Real map-germs with good perturbations

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Abstract This paper is concerned with the relation between the topology of certain real algebraic sets and that of their complexification.

Our aim is to determine for which singularities of mappings from surfaces to 3-space can the changes in the homology of the complex image resulting from a deformation of the mapping be observed in the real image. More precisely, we determine all right-left equivalence classes of map-germs $\mathbf{C}^2, 0 \to \mathbf{C}^3, 0$ for which it is possible to find a real form with a real stable perturbation whose image carries the vanishing cohomology of the image of a complex stable perturbation (thus, a "good real perturbation"). In fact, the only such classes are the singularities S_1 and H_k ($k \geq 2$) (see below for their definition). We exhibit real stable perturbations of these with the required property, and give drawings of their images in $\mathbf{R}^3$ (in Section 3).

This relative scarcity of singularities with good real perturbations is in sharp contrast to the case of map-germs $\mathbf{R}, 0 \to \mathbf{R}^2, 0$; here it was shown by N. A'Campo and S.M. Gusein-Sade (independently) in [1] and [6] that such stable perturbations always exist.

1 Introduction

This paper is motivated by curiosity as to how much of the vanishing homology of complex singularities can be observed over $\mathbf{R}$. The theory of vanishing cycles associated to singularities of complex analytic mappings is comparable to the theory of vanishing cycles associated to isolated hypersurface singularities, as described by Milnor in [9]: suppose that $U \subseteq \mathbf{C}^n$ and that $f : U \to \mathbf{C}^{n+1}$ is a finite analytic mapping. If the germ of f at $0 \in U$ has "finite codimension" (i.e. has instability confined to the origin - see [13] 2.1), and B_ϵ is a neighbourhood of 0 in $\mathbf{C}^{n+1}$ so small that $f(U) \cap B_\epsilon$ is a cone on its boundary, then as we deform f to f_t (here t is a complex parameter) so that it becomes stable, its image within B_ϵ acquires cohomology in dimension n ([11]) .

Let $U_t = f_t^{-1}(B_\epsilon)$, and choose $\eta > 0$ so small that as for all t with $\mid t \mid \leq \eta$ f_t is stratified transverse to ∂B_ϵ. Then up to homeomorphism in source and target, the map $f_t : U_t \to B_\epsilon$ is independent of $t \neq 0$. It is known as a *stable perturbation* of f, and it can be shown that, again up to homeomorphism, it depends only on the original germ f (cf [7]). One should think of it (or its image) as the analogue of the Milnor fibre of an isolated hypersurface singularity.

The rank of $H^n(\tilde{f}_t(U) \cap B_\epsilon, \mathbf{Z})$ is an analytic invariant of f, which we shall refer to as the *image Milnor number*. For example, in the case of irreducible parametrised plane curve singularities, it is easy to see that the image Milnor number is just the δ-invariant, the number of nodes which appear in a generic perturbation. Indeed, here the image is evidently homotopy- equivalent to a wedge of δ circles. The image Milnor number is not the same as the Milnor number of the image: this latter is finite only when $n = 1$, and in this case is equal to twice the image Milnor number (by Milnor's formula (cf [9]), $\mu = 2\delta - r + 1$, and here the number of branches r is equal to 1).

In 1974 S.M. Gusein-Sade and N.A'Campo independently showed that for every real curve-germ one could find a real deformation with δ real nodes; inclusion of the real image in its complexification is then a homotopy-equivalence ([1],[6]).In his 1989 Warwick thesis the first named author found real stable perturbations of certain map-germs $\mathbf{R}^2, 0 \to \mathbf{R}^3, 0$, with a similar property: the 2-dimensional homology of the image has rank equal to the image Milnor number of the complexification. The germs which he dealt with are S_1, given by

$$(x, y) \to (x, y^2, y^3 + x^2 y)$$

and the members of the series H_k $(k \geq 2)$, given by

$$(x, y) \to (x, y^3, xy + y^{3k-1}).$$

Drawings of the images of these "good real perturbations" are shown in the last section of this paper.

Our main result here is that these are (up to right-left equivalence) the only germs $f : \mathbf{C}^2, 0 \to \mathbf{C}^3, 0$ to have good real perturbations. To prove this we make use of a spectral sequence relating the rational cohomology of the image of a mapping to the cohomology of its multiple- point spaces. We show (2.3) that the assumption that the image of a real stable perturbation has 2nd cohomology with rank equal to the image Milnor number of the complexification implies an isomorphism of spectral sequences for real and complex mappings. With a little extra work, we deduce that the multiple point spaces of the real mapping must have the same cohomology as those of the complex mapping. Since these multiple point spaces are in fact Milnor fibres of the (in general singular) multiple point spaces of the original map-germ f_0, we deduce that only the triple point space of f_0 (which unlike the double point space is 0-dimensional) can have Milnor number greater than 1; from this the conclusion follows from the classification ([10]).

All of this is proved also in the case of multi-germs $\mathbf{C}^2, S \to \mathbf{C}^3, 0$, where S is a finite set. Here of course there are many more germs with good real perturbations, among which it is worth mentioning two classes:
i) all germs whose image Milnor number is 1 (cf [4], [12]); this seems to be a general phenomenon, and to hold in other dimensions also;
ii) germs whose singularity is entirely combinatorial; that is, multi-germs consisting of k immersed planes, each three of which meet in general position. Here it is clear that the spectral sequence for *any* real stable perturbation is the same, from E_1 on, as the spectral sequence for its complexification. Again, this is a general phenomenon.

We do not attempt any further classification here.

The isomorphism of spectral sequences (which is a consequence of nothing more than the assumption that the real and complex images have equal second Betti numbers) also implies that all of the salient features of the complex image (triple points, pinch points...) are present in the real image.

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2 Section 2

Definition: Suppose that $f : \mathbf{C}^2, S \to \mathbf{C}^3, 0$ is real (i.e. defined by real analytic functions), where $S \subset \mathbf{R}^2$ is a finite set, and let $f_{\mathbf{C},t} : U \to \mathbf{C}^3$ be a real stable perturbation of f. Let $Y_{\mathbf{C},t}$ and $Y_{\mathbf{R},t}$ be the images of f_t and of the restriction $f_{\mathbf{R},t}$ of $f_{\mathbf{C},t}$ to $U \cap \mathbf{R}^3$. We say that $f_{\mathbf{C},t}$ is a *good real perturbation* if $rank\ H_2(Y_{\mathbf{R},t};\mathbf{Z}) = rank\ H_2(Y_{\mathbf{C},t};\mathbf{Z})$.

In Theorem 1.1 of [12], it is shown that if $f_{\mathbf{C},t}$ is a good real perturbation then the inclusion of real image in complex image induces an isomorphism on second cohomology, so that the vanishing 2-cycles one sees in the real picture really are the complex vanishing 2-cycles. The argument is very straightforward: let g_t be a real equation for $Y_{\mathbf{C},t}$. Then $Y_{\mathbf{C},t} = \mid g_t \mid^{-1}(0) \cap B_\epsilon$, while for large enough η, $B_\epsilon = \mid g_t \mid^{-1}([0,\eta]) \cap B_\epsilon$. By Morse theory, B_ϵ is obtained from $Y_{\mathbf{C},t}$, up to homotopy, by gluing in 3-cells as η passes through positive critical values of $\mid g_t \mid$ (we may suppose that the corresponding critical points are non-degenerate). As B_ϵ is contractible, the boundaries of these 3-cells freely generate $H_2(Y_{\mathbf{C},t};\mathbf{Z})$. The 3-cells are contained in the stable manifolds of the critical points. Since the inclusion of $Y_{\mathbf{R},t}$ in $Y_{\mathbf{C},t} \cap \mathbf{R}^3$ induces an isomorphism on H_2, it follows that if f_t is a good real perturbation, then $rank\ H_2(Y_{\mathbf{C},t} \cap \mathbf{R}^3;\mathbf{Z}) = rank\ H_2(Y_{\mathbf{C},t};\mathbf{Z})$. Applying the Morse-theoretic analysis now in $\mathbf{R}^3$, we conclude from equality of the rank of the second homology that each of the critical points of $\mid g_t \mid$ in B_ϵ is real, and that each stable manifold is contained in $\mathbf{R}^3$. Thus the same 3-cells kill the homotopy of both $Y_{\mathbf{C},t} \cap \mathbf{R}^3$ and $Y_{\mathbf{C},t}$. This implies that the inclusion of the former in the latter is a homotopy equivalence.

The main technical tool we shall use in classifying germs with good real perturbations is a spectral sequence which relates the cohomology of the image of a mapping to the *alternating cohomology* of its multiple point spaces, described in [5], and which we now recall. If $f : X \to Y$ is a surjective finite-to-one map of topological spaces, we define $D^k(f)$ to be the closure, in X^k, of the set $\{(x_1,...,x_k) : x_i \neq x_j \text{ if } i \neq j, f(x_i) = f(x_j) \forall i,j\}$. The symmetric group S_k acts on $D^k(f)$, permuting the copies of X, and we denote by $Alt_k H^q(D^k(f);\mathbf{Q})$ the subspace of $H^q(D^k(f);\mathbf{Q})$ on which S_k acts via its sign representation:

$$Alt_k H^q(D^k(f);\mathbf{Q}) = \{c \in H^q(D^k(f);\mathbf{Q}) : \sigma \cdot c = sign(\sigma)c\ \forall \sigma \in S_k\}.$$

Then the main result of Section 2 of [5] is the following:

Theorem 2.1 *There is a spectral sequence with* $E_1^{p,q} = Alt_q H^{p+1}(D^q(f);\mathbf{Q})$, *converging to* $H^{p+q}(Y;\mathbf{Q})$.

In order to use this result, we need an elementary observation:

Lemma 2.2 *If $X_1 \subset X_2$, $Y_1 \subset Y_2$, $f_i : X_i \to Y_i$ is surjective, and the f_i together with the inclusions form a commutative diagram, then the inclusions $D^k(f_1) \subset D^k(f_2)$ and $Y_1 \subset Y_2$ induce a morphism of spectral sequences.*

Proof Denote by j_k the inclusion $X_1^k \hookrightarrow X_2^k$; for each $\sigma \in S_k$, we have (as maps on X_1^k) $\sigma \circ j_k = j_k \circ \sigma$, so that on $H^q(D^k(f_2); \mathbf{Q})$, we have $H^q(j_k) \circ H^q(\sigma) = H^q(\sigma) \circ H^q(j_k)$. Now define the operator Alt_k by

$$Alt_k = (1/k!) \sum_{\sigma \in S_k} sgn(\sigma) H^q(\sigma).$$

Then Alt_k is an idempotent operator on $H^q(D^k(f_i); \mathbf{Q})$, with image $Alt_k H^q(D^k(f_i); \mathbf{Q})$. It follows immediately from the commutativity relations that $H^q(j_k)$ maps $Alt_k H^q(D^k(f_2); \mathbf{Q})$ to $Alt_k H^q(D^k(f_1); \mathbf{Q})$. $\square$

In what concerns us, X_2 will of course be the domain U of the stable perturbation $f_{\mathbf{C},t}$ and X_1 will be $U \cap \mathbf{R}^2$.

Theorem 2.3 *If $f_{\mathbf{C},t}$ is a good real perturbation of $f : \mathbf{C}^2, S \to \mathbf{C}^3, 0$, then the inclusion of real spaces in complex spaces induces an isomorphism of the spectral sequences at E_1: for all p and q*

$$H^q(j_k) : Alt_p H^q(D^p(f_t); \mathbf{Q}) \to Alt_p H^q(D^p(f_{\mathbf{R},t}); \mathbf{Q})$$

is an isomorphism.

Proof As decribed above, the inclusion j of real in complex induces an isomorphism $j^* : H^2(Y_{\mathbf{C}}) \simeq H^2(Y_{\mathbf{R}})$. In order to show that this induces an isomorphism of spectral sequences, we examine the E_2 term of each. Since at E_1 the only non-zero terms are

$$\begin{array}{ll} E_1^{0,0} & = H^0(U; \mathbf{Q}) \\ E_1^{1,0} & = Alt_2 H^0(D^2(f_t); \mathbf{Q}) \\ E_1^{1,1} & = Alt_2 H^1(D^2(f_t); \mathbf{Q}) \\ E_1^{2,0} & = Alt_3 H^0(D^3(f_t); \mathbf{Q}) \end{array},$$

the spectral sequences both collapse at E_2, and thus the isomorphism $H_2(Y_{\mathbf{C}}) \simeq H_2(Y_{\mathbf{R}})$ restricts to an injection $E_{2,\mathbf{C}}^{2,0} \hookrightarrow E_{2,\mathbf{R}}^{2,0}$. On the other hand $E_2^{2,0}$ (real or complex) is a quotient of $E_1^{2,0} = Alt_3 H^0(D^3(f_t); \mathbf{Q})$, and since each real triple point is also a complex triple point, the inclusion of real in complex induces an *epimorphism* $Alt_3H^0(D^3(f_{\mathbf{C},t});\mathbf{Q}) \to Alt_3H^0(D^3(f_{\mathbf{R},t});\mathbf{Q})$. It follows that the map $E^{2,0}_{2,\mathbf{C}} \to E^{2,0}_{2,\mathbf{R}}$ is an isomorphism. Since

$$H^2(Y;\mathbf{Q})/E_2^{2,0} \simeq E_2^{1,1} = Alt_2(H^1(D^2(f_t);\mathbf{Q})$$

in both spectral sequences, it follows that j^* induces an isomorphism

$$Alt_2(H^1(D^2(f_{\mathbf{C},t});\mathbf{Q}) \simeq Alt_2(H^1(D^2(f_{\mathbf{R},t});\mathbf{Q}).$$

This shows that j^* induces isomorphisms on all terms $E_2^{p,q}$ with $p+q=2$. For the remaining terms in E_2, we note that by Lemma 2.2 of [12], the inclusion of $Y_{\mathbf{C}} \cap \mathbf{R}^3$ in $Y_{\mathbf{C}}$ is a homotopy equivalence. Now the complement of $Y_{\mathbf{R}}$ in $Y_{\mathbf{C}} \cap \mathbf{R}^3$ consists of a finite collection of "whiskers", real curves with one or both ends at cross-caps in the image. We claim that removing these affects neither H^0 nor H^1. For the latter, this is clear: $Y_{\mathbf{C}} \cap \mathbf{R}^3$ is already simply connected; for H^0, the argument is scarcely more complicated; we prove it with a lemma which we will use again later:

Lemma 2.4 *Under the hypothesis of Theorem 2.3, each pair of irreducible components of* $Y_{\mathbf{R}}$ *meet in* $\mathbf{R}^3$.

Proof Let $U_1, ..., U_r$ be the connected components of the domain of $f_{\mathbf{C},t}$ corresponding to the distinct points of the set S at which f is defined. If two components of the image of $f_{\mathbf{R}}$ can be pulled apart in a perturbation, they cannot be transverse. Then because of the (non-transverse) contact of the (complex) images of $f_{\mathbf{C}}(\mathbf{C}^2, x_i)$ and $f_{\mathbf{C}}(\mathbf{C}^2, x_j)$, the homology group $H^1(D^2(f_{\mathbf{C},t}) \cap (U_i \times U_j);\mathbf{Q})$ is non-trivial. This group is isomorphic to a direct summand of $Alt_2H^1(D^2(f_{\mathbf{C},t})$, namely the subspace of

$$H^1(D^2(f_{\mathbf{C},t}) \cap (U_i \times U_j);\mathbf{Q}) \oplus H^1(D^2(f_{\mathbf{C},t}) \cap (U_j \times U_i);\mathbf{Q})$$

consisting of elements $(c, -\sigma^*(c))$, (where σ is the generator of $S_2 = \mathbf{Z}_2$) which must be mapped by the isomorphism j^* to the corresponding summand of $Alt_2H^1(D^2(f_{\mathbf{R},t});\mathbf{Q})$. Therefore this summand is not zero, and so $f_{\mathbf{R},t}(U_i \cap \mathbf{R}^2) \cap f_{\mathbf{R},t}(U_j \cap \mathbf{R}^2) \neq \emptyset$. $\square$

This completes the proof that j^* induces an isomorphism of the spectral sequences at E_2. It follows that it induces an isomorphism at E_1 also. To see this, note that since in both spectral sequences $E_2^{1,1} = E_1^{1,1}$, it is necessary only to show that the quasi-isomorphism induced by j^* from the complex $E_{1,\mathbf{C}}^{\cdot,0}$ to the complex $E_{1,\mathbf{R}}^{\cdot,0}$ is an isomorphism. We prove this by the 5-lemma: adding to the complex $E_1^{\cdot,0}$ (real and complex) the kernel of the first arrow and the cokernel of the last arrow, we obtain two 5-term exact sequences (recall that in both spectral sequences $E_2^{1,0} = 0$) with a morphism between them. In fact, proceeding from left to right, the first, second and fifth arrows are isomorphisms, and the fourth ($E_{1,\mathbf{C}}^{2,0} \to E_{1,\mathbf{R}}^{2,0}$) is an epimorphism. We claim that the third is a monomorphism (which will be enough to finish the proof). In fact this follows immediately from the lemma above: $E_{1,\mathbf{C}}^{1,0} = Alt_2 H^0(D^2(f_{\mathbf{C},t};\mathbf{Q}))$ is a direct sum of 1-dimensional spaces generated by alternating differences of characteristic functions $\chi_{i,j} =: \chi_{D^2(f_t)\cap U_i\times U_j} - \chi_{D^2(f_t)\cap U_j\times U_i}$; the morphism of spectral sequences respects this direct sum structure. If $\chi_{i,j}$ is sent to zero, then the images of $U_i \cap \mathbf{R}^2$ and $U_j \cap \mathbf{R}^2$ do not meet, and this contradicts the lemma. $\square$

Corollary 2.5 *With the hypothesis of the previous theorem, we have*

i) all complex triple points and cross-caps are real i.e. are triple points and cross-caps of $f_{\mathbf{R},t}$;

ii) for each pair x_i, x_j of distinct members of S, we have $\mu(D^2(f) \cap (\mathbf{C}^2, x_i) \times (\mathbf{C}^2, x_j)) \leq 1$;

iii) for each point x_i of S, if f_i is the germ of f at x_i, then $\mu(D^2(f_i)) \leq 1$. Hence the germ of f_i is either an immersion, or is equivalent to one of

cross-cap	$(x,y) \to (x, y^2, xy)$
S_1	$(x,y) \to (x, y^2, y^3 + x^2 y)$
$H_k (k \geq 2)$	$(x,y) \to (x, xy + y^{3k-1}, y^3)$.

Proof i) This is immediate from the theorem ii) As described in the proof of the theorem, the isomorphism

$$Alt_2(H^1(D^2(f_{\mathbf{C},t});\mathbf{Q}) \simeq Alt_2(H^1(D^2(f_{\mathbf{R},t});\mathbf{Q}).$$

induces isomorphisms

$$H^1(D^2(f_{\mathbf{C},t}) \cap U_i \times U_j) \simeq H^1(D^2(f_{\mathbf{R},t}) \cap U_i \cap \mathbf{R}^2 \times U_j \cap \mathbf{R}^2)).$$

The rank of the latter group is simply the number of connected components of $D^2(f_{\mathbf{R},t}) \cap ((U_i \cap \mathbf{R}^2) \times (U_j \cap \mathbf{R}^2))$ diffeomorphic to the circle S^1. Now by Smith theory ([2]), which can be applied as $D^2(f_{\mathbf{R},t}) \cap (U_i \cap \mathbf{R}^2) \times (U_j \cap \mathbf{R}^2))$ is the fixed set of the $\mathbf{Z}_2$ action on $D^2(f_{\mathbf{C},t}) \cap (U_i \times U_j))$ generated by complex conjugation, the sum of the Betti numbers of the space $D^2(f_{\mathbf{R},t}) \cap (U_i \cap$

$\mathbf{R}^2) \times (U_j \cap \mathbf{R}^2))$ is less than or equal to the sum of the Betti numbers of the space $D^2(f_{\mathbf{C},t}) \cap (U_i \times U_j))$. Writing $\mu_{i,j}$ for $\mu(D^2(f) \cap (\mathbf{C}^2, x_i) \times (\mathbf{C}^2, x_j))$, which is of course equal to the rank of $H^1(D^2(f_{\mathbf{C},t}) \cap U_i \times U_j)$, we deduce that $2\mu_{i,j} \leq 1 + \mu$, and so $\mu_{i,j} \leq 1$.

iii) The space $D^2(f_{\mathbf{C},t}) \cap U_i \times U_i$ is the Milnor fibre of the isolated curve singularity $(D^2(f_i), 0)$ and hence is connected (cf [3]). It follows that its 0'th cohomology has no alternating part. The same must therefore be true of $D^2(f_{\mathbf{R},t}) \cap (U_i \cap \mathbf{R}^2 \times U_i \cap \mathbf{R}^2)$. Hence this space is a collection of (disjoint) components homeomorphic to circles and intervals, each of which is mapped to itself by the involution which exchanges copies of U_i. On each of these circles the involution is either reflection in a diameter, or the antipodal map. Suppose that on c of the circles, the involution consists of reflection in a diameter, and that the number of intervals is r. Then the involution has $2c + r$ fixed points. These fixed points map of course to Whitney umbrella points in the real image. Let W_i be the number of Whitney umbrellas in the complex image of U_i. Again, each complex Whitney umbrella corresponds to a fixed point of the involution on $D^2(f_{\mathbf{C},t}) \cap (U_i \times U_i)$. By the Lefschetz fixed point theorem, this number is equal to $\sum_k (-1)^k Trace(H^k(j))$. As $D^2(f_{\mathbf{C},t}) \cap (U_i \times U_i)$ is connected, $Trace(H^0(j)) = 1$; and since $H^1(D^2(f_{\mathbf{C},t}) \cap (U_i \times U_i); \mathbf{Q})$ decomposes as a direct sum of eigenspaces of $H^1(j)$, $H^1_+ \oplus H^1_-$, we have

$$\begin{aligned} Trace(H^1(j)) &= dim(H^1_+) - dim(H^1_-) \\ &= dim(H^1(D^2(f_{\mathbf{C},t}) \cap U_i \times U_i; \mathbf{Q}) - 2dim(H^1_-) \\ &= \mu(D^2(f_i)) - 2c. \end{aligned}$$

Thus, $W_i = 1 - \mu(D^2(f_i)) + 2c$ and hence since each real fixed point is also a complex fixed point, we have $2c + r \leq 1 - \mu(D^2(f_i)) + 2c$, and therefore $\mu(D^2(f_i)) \leq 1$.

The remaining assertion follows from the classification (see [10]): if $\mu(D^2(f_i)) = 0$ then $D^2(f_i)$ is smooth and so f_i must be a cross-cap; if $\mu(D^2(f_i)) = 1$ then $D^2(f_i)$ must be a plane curve singularity, so the 2-jet of f_i is equivalent either to $(x,y) \to (x, xy, 0)$ or to $(x,y) \to (x, y^2, 0)$; in the former case, $D^2(f_i)$ can only have Milnor number 1 if the 3-jet of f_i is equivalent to $(x,y) \to (x, xy, y^3)$, from which it follows that f_i is of type H_k; in the latter case, f_i is equivalent to a germ of the form $(x,y) \to (x, y^2, yp(x, y^2))$, and then $D^2(f_i)$ is isomorphic to the plane curve with equation $p(x, y^2) = 0$; as $\mu(D^2(f_i) = 1$ then $p(x, y^2)$ must define a Morse singularity, and it follows that it is equivalent to $x^2 + y^2$, from which f_i is equivalent to the singularity S_1. $\square$

3 Good real perturbations of S_1 and H_k

Recall that H_k is defined by the polynomial map

$$f_0(x,y) = (x, y^3, xy + y^{3k-1}).$$

If we take $k = 1$ we obtain a singularity equivalent to S_1, and this fact enables us to treat S_1 as one of the H series, which we now do. Consider the perturbation f_t of this map given by

$$(x,y) \to (x, y^3 - ty, xy + y^2(\prod_{i=1}^{k-1}(y^3 - ty - c_i))$$

where $t > 0$ and the c_i are real numbers with $\mid c_i \mid < \frac{2}{3}t^{\frac{3}{2}}$, and $c_i \neq c_j$ for $i \neq j$. Clearly this reduces to the original map f when $t = 0 = c_i$ for all i. We claim that

i) $D^2(f_{\mathbf{R},t})$ is homeomorphic to a circle, and the involution generating the $\mathbf{Z}_2$-action is conjugate to a reflection in a diameter, and

ii) $f_{\mathbf{R},t}$ has $k-1$ ordinary triple points.

To see that (i) holds, recall that as defining equations for $D^2(f_{\mathbf{C},t})$ in $\mathbf{C}^3 = x, y_1, y_2$-space we can take the quotient, after division by $y_1 - y_2$, of the difference between the values at (x, y_1) and (x, y_2) of each of the second and third components of $f_{\mathbf{C},t}$; it is easy to see from these equations that $D^2(f_{\mathbf{C},t})$ is smooth. As $D^2(f_{\mathbf{R}})$ is empty (if one defines it as the closure of the set of genuine double points, or just a point, if one uses instead the non-reduced structure given by these equations), it follows that $D^2(f_{\mathbf{R},t})$ is compact, and thus a disjoint union of circles. As the Milnor number of $D^2(f_{\mathbf{C}})$ is 1, there can be only one of these.

The assertion about the involution follows from the fact that it has two fixed-points - the two non-immersive points of $f_{\mathbf{R},t}$, which we leave the reader to locate.

To find the the triple points of $f_{\mathbf{R},t}$, we proceed as follows: let $h : \mathbf{R} \to \mathbf{R}$ be defined by $h(y) = y^3 - ty$, with $t > 0$. The assumption on c_i guarantees that it lies between the local maximum and local minimum values of h, and so the equation $y^3 - ty - c_i = 0$ has three distinct real roots, say $y_{i,1}, y_{i,2}, y_{i,3}$. Then the three points $(0, y_{i,1}), (0, y_{i,2}), (0, y_{i,3})$ have the same image $(0, c_i, 0)$ under f_t, and thus there are $k-1$ triple points. They must be ordinary: $D^3(f_t)$ consists of $6(k-1)$ points (one faithful S_3 orbit for each triple point) and so must be smooth, since $6(k-1)$ is the multiplicity of the (non-reduced) singularity of $D^3(f)$ at 0 (defined by divided difference polynomials, cf [8]).

Proposition 3.1 *The image X_t of $f_{\mathbf{R},t}$ has the following rational cohomology groups:*

$$H^0(X_t) = \mathbf{Q}, H^1(X_t) = 0, H^2(X_t) = \mathbf{Q}^k,$$

so that $f_{\mathbf{R},t}$ is a good real perturbation of f_0.

Proof We use the spectral sequence described in section 2. We have

$$H^0(\mathbf{R}^2) = \mathbf{Q}, Alt_2 H^1(D^2(f_t)) = \mathbf{Q}, Alt_3 H^0(D^3(f_t)) = \mathbf{Q}^{k-1},$$

and all other terms at E_1 are zero. It follows that the spectral sequence collapses at E_1, and the proposition follows. □

Figure 1: Source double point curves $D_1^2(f_{\mathbf{R},t})$ for good perturbations of H_k, for $1 \leq k \leq 4$

Observe that the double curve $D_1^2(f_{\mathbf{R},t}) \subset \mathbf{R}^2$ is a real Morsification of the A_{6k-5} singularity of $D_1^2(f)$ at 0 (Figure 1); for it is a compact curve with $3(k-1)$ nodes (the double points of the projection $D^2(f_{\mathbf{R},t}) \to \mathbf{R}^2$ correspond of course to the triple points of $f_{\mathbf{R},t}$).

Figure 2: Image double curves for good real perturbations of H_k, for $1 \leq k \leq 5$

This curve is mapped 2-to-1 by $f_{\mathbf{R},t}$, except for the non-immersive points, which have no companion with the same image, and the nodes, which are mapped 3-to-1. It is a pleasant exercise to determine where on $D_1^2(f_{\mathbf{R},t})$ the two non-immersive points must be located; up to obvious symmetries, there is only one correct answer, consistent with the identifications which must take place.

Figure 2 shows the image double curve $f_{\mathbf{R},t}(D_1^2(f_{\mathbf{R},t}))$ for the cases $1 \leq k \leq 5$, and Figure 3 shows the image X_t being built up from prefabricated components - triple points and cross-caps.

Figure 3: Images of good perturbations of H_k, $1 \leq k \leq 3$

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