$$

Finally, if f has corank 1, then $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ is regular, and so in particular Gorenstein, and $\omega_{\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f} \simeq \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$. The lemma is proved. □

The lemma suggests that if f has corank 1 then $\mathrm{Der}(\log h)$ is self-dual over $\mathcal{O}_{\mathbb{C}^{n+1}}$. In the remainder of this section we prove that this is the case and moreover that there is an isomorphism $\mathrm{Der}(\log h) \to \mathrm{Der}(\log h)^*$ such that the induced pairing $\mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \to \mathcal{O}_{\mathbb{C}^{n+1}}$ is skew-symmetric. We go on to show that it is induced by a 2-form $\omega_r \in \Omega^2(\log D)$. However, before specialising to the corank 1 case, we prove:

Proposition 3.4 *Suppose that $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ is a stable map of corank greater than 1. Then $\mathrm{Der}(\log h) \not\simeq \mathrm{Der}(\log h)^*$ for any defining equation h of Image(f).*

Proof Every stable map of corank bigger than 1 is adjacent to a germ isomorphic to $\mathrm{id} \times g_0$, where g_0 is the map-germ $\mathbb{C}^6, 0 \to \mathbb{C}^7, 0$ defined by

$$g_0(x, y, a, b, c, d) = (x^2 + ay, xy + bx + cy, y^2 + dx, a, b, c, d);$$

in other words, $0 \in \mathbb{C}^n$ lies in the closure of the set of points where the germ of f is left-right equivalent to $\mathrm{id} \times g_0$. This is simply because every Boardman stratum of type $\Sigma^{i_1,i_2,\cdots,}$ is in the closure of the Boardman stratum of type $\Sigma^{2,0}$, and up to left-right equivalence, $\mathrm{id} \times g_0$ is the only stable germ of type $\Sigma^{2,0}$ in these dimensions, as any germ g with $j^1 g$ transverse to $\Sigma^{2,0}$ is left-right equivalent to g_0. One calculates that $\mathcal{O}_{\mathbb{C}^6,0}/\mathcal{R}_{g_0}$ and its dualising module are not isomorphic as $\mathcal{O}_{\mathbb{C}^7,0}$-modules; for example, the former is minimally generated by 3 elements, the latter by 4. It follows that $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_{1\times g_0}$ and its dualising module are not isomorphic as $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules. As $1 \times g_0$ is weighted homogeneous, Lemma 3.3 applies and so for no defining equation h of $\mathrm{Image}(1 \times g)$ is $\mathrm{Der}(\log h)$ isomorphic to $\mathrm{Der}(\log h)^*$. If h is the defining equation of the image of f, then an isomorphism $\mathrm{Der}(\log h)_0 \simeq \mathrm{Der}(\log h)_0^*$ would give isomorphisms $\mathrm{Der}(\log h)_y \simeq \mathrm{Der}(\log h)_y^*$ at all neighbouring points y, and we have seen that this is impossible for points y such that $f : \mathbb{C}^n, f^{-1}(y) \to \mathbb{C}^{n+1}, y$ is left-right equivalent to $1 \times g_0$. Since there are points of this type in every neighbourhood of 0, we conclude that $\mathrm{Der}(\log h)_0 \not\simeq \mathrm{Der}(\log h)_0^*$. $\square$

For the rest of this section (and the paper) we suppose that $f : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ is a minimal stable map-germ of corank one. Lemma 3.3 says that (up to a free direct summand) in order to find a presentation of $\mathrm{Der}(\log h)$, it is enough to find a minimal presentation of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ and transpose it. We shall see that much more information about $\mathrm{Der}(\log h)$ is available from a projective resolution over $\mathcal{O}_{\mathbb{C}^{n+1}}$ of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$. To construct such a resolution we use a trick previously used in [17] to show that $f_*(\mathcal{O}_{\mathbb{C}^n})$ has a symmetric presentation over $\mathcal{O}_{\mathbb{C}^{n+1}}$.

Embed $\mathbb{C}^n$ as $\{t = 0\}$ in $\mathbb{C}^{n+1}$, and consider the map F from this bigger space to $\mathbb{C}^{n+1}$ defined by $(x, t) \mapsto (f_1(x), ..., f_n(x), f_{n+1}(x) + t)$. Since there are now two copies of $\mathbb{C}^{n+1}$, we distinguish them by denoting the domain of F by S and its codomain by T. Now $\mathcal{O}_S$ is a free module over $\mathcal{O}_T$ on $k + 1$ generators; in fact the local algebra $\mathcal{O}_S/m_T \cdot \mathcal{O}_T$ is isomorphic to $\mathbb{C}\{x\}/(x^{k+1})$, so we can take as free basis for $\mathcal{O}_S$ over $\mathcal{O}_T$ the geometric progression $1, x, x^2, ..., x^k$. By [4, Theorem 3.3.7], there are isomorphisms of $\mathcal{O}_S$-modules

$$\mathcal{O}_S \simeq \omega_S \simeq \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \omega_T) \simeq \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T).$$

Let ϕ be an $\mathcal{O}_S$-generator of $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$. Then $\langle g, h\rangle = \phi(gh)$ defines an $\mathcal{O}_T$-perfect pairing $\mathcal{O}_S \times \mathcal{O}_S \to \mathcal{O}_T$. Let $\theta : \mathcal{O}_S \to \mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ be the $\mathcal{O}_S$-isomorphism $\theta(s) = s \cdot \phi$.

Let E denote the $\mathcal{O}_T$ basis $1, x, ..., x^k$ for $\mathcal{O}_S$; if $\psi_1, ..., \psi_{k+1}$ is the dual basis of $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ let $\hat{g}_i = \theta^{-1}(\psi_i)$, for $i = 1, ..., k + 1$, and let $\hat{E}$ denote the $\mathcal{O}_T$-basis $\hat{g}_1, ..., \hat{g}_{k+1}$ of $\mathcal{O}_S$.

If $s \in \mathcal{O}_S$, let Ψ_s denote the $\mathcal{O}_T$-linear morphism $\mathcal{O}_S \to \mathcal{O}_S$ induced by multiplication by s, and denote the matrix of Ψ_s with respect to the bases E (in the domain) and $\hat{E}$ (in the codomain) by $(\Psi_s)^E_{\hat{E}}$.

Lemma 3.5 *The matrix $(\Psi_s)^E_{\hat{E}}$ is* persymmetric*; that is, its i, j-th entry depends only on $i + j$. In particular, it is symmetric.*

Proof If $sx^{j-1} = \sum_{i=1}^{k+1} \lambda^i_j \hat{g}_i$, then evidently $\lambda^i_j = \langle sx^{j-1}, x^{i-1}\rangle = \phi(sx^{i+j-2})$. □

Note that $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f = \mathcal{O}_{\mathbb{C}^n}/(\partial f_n/\partial x, \partial f_{n+1}/\partial x) \simeq \mathcal{O}_S/(t, \partial f_n/\partial x, \partial f_{n+1}/\partial x)$ and so $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ has (Koszul) resolution over $\mathcal{O}_S$

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\alpha} \mathcal{O}_S^3 \xrightarrow{\beta} \mathcal{O}_S^3 \xrightarrow{\alpha^t} \mathcal{O}_S \longrightarrow \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \longrightarrow 0 \qquad (3)$$

where α is the matrix

$$\begin{pmatrix} t \\ \partial f_n/\partial x \\ \partial f_{n+1}/\partial x \end{pmatrix},$$

β is the matrix

$$\begin{pmatrix} 0 & \partial f_{n+1}/\partial x & -\partial f_n/\partial x \\ -\partial f_{n+1}/\partial x & 0 & t \\ \partial f_n/\partial x & -t & 0 \end{pmatrix}.$$

and α^t is the transpose of α.

Now identify the free $\mathcal{O}_S$-modules in the sequence (3) with direct sums of the appropriate number of copies of $\mathcal{O}_T$, alternating the use of the free bases E and $\hat{E}$ to carry out these identifications. Thus, the left-most copy of $\mathcal{O}_S$ is identified with $\mathcal{O}_T^{k+1}$ by means of the basis $\hat{E}$, the left-most copy of $(\mathcal{O}_S)^3$ with $(\mathcal{O}_T{}^{k+1})^3$ by means of the basis E, and so on. These identifications now turn (3) into the following exact sequence of $\mathcal{O}_T$-modules:

$$0 \longrightarrow \mathcal{O}_T^{k+1} \xrightarrow{r} (\mathcal{O}_T^{k+1})^3 \xrightarrow{M} (\mathcal{O}_T^{k+1})^3 \xrightarrow{r^t} \mathcal{O}_T^{k+1} \longrightarrow \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \longrightarrow 0, \tag{4}$$

where r is the matrix

$$\begin{pmatrix} (\Psi_t)_E^{\hat{E}} \\ (\Psi_{\partial f_n/\partial x_n})_E^{\hat{E}} \\ (\Psi_{\partial f_{n+1}/\partial x_n})_E^{\hat{E}} \end{pmatrix},$$

and M is the matrix

$$\begin{pmatrix} 0 & (\Psi_{\partial f_{n+1}/\partial x_n})_{\hat{E}}^E & -(\Psi_{\partial f_n/\partial x_n})_{\hat{E}}^E \\ -(\Psi_{\partial f_{n+1}/\partial x_n})_{\hat{E}}^E & 0 & (\Psi_t)_{\hat{E}}^E \\ (\Psi_{\partial f_n/\partial x_n})_{\hat{E}}^E & -(\Psi_t)_{\hat{E}}^E & 0 \end{pmatrix}.$$

The matrix M is skew-symmetric, because of the skew-symmetry of β and the symmetry of each of $(\Psi_{\partial f_n/\partial x_n})_{\hat{E}}^E$, $(\Psi_{\partial f_{n+1}/\partial x_n})_{\hat{E}}^E$ and $(\Psi_t)_{\hat{E}}^E$.

The resolution (4) is not minimal. For, one has the equation

$$\partial f_n/\partial x = (k+1)x^k + \sum_{i=1}^{k-1}(k-i)u_i x^{k-i-1}$$

which expresses $\partial f_n/\partial x$ as an $\mathcal{O}_T$-linear combination of the generators of $\mathcal{O}_{\mathbb{C}^n}$, with coefficient of x^k a unit. It follows that $\partial f_n/\partial x$ generates the socle in $\mathcal{O}_S/m_T\mathcal{O}_S$. In particular, one can choose ϕ to take the value 1 on $\partial f_n/\partial x$. Thus, the $1,1$'th entry $\langle \partial f_n/\partial x, 1\rangle$ in the matrix $(\Psi_{\partial f_n/\partial x})_{\hat{E}}^E$ is a unit. However, reducing the resolution to a minimal one by means of row and column operations does not alter its symmetries; in the process, the ranks of the non-trivial free modules in the resolution are reduced by 1,3,3 and 1 respectively. Thus, we have proved:

Proposition 3.6 *There exists a minimal resolution of $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f$ of the form*

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{C}^{n+1}} \xrightarrow{r} \mathcal{O}^{3k}_{\mathbb{C}^{n+1}} \xrightarrow{m} \mathcal{O}^{3k}_{\mathbb{C}^{n+1}} \xrightarrow{r^t} \mathcal{O}^k_{\mathbb{C}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \longrightarrow 0,$$

in which m is skew symmetric. □

We can now establish the principal element in the proof of Theorem 1.1.

Theorem 3.7 *There exists a perfect, skew-symmetric pairing*

$$(\ ,\ ) : \mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}.$$

Proof From Lemmas 3.1 and 3.3, together with the uniqueness of minimal resolutions, it follows that there is a projective resolution of $\mathrm{Der}(\log h)$ of the form

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{C}^{n+1}} \xrightarrow{r\oplus 0} \mathcal{O}^{3k+\ell}_{\mathbb{C}^{n+1}} \xrightarrow{p} \mathrm{Der}(\log h) \longrightarrow 0 \tag{5}$$

where r is the $\mathcal{O}_{\mathbb{C}^{n+1}}$-linear map of Proposition 3.6. On the other hand, $\mathrm{Der}(\log D)$ has rank $n+1$, since it coincides with $\theta_{\mathbb{C}^{n+1}}$ outside D. It follows that $\mathrm{Der}(\log h)$ has rank n and so we must have $\ell = 0$. Hence

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}) \simeq \mathrm{Ker}\, r^t = \mathrm{Im}\, m,$$

and since the skew matrix m passes to the quotient to define an isomorphism $\bar{m} : (\mathcal{O}_{\mathbb{C}^{n+1}})^{3k}/\mathrm{Ker}\, m \to \mathrm{Im}\, m$, we have a chain of isomorphisms

$$\mathrm{Der}(\log h) \cong \mathrm{Coker}\, r = (\mathcal{O}_{\mathbb{C}^{n+1}})^{3k}/\mathrm{Im}\, r = (\mathcal{O}_{\mathbb{C}^{n+1}})^{3k}/\mathrm{Ker}\, m$$

and

$$(\mathcal{O}_{\mathbb{C}^{n+1}})^{3k}/\mathrm{Ker}\, m \xrightarrow{\bar{m}} \mathrm{Im}\, m \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}}).$$

From this it follows that the matrix m defines a perfect skew-symmetic pairing $(\ ,\ ) : \mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \to \mathcal{O}_{\mathbb{C}^{n+1}}$: given $\chi_1, \chi_2 \in \mathrm{Der}(\log h)$, choose $a_1, a_2 \in (\mathcal{O}_{\mathbb{C}^{n+1}})^{3k}$ such that $p(a_1) = \chi_2, p(a_2) = \chi_2$; then $(\chi_1, \chi_2) = a_1^t \cdot m \cdot a_2$. □

Proof of Theorem 1.1 Firstly, we show that the pairing $(\ ,\ ) : \mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \to \mathcal{O}_{\mathbb{C}^{n+1}}$ of Theorem 3.7 is induced by a logarithmic 2-form. We say that a divisor D in the complex manifold X is *locally quasihomogeneous* if for each $x \in D$ there is a neighbourhood U of x in X in which there are local coordinates, centred on x, with respect to which D has a weighted homogeneous defining equation (with all weights positive). This definition is taken from [6], where the term *strongly quasihomogeneous* is used in place of locally quasihomogeneous. Our next lemma is taken from [9, Proposition 1.5].

Lemma 3.8 *Let D be a locally quasihomogeneous hypersurface in $\mathbb{C}^{n+1}$. Then contraction of differential forms by vector fields defines an isomorphism*

$$\phi : \Omega^p(\log D) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\wedge^p \mathrm{Der}(\log D), \mathcal{O}_{\mathbb{C}^{n+1}}).$$

□

Now extend the pairing $(\ ,\ ) : \mathrm{Der}(\log h) \times \mathrm{Der}(\log h) \to \mathcal{O}_{\mathbb{C}^{n+1}}$ to a pairing $[\ ,\ ] : \mathrm{Der}(\log D) \times \mathrm{Der}(\log D) \to \mathbb{C}^{n+1}$ by setting $[\chi_0,\ ] = [\ ,\chi_0] = 0$ (where χ_0 is some chosen generator of the complementary summand to $\mathrm{Der}(\log h)$ in $\mathrm{Der}(\log D)$). By the lemma, $[\ ,\ ]$ is induced by a form $\omega_r \in \Omega^2(\log D)$; restriction of ω to $\mathrm{Der}(\log h) \times \mathrm{Der}(\log h)$ thus induces $(\ ,\ )$.

Finally, we come to the last statement in Theorem 1.1. Recall that the statement is *global*, whereas all the previous considerations of this section have been *local*. In order to globalise, notice that following through the arguments of this section ω_r can be taken to be a homogeneous (and in particular, algebraic) logarithmic 2-form. Now let $\mathrm{Der}_{\mathrm{alg}}(\log h)$ denote the graded module over the polynomial ring $\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^{n+1})$ consisting of the globally defined algebraic vector fields which annihilate the weighted homogenenous defining equation h. There is a natural functor from such graded modules to $\mathcal{O}_{\mathbb{C}^{n+1},0}$-modules given by

$$\mathcal{O}_{\mathbb{C}^{n+1},0} \otimes_{\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^{n+1})} \text{—}.$$

This functor is faithfully exact (combine [4, Proposition 1.5.15] and [21, Corollary 2.6.1, Annexe: Definition 4, Proposition 22]) and, evidently, maps $\mathrm{Der}_{\mathrm{alg}}(\log h)$ to $\mathrm{Der}(\log h)_0$. It follows immediately that ω_r induces a perfect pairing on $\mathrm{Der}_{\mathrm{alg}}(\log h)$. Now, consider the natural functor from graded $\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^{n+1})$-modules to sheaves of $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules defined by

$$\mathcal{O}_{\mathbb{C}^{n+1}} \otimes_{\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^{n+1})} \text{—}.$$

This functor maps $\mathrm{Der}_{\mathrm{alg}}(\log h)$ to the sheaf of germs of holomorphic vector fields on $\mathbb{C}^{n+1}$ which annihilate h. This establishes the proof of the theorem. $\square$

The relative form ω_r above is easy to describe if $k = 1$, one can take $\omega_r = \iota_{\chi_e}(dV \wedge dZ \wedge dT)$. For larger k we cannot give such a succinct formula. However, using Macaulay as in Section 5, below, one can check:

Example 3.9 *In the case $k = 2$ one can take*

$$\begin{aligned}
\omega_r \;\; = \;\; & (9V_2^2Z - 7UV_2T + 15V_1ZT)\, dU \wedge dV_1 \\
& -(6V_1V_2Z + 5UV_1T + 9T^2)\, dU \wedge dV_2 \\
& +(V_1V_2^2 + 4V_1^2T)\, dU \wedge dZ \\
& -(UV_1V_2 - 9V_1^2Z - 6V_2T)\, dU \wedge dT \\
& +(6UV_2Z + 9V_1Z^2 - 4U^2T)\, dV_1 \wedge dV_2 \\
& +(2UV_2^2 - 6V_1V_2Z + 13UV_1T + 9T^2)\, dV_1 \wedge dZ \\
& -(2U^2V_2 + 3UV_1Z + 9ZT)\, dV_1 \wedge dT \\
& -(2UV_1V_2 - 3V_1^2Z - 9V_2T)\, dV_2 \wedge dZ \\
& +(2U^2V_1 - 9V_2Z + 6UT)\, dV_2 \wedge dT \\
& +(7UV_1^2 + 6V_2^2 + 3V_1T)\, dZ \wedge dT.
\end{aligned}$$

4 Torus actions

The 2-form ω_r described in the proof of Theorem 1.1 induces the perfect pairing of Theorem 3.7, and also satisfies $\iota_{\chi_e}(\omega_r) = 0$, by construction. Evidently ω_r may be taken to be bi-homogeneous with respect to the torus action described in Section 2. We now investigate some of the consequences of this choice. In this section it is convenient to work in the global algebraic category of $\mathcal{O}_{\mathrm{alg}}(\mathbb{C}^{n+1})$-modules. Let h be a weighted homogenenous defining equation for D. To simplify notation we will denote the ring of polynomial functions on $\mathbb{C}^{n+1}$ by $\mathcal{O}_{\mathbb{C}^{n+1}}$, write $\mathrm{Der}(\log h)$ for the globally defined algebraic vector fields which annihilate h, etc.

Let us describe a direct sum decomposition of the exterior algebra $\oplus\Omega^p(\log D)$ (though not of the complex $\Omega^\bullet(\log D)$). Recall from the start of Section 3 that the choice of an Euler vector field in $\mathbb{C}^{n+1}$ determines a free complementary summand to $\mathrm{Der}(\log h)$ in $\mathrm{Der}(\log D)$. This choice thus determines an isomorphism of $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}})$ with the submodule of

$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log D), \mathcal{O}_{\mathbb{C}^{n+1}})$ consisting of those homomorphisms killing the Euler field. Now $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log D), \mathcal{O}_{\mathbb{C}^{n+1}}) = \Omega^1(\log D)$ (with the pairing induced by contraction of differential forms by vector fields; see [20, Lemma 1.6]), and so $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathrm{Der}(\log h), \mathcal{O}_{\mathbb{C}^{n+1}})$ can be identified with $\{\omega \in \Omega^1(\log D) : \iota_{\chi_e}(\omega) = 0\}$.

Definition 4.1 $\Omega^p(\log h) = \{\omega \in \Omega^p(\log D) : \iota_{\chi_e}(\omega) = 0\}$.

The following properties are obvious:

Proposition 4.2 *Let D be a weighted homogeneous divisor in $\mathbb{C}^{n+1}$ with equation h and Euler field χ_e, and suppose that h has weighted degree $w \neq 0$. Then*

1. *the wedge of forms in $\Omega^p(\log h)$ and $\Omega^q(\log h)$ lies in $\Omega^{p+q}(\log h)$.*
2. *the operators on $\Omega^p(\log D)$ defined by*
$$\omega \mapsto \iota_{\chi_e}((dh/wh) \wedge \omega)$$
and
$$\omega \mapsto (dh/wh) \wedge \iota_{\chi_e}\omega$$
are projections onto $\Omega^p(\log h)$ and its complementary summand $(dh/h)\wedge \Omega^{p-1}(\log D) = (dh/h) \wedge \Omega^{p-1}(\log h)$ in $\Omega^p(\log D)$.
3. *$\Omega^{n+1}(\log h) = 0$*
4. *$\Omega^n(\log h) = \iota_{\chi_e}(\Omega^{n+1}(\log h))$ is freely generated over $\mathcal{O}_{\mathbb{C}^{n+1}}$ by $\iota_{\chi_e}(dX_1\wedge \cdots \wedge dX_{n+1}/h)$.*
5. *the map $d_r : \Omega^p(\log h) \to \Omega^{p+1}(\log h)$ defined by $d_r(\omega) = \iota_{\chi_e}\{(dh/wh)\wedge d\omega\}$ makes $\Omega^\bullet(\log h)$ into a complex.*
6. *For any form $\omega \in \Omega^p(\log D)$ and any $t \neq 0$, $d_r(\omega)|_{X_t} = d(\omega|_{X_t})$, so that if $\omega \in \Omega^p(\log h)$ is d_r-closed, then $\omega|_{X_t}$ is closed for all $t \neq 0$.*

It is convenient to record here a consequence of Theorem 1.1. The theorem sets up a perfect pairing $\mathrm{Der}(\log D) \times \mathrm{Der}(\log D) \to \mathcal{O}_{\mathbb{C}^{n+1}}$ and, dually, a perfect pairing $\Omega^1(\log D) \times \Omega^1(\log D) \to \mathcal{O}_{\mathbb{C}^{n+1}}$. In fact, this can be extended to higher degrees.

Corollary 4.3 *There is a perfect pairing $\Omega^p(\log D) \times \Omega^p(\log D) \to \mathcal{O}_{\mathbb{C}^{n+1}}$. In particular, there are isomorphisms $\Omega^p(\log D) \cong \Omega^{n+1-p}(\log D)$.*

Proof Denote the duality functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\_, \mathcal{O}_{\mathbb{C}^{n+1}})$ by $*$. Suppose that we are given some finitely generated $\mathcal{O}_{\mathbb{C}^{n+1}}$-module M together with an isomorphism of $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules $\alpha : M \to M^*$. Thus, taking pth exterior powers and then double duals we obtain an isomorphism of $\mathcal{O}_{\mathbb{C}^{n+1}}$-modules:

$$(\wedge^p \alpha)^{**} : (\wedge^p M)^{**} \longrightarrow (\wedge^p(M^*))^{**}.$$

On the other hand, dualising the natural map $\wedge^p M \to (\wedge^p(M^*))^*$ gives a map $\beta : (\wedge^p(M^*))^{**} \to (\wedge^p M)^*$. If p is a height one prime of $\mathcal{O}_{\mathbb{C}^{n+1}}$ then M_p is a free module. It follows that β_p is an isomorphism. Since β is a map of reflexive modules then it is, in fact, determined in codimension one. Thus, β is an isomorphism and hence we obtain an isomorphism $((\wedge^p M)^*)^* \cong ((\wedge^p M^*)^*)^*$.

Now, setting $M = \mathrm{Der}(\log h)$, $\alpha = \omega_r$ and recalling from Lemma 3.8 that $\Omega^p(\log h) = (\wedge^p \mathrm{Der}(\log h))^*$ we obtain that $\Omega^p(\log h) \cong \Omega^p(\log h)^*$. By Proposition 4.2.2 there is a direct sum decomposition $\Omega^p(\log D) \cong \Omega^p(\log h) \oplus \Omega^{p-1}(\log h)$ and, by [9, Proposition 1.5], the wedge product gives a perfect pairing $\Omega^p(\log D) \times \Omega^{n+1-p}(\log D) \to \mathcal{O}_{\mathbb{C}^{n+1}}$. The result follows. $\square$

Now we return to a discussion of the bi-homogeneous 2-form ω_r inducing the perfect pairing on $\mathrm{Der}(\log h)$. Note that the generator of $\Omega^{2k+1}(\log D)$ should now be referred to as $dU_1 \wedge \cdots \wedge dT/h$ in adherence to the notation introduced in Section 2. By 4.2.1, $\omega_r^{\wedge k} \in \Omega^{2k}(\log h)$ and thus $\omega_r^{\wedge k} = \alpha \iota_{\chi_e}(dU_1 \wedge \cdots \wedge dT/h)$ for some bi-homogeneous polynomial α. If the weight of α with respect to the good $\mathbb{C}^*-$ action generated by χ_e is not 0, then $\omega_r^{\wedge k}$ vanishes on the non-empty set $\{\alpha = 0\}$, contradicting the relative non-degeneracy of ω_r. Thus,

Proposition 4.4 *$\omega_r^{\wedge k} = \alpha \iota_{\chi_e}(dU_1 \wedge \cdots \wedge dT/h)$ for some non-zero constant α.*

Proposition 4.4 allows us to determine the weight of ω_r, and from this several interesting consequences flow. In particular, $\omega_r|_{X_t}$ (assumed closed) is cohomologically trivial.

Proposition 4.5 *1. With respect to both generators of the torus action described in Section 2, ω_r has weight 1.*

2. *ω_r is d_r-closed if and only if $d\omega_r = (dh/wh) \wedge \omega_r$.*

3. *ω_r is d_r-closed if and only if $\omega_r = (d - (dh/wh)\wedge)\iota_{\chi_r}(\omega_r)$*

Proof By inspection, the generator $dU_1 \wedge \cdots \wedge dT/h$ of $\Omega^{2k+1}(\log D)$ has weight k with respect to both χ_e and χ_r, this weight being just the sum of the weights of the variables minus the weight of h. Thus (1) follows from Proposition 4.4.

For (2), note that since ω_r has weight 1 with respect to χ_e, the Lie derivative $L_{\chi_e}(\omega_r)$ is equal to ω_r. By Cartan's formula, $L_{\chi_e}(\omega_r) = d\iota_{\chi_e}(\omega_r)+\iota_{\chi_e}d\omega_r$, and since by construction $\iota_{\chi_e}(\omega_r) = 0$, we get $\omega_r = \iota_{\chi_e}(d\omega_r)$. Since

$$d_r\omega_r = 0 \Leftrightarrow \iota_{\chi_e}\{(dh/wh) \wedge d\omega_r\} = 0 \Leftrightarrow d\omega_r - (dh/wh) \wedge \iota_{\chi_e}(d\omega_r) = 0$$

(2) follows.

For (3), once again by (1) of this proposition we have $L_{\chi_r}\omega_r = \omega_r$, so Cartan's formula gives $\omega_r = d\iota_{\chi_r}\omega_r + \iota_{\chi_r}d\omega_r$. By (2), we deduce that ω_r is d_r-closed if and only if $\omega_r = d\iota_{\chi_r}\omega_r+\iota_{\chi_r}\{(dh/wh)\wedge\omega_r\}$. But $\iota_{\chi_r}(dh/wh) = 0$ (i.e. the weight of h with respect to χ_r is 0), so this reads $\omega_r = d\iota_{\chi_r}\omega_r - (dh/wh) \wedge \iota_{\chi_r}\omega_r$. $\square$

If $k = 1$ it is trivial that ω_r is d_r-closed. To end this section, we treat the next case when $k = 2$. Firstly, though, we need a result concerning the weights of the vector fields in $\mathrm{Der}(\log h)$.

Proposition 4.6 *All non-zero homogeneous elements of* $\mathrm{Der}(\log h)$ *have non-negative weight with respect to χ_e.*

Proof We compare the projective resolutions

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{C}^{n+1}} \xrightarrow{r} \mathcal{O}^{3k}_{\mathbb{C}^{n+1}} \xrightarrow{p} \theta_{\mathbb{C}^{n+1}} \xrightarrow{\cdot h} \mathcal{O}_{\mathbb{C}^{n+1}} \longrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}/J_h \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}^k_{\mathbb{C}^{n+1}} \xrightarrow{r} \mathcal{O}^{3k}_{\mathbb{C}^{n+1}} \xrightarrow{m} \mathcal{O}^{3k}_{\mathbb{C}^{n+1}} \xrightarrow{r^t} \mathcal{O}^k_{\mathbb{C}^{n+1}} \longrightarrow \mathcal{F}_1/J_h \simeq \mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \longrightarrow 0.$$

The inclusion $i : \mathcal{F}_1/J_h \hookrightarrow \mathcal{O}_{\mathbb{C}^{n+1}}/J_h$ lifts to give a morphism of resolutions which induces an isomorphism

$$\mathrm{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{O}_{\mathbb{C}^{n+1}}/J_h, \mathcal{O}_{\mathbb{C}^{n+1}}) \to \mathrm{Ext}^3_{\mathcal{O}_{\mathbb{C}^{n+1}}}(\mathcal{F}_1/J_h, \mathcal{O}_{\mathbb{C}^{n+1}})$$

(see the first paragraph of the proof of Lemma 3.3) and so by minimality of the two resolutions the lifting of i from the left most copy of $\mathcal{O}^k_{\mathbb{C}^{n+1}}$ in the lower sequence to $\mathcal{O}^k_{\mathbb{C}^{n+1}}$ in the upper sequence must be an isomorphism. After a change of basis it may thus be assumed to be the identity. Since the maps r in the upper and lower sequence are the same, it follows that the degrees of the homogeneous generators of the summands in $\mathcal{O}^{3k}_{\mathbb{C}^{n+1}}$ in the upper sequence must be the same as the corresponding degrees in the left-most $\mathcal{O}^{3k}_{\mathbb{C}^{n+1}}$ in the lower sequence.

Note that the isomorphism $\mathcal{O}_{\mathbb{C}^n}/\mathcal{R}_f \xrightarrow{c} \mathcal{F}_1/J_h$ (multiplication by a homogeneous $\mathcal{O}_{\mathbb{C}^n}$-generator of the conductor) is homogeneous of degree k^2; for the quotient of $\partial h/\partial T$ by the minor $\partial(f_1,\dots,f_{2k})/\partial(x_1,\dots,x_{2k}) = \partial(f_{2k})/\partial x$ of the matrix of df is such a generator (see [19, Theorem 1, Example 1]). By following through the construction of the lower sequence in the proof of 3.7 and using this fact, it is possible to check that the degrees referred to at the end of the last paragraph are

$$k^2+2k+1,\dots,k^2+3k,k^2+2k+1,\dots,k^2+3k,k^2+2k+2,\dots,k^2+3k+1.$$

The degrees of the generators

$$\partial/\partial U_1,\dots,\partial/\partial U_{k-1},\partial/\partial V_1,\dots,\partial/\partial V_k,\partial/\partial Z,\partial/\partial T$$

of $\theta_{\mathbb{C}^{n+1}}$ are

$$(k+1)^2-2,\dots,(k+1)^2-k,(k+1)^2-1,\dots,(k+1)^2-k,(k+1)^2-k-1,(k+1)^2-k-1;$$

here, the usual weighting of $\theta_{\mathbb{C}^{n+1}}$ is shifted by $\deg h = (k+1)^2$ in order that the map $\theta \xrightarrow{\cdot h} \mathcal{O}_{\mathbb{C}^{n+1}}$ should be homogeneous of degree 0.

It follows that the $3k$ generators of $\mathrm{Der}(\log h)$ listed by the columns of the matrix p have weight

$$0,\dots,k-1,0,\dots,k-1,1,\dots,k.$$

Note that the weight of the ith vector field differs from the weight of the elements in the ith column of the matrix p by the weights of the variables. The proposition is proved. $\square$

Finally, as promised, we prove Conjecture 1.2 in the first non-trivial case.

Proposition 4.7 *If $k=2$, then ω_r is d_r-closed.*

Proof We have $\Omega^4(\log D) = \{\iota_\chi(dU_1 \wedge \cdots \wedge dT/h) : \chi \in \mathrm{Der}(\log D)\}$, for the condition that $\sum_i (-1)^i c_i dX_1 \wedge \cdots \wedge d\hat{X}_i \wedge \cdots \wedge dX_{n+1}/h$ should be logarithmic is the same as the condition that $\sum_i c_i \partial/\partial X_i \cdot h \in (h)$. Now none of the generators of $\mathrm{Der}(\log D)$ has negative weight, and since the weight (with respect to χ_e) of $dU_1 \wedge \cdots \wedge dT/h$ is 2, it follows that there is no non-zero form of weight 1 in $\Omega^4(\log D)$. Hence, $(dh/h) \wedge d\omega_r$ must be equal to 0, and $d_r\omega_r = 0$. □

5 Final Remarks

Assume now that Conjecture 1.2 is correct and ω_r is relatively closed.
1. Then the total space $\mathbb{C}^{n+1} \setminus D$ of the Milnor fibration is foliated by algebraic symplectic manifolds. However, the fibration does not seem to be a symplectic fibration in the sense of [15], pp 191-214: the natural local trivialisations (coming from the $\mathbb{C}^*$-action) are only *conformally* symplectic. This can be seen as follows: let $w_1, \cdots, w_{n+1}$ be the weights of the variables, let d be the degree of the weighted homogenenous defining equation h, let $X_u = h^{-1}(u)$ and $X_v = h^{-1}(v)$ be nonsingular fibres, and let ξ be a d-th root of u/v. Then the linear isomomorphism A_ξ with matrix

$$\mathrm{diag}(\xi^{w_1}, \cdots, \xi^{w_{n+1}})$$

maps X_v to X_u. As ω_r has weight 1 (by 4.5), we have $A_\xi^*(\omega_r) = \xi\omega_r$. Thus $A_\xi : X_v \to X_u$ is not a symplectomorphism, but a conformally symplectic diffeomorphism — pulling back the symplectic form on X_u to a scalar multiple of the symplectic form on X_v. Nevertheless, the structure is still an interesting one. For example, the $\mathbb{C}^*$ action equips each fibre with d canononical conformally symplectic automorphisms, induced by the isomorphisms A_ξ with ξ a d-th root of unity.

Moreover, the form ω_r determines a conformally symplectic connection on the bundle $\mathbb{C}^{n+1} \setminus D \to \mathbb{C} \setminus 0$, with associated horizontal distribution

$$\mathrm{Hor}_x = \{v \in T_x\mathbb{C}^{n+1} : \omega_r(u, v) = 0 \ \forall \ u \in T_x X_{h(x)}\}$$

spanned by the Euler vector field χ_e.

2. The relative symplectic form ω_r gives a Poisson bracket on $\mathcal{O}_{\mathbb{C}^{n+1}}$. Further, this restricts to the divisor D. In other words there is a Lie algebra structure on $\mathcal{O}_{\mathbb{C}^{n+1}}$ with bracket $\{\ ,\ \}$ for which $h\mathcal{O}_{\mathbb{C}^{n+1}}$ is a Lie-ideal and

$$\{a, bc\} = \{a, b\}c + \{a, c\}b,$$

for all $a, b, c \in \mathcal{O}_{\mathbb{C}^{n+1}}$. We just show how to construct the bracket and leave to the reader the task of verifying that it has the claimed properties. For this is similar to the standard fact that a symplectic manifold has a Poisson bracket on its algebra of functions (see, for example, [1]).

We have an isomorphism $\flat : \mathrm{Der}(\log h) \to \Omega^1(\log h)$ given by $X^\flat = \iota_X \omega_r$. Denote the inverse isomorphism by $\sharp$ and extend its domain to $\Omega^1(\log D)$ by composing with the projection

$$\Omega^1(\log D) \to \Omega^1(\log h) \qquad \omega \mapsto \iota_{\chi_e}((dh/wh) \wedge \omega).$$

Finally, we define the Poisson bracket by

$$\{f, g\} = \omega_r((dg)^\sharp, (df)^\sharp).$$

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