

Logarithmic vector fields and the Severi strata in the discriminant

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Abstract. The discriminant, D , in the base of a miniversal deformation of an irreducible plane curve singularity, is partitioned according to the genus of the (singular) fibre, or, equivalently, by the sum of the delta invariants of the singular points of the fibre. The members of the partition are known as the *Severi strata*. The smallest is the δ -constant stratum, $D(\delta)$, where the genus of the fibre is 0. It is well known, by work of Givental' and Varchenko, to be Lagrangian with respect to the symplectic form Ω obtained by pulling back the intersection form on the cohomology of the fibre via the period mapping. We show that the remaining Severi strata are also co-isotropic with respect to Ω , and moreover that the coefficients of the expression of Ω with respect to a basis of $\Omega^2(\log D)$ are equations for $D(\delta)$. Similarly, the coefficients of $\Omega^{\wedge k}$ with respect to a basis for $\Omega^{2k}(\log D)$ are equations for $D(\delta - k + 1)$. These equations allow us to show that for E_6 and E_8 , $D(\delta)$ is Cohen-Macaulay (this was already shown by Givental' for A_{2k}), and that, as far as we can calculate, for A_{2k} all of the Severi strata are Cohen-Macaulay.

1. Introduction: the discriminant and its Severi strata

Two of the most basic invariants of a plane curve singularity $(C, 0)$ are its *Milnor number* μ and its *delta invariant* δ . If $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is a holomorphic map defining $(C, 0) = f^{-1}(0)$, then $\mu(C)$ is the dimension of the jacobian algebra $\mathcal{O}_{\mathbb{C}^2, 0}/J_f$ and equals the dimension of the vanishing cohomology. If $n : \tilde{C} \rightarrow C$ denotes the normalisation of $(C, 0)$, then $\delta(C)$ is the dimension $n_* \mathcal{O}_{\tilde{C}} / \mathcal{O}_C$ and equals the number of double points appearing in a generic perturbation of the map n . These invariants are related by the relation

$$\mu = 2\delta + r - 1$$

where r denotes the number of branches of $(C, 0)$. The number μ also appears as the number of parameters of an \mathcal{R}_e miniversal deformation $F : (\mathbb{C}^2 \times \mathbb{C}^\mu, 0) \rightarrow (\mathbb{C}, 0)$ of the function $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ defining $(C, 0)$. The restriction $\pi : X :=$

$F^{-1}(0) \rightarrow S = (\mathbb{C}^\mu, 0)$ is a versal deformation of the plane curve singularity $(C, 0)$. The fibre C_u over $u \in S$ is the curve defined by zero level of the deformed function $f_u := F(\cdot, u)$ and *discriminant* $D \subset S$ is the set of parameter values u for which the fibre C_u is singular. This set is stratified by the types of singularities that appear in the fibres. In this paper we will focus on the so-called *Severi strata*, where the sum of the delta-invariants add up to a value $\geq k$:

$$D(k) = \{u \in S : \delta(C_u) \geq k\}$$

where $\delta(C_u) = \sum_{x \in C_u} \delta(C_u, x)$. Clearly $D(0) = S$ and $D(1) = D$, and as $D(i)$ is contained in $D(i-1)$ we obtain a chain

$$D(\delta) \subset D(\delta-1) \subset \dots \subset D(1) \subset D(0) = S$$

The smallest non-empty Severi stratum, $D(\delta)$, is the classical δ -*constant stratum*. The term “stratum” here is a bit of a misnomer, since the Severi strata are not in general smooth.

It is a classical fact, going back at least to Cayley [Cay66], that any curve singularity with $\delta = k$ can be deformed into a curve with precisely k A_1 points, a fact which explains the name *virtual number of double points* for the δ -invariant. For a very nice proof see the paper by C. Scott [Sco92]. Thus the set $D(kA_1)$ of parameter values u for which C_u has precisely k A_1 singularities is *dense* in $D(k)$. Moreover, $D(k)$ is smooth at such points, for there, by openness of versality, $D(k)$ is a normal crossing of k local smooth components of the discriminant D . A curve singularity with δ -invariant $k > 1$ is also adjacent to a curve with one A_2 singularity and $k-1$ A_1 singularities. Hence $D(k)_{\text{reg}} = D(kA_1)$. We refer to [Tei80] for more background on this.

In the famous Anhang F to his *Vorlesungen über Algebraische Geometrie* [Sev21], Severi considered the variety of plane curves of degree d with a given number of nodes which he used to argue for the irreducibility of the space of all curves of a given genus. A complete argument along these lines with given much later by J. Harris, [Har86], and by Harris and Diaz in [DH88], which started the interest in the local case. This seems to justify the name *Severi-strata* for the $D(k)$'s, which was introduced in [She12]. Recently, these strata have been the subject of several papers and their geometry appears to hide some great mysteries. In [FGvS99] the multiplicity of $D(\delta)$ was shown to be equal to the Euler number of the compactified Jacobian of $(C, 0)$. This was further explored in [She12], where multiplicities of the other $D(k)$ were related to the puntual Hilbert-schemes $\text{Hilb}^n(C, 0)$. Most surprisingly, these invariants turn out to be related to the HOMFLY-polynomial of the knot in the 3-sphere defined by $(C, 0)$, [OS12].

If the curve $(C, 0)$ is irreducible, its Milnor fibre C_u has just one boundary component, and it follows that the intersection form I_u on $H^1(C_u; \mathbb{C})$ is non-degenerate. In [GV82], Givental' and Varchenko used the period map associated to a non-degenerate 1-form η on the total space of the Milnor fibration of F , together with the Gauss-Manin connection, to pull back the intersection form

from the cohomology bundle \mathcal{H}^* over S to get a symplectic form Ω on $S \setminus D$, and proved

Theorem 1.1. (a) Ω extends to a symplectic form on S , and
 (b) the δ -constant stratum $D(\delta)$ in the discriminant is Lagrangian with respect to Ω .

Below we complement their results and show the following theorems.

Theorem 1.2. All of the Severi strata are coisotropic with respect to Ω .

The form Ω can also be used to obtain equations defining the Severi-strata. Let $\wedge^k \Omega$ be the k -fold wedge product of Ω . Although it is a regular form, it can be considered as an element of $\Omega_S^{2k}(\log D)$. Let I_k be the ideal generated by its coefficients with respect to a basis of $\Omega_S^{2k}(\log D)$.

Theorem 1.3. For $k = 1, \dots, \delta$, the Severi stratum $D(k)$ is defined by the ideal $I_{\delta-k+1}$:

$$D(k) = V(I_{\delta-k+1}).$$

Equivalently, if χ_1, \dots, χ_μ form a basis for the free module of logarithmic vector fields $\Theta_S(-\log D)$, then $D(k)$ is defined by the ideal generated by the Pfaffians of size $2\delta - 2k + 2$ of the skew matrix $(\Omega(\chi_i, \chi_j))_{1 \leq i, j \leq \mu}$.

We do not know whether in general the ideals I_k are radical. Our calculations suggest that they are, but we have not been able to show this.

Givental' proved in [Giv88] that for curve singularities of type A_{2k+1} , $D(\delta)$ is Cohen-Macaulay and it can be conjectured that this is always the case, [vS06]. In the first author's PhD thesis, [Cad11], Theorem 1.3 was used to show that $D(\delta)$ is Cohen Macaulay also for E_6 and E_8 . Calculations using Theorem 1.3 suggest that the remaining Severi strata are Cohen-Macaulay in the case of A_{2k} , but show that for E_6 the stratum $D(2)$ is not Cohen-Macaulay.

In the process of proving these theorems we noticed that Ω determines a natural rank 2 maximal Cohen-Macaulay module over the discriminant D , which seems to be of independent interest.

2. Preliminaries

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ define an isolated singularity $(C, 0)$ and let

$$g_1, g_2, \dots, g_\mu = 1 \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$$

be functions giving a basis for the jacobian algebra \mathcal{O}/J_f . We consider a good representative of a miniversal deformation of f of the form

$$F : B \times S \rightarrow \mathbb{C}, \quad F(x, u) = f(x) + \sum_{i=1}^{\mu} u_i g_i(x),$$

where B is a Milnor ball for C and S is a sufficiently small ball in \mathbb{C}^μ , [Loo84]. The set $X := F^{-1}(0)$ comes with a map $\pi : X \rightarrow S$, with C_u as fibre over $u \in S$.

2.1. The critical space Σ

The relative critical set Σ of F is defined to be

$$\Sigma = \left\{ (x, u) \in B \times S : \frac{\partial F}{\partial x_i}(x, u) = 0, i = 0, \dots, n \right\}.$$

It is smooth and the projection $\pi : \Sigma \rightarrow S$ is a μ -fold branched cover: its fibre over $u \in S$ consists of the critical points of $F(-, u)$. As the partial derivatives form a regular sequence,

$$\mathcal{O}_\Sigma = \mathcal{O}_{B \times S} / (\partial F / \partial x_0, \dots, \partial F / \partial x_n)$$

is a free \mathcal{O}_S -module of rank μ . Miniversality of F is equivalent to the statement that the Kodaira-Spencer map

$$dF : \Theta_S \rightarrow \mathcal{O}_\Sigma, \quad \vartheta \mapsto \vartheta(F) = dF(\vartheta)$$

is an isomorphism. The set $X \cap \Sigma$ is the union over $u \in S$ of the set of singular points of C_u , and its image under π is the discriminant, D . For brevity we denote $X \cap \Sigma$ by \tilde{D} . It is indeed the normalisation of D .

2.2. D as a free divisor

Let $\bar{F} : (B \times S, (0, 0)) \rightarrow (\mathbb{C} \times S, (0, 0))$ be the unfolding of f corresponding to the deformation F . Then $\Sigma \subset B \times \mathbb{C}^\mu$ is the (absolute) critical locus of \bar{F} . We write $\Delta = \bar{F}(\Sigma) \subset \mathbb{C} \times S$ for the set of critical values of \bar{F} . It is well known that Σ is the normalisation of Δ : it is smooth, and the map $\bar{F}| : \Sigma \rightarrow \Delta$ is generically one-to-one. Then $D = \Delta \cap \{0\} \times S$. As usual, $\Theta_{\mathbb{C} \times S}(-\log \Delta)$ denotes the $\mathcal{O}_{\mathbb{C} \times S}$ -module of vector fields on $\mathbb{C} \times S$ which are tangent to Δ , and $\Theta_S(-\log D)$ denotes the \mathcal{O}_S -module of vector fields on S which are tangent to D .

Proposition 2.1. (i) $\Theta_{\mathbb{C} \times S}(-\log \Delta)$ is the $\mathcal{O}_{\mathbb{C} \times S}$ -module of vector fields on $\mathbb{C} \times S$ which are \bar{F} -liftable to $B \times S$.

(ii) $\Theta_S(-\log D)$ is the \mathcal{O}_S -module of vector fields on S which are π -liftable to $V(F)$.

Proof. ([Loo84]) (i) Let $\vartheta \in \Theta_{\mathbb{C} \times S}(-\log \Delta)$. Since $F| : \Sigma \rightarrow \Delta$ is the normalisation of Δ , there is a vector field $\tilde{\vartheta}_0$ on Σ lifting ϑ . For any extension $\tilde{\vartheta}_1$ of $\tilde{\vartheta}_0$ to $B \times S$, $\omega F(\tilde{\vartheta}) - tF(\tilde{\vartheta}_1)$ vanishes on Σ , and since the jacobian ideal $(\partial F / \partial x_0, \dots, \partial F / \partial x_n)$ is radical, there exists a second vector field ξ on $B \times S$ such that $\omega F(\tilde{\vartheta}_1) - tF(\tilde{\vartheta}_1) = tF(\xi)$. Then $\tilde{\vartheta}_1 + \xi$ is an \bar{F} -lift of ϑ .

Conversely, suppose $\tilde{\vartheta}$ is a \bar{F} -lift of ϑ . Then $\tilde{\vartheta}$ must be tangent to Σ , for the integral flows $\tilde{\Phi}_t$ and Φ_t of $\tilde{\vartheta}$ and ϑ satisfy $\Phi_1 \circ \bar{F} = \bar{F} \circ \tilde{\Phi}_1$, showing that $\tilde{\Phi}_t$ defines an isomorphism $\bar{F}^{-1}(u) \rightarrow \bar{F}^{-1}(\Phi_t(u))$, and must therefore map singular points of $\bar{F}^{-1}(u)$ to singular points of $\bar{F}^{-1}(\Phi_t(u))$. It follows that $\tilde{\vartheta}$ is tangent to Δ .

(ii) Let $i_0 : S \rightarrow \mathbb{C} \times S$ be the inclusion $u \mapsto (0, u)$. Then $D = i_0^{-1}(\Delta)$. Now i_0 is logarithmically transverse to Δ , i.e. transverse to the distribution $\Theta_{\mathbb{C} \times S}(-\log \Delta)$. If F is the standard deformation $f(x) + \sum_i u_i g_i$, with $g_\mu = 1$, then this transversality is obvious: the vector field $\partial / \partial t + \partial / \partial u_\mu$ on $\mathbb{C} \times S$ has \bar{F} -lift $\partial / \partial u_\mu$, and

therefore lies in $\Theta_{\mathbb{C} \times S}(-\log \Delta)$. Any other miniversal deformation is parametrised \mathcal{R} -equivalent to the standard deformation, so the transversality holds there too.

Identifying \mathbb{C}^μ with $\{0\} \times \mathbb{C}^\mu$, from the logarithmic transversality of i_0 to Δ it follows that $\Theta_S(-\log D)$ is equal to $\theta_{\mathbb{C} \times S}(-\log \Delta) \cap \theta_{\mathbb{C} \times S}(-\log(\{0\} \times S))$ restricted to \mathbb{C}^μ , and that every vector field in $\Theta_S(-\log D)$ extends to a vector field in $\Theta_{\mathbb{C} \times S}(-\log \Delta)$. Clearly, any lift to $\mathbb{C}^{n+1} \times S$ of a vector field in $\theta_{\mathbb{C} \times S}(-\log \Delta) \cap \theta_{\mathbb{C} \times S}(-\log(\{0\} \times S))$ must be tangent to $V(F)$, and any vector field whose \bar{F} -lift is tangent to $V(F)$ must lie in $\theta_{\mathbb{C} \times S}(-\log \Delta) \cap \theta_{\mathbb{C} \times S}(-\log(\{0\} \times S))$.

□

Therefore we have a diagram

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \mathcal{O}_{\bar{D}} & \longleftarrow & \Theta_S & \longleftarrow & \Theta_S(-\log D) \\
 & & \downarrow = & & \downarrow dF & & \downarrow \frac{dF}{\bar{F}} \\
 0 & \longleftarrow & \mathcal{O}_{\bar{D}} & \longleftarrow & \mathcal{O}_\Sigma & \xleftarrow{F} & \mathcal{O}_\Sigma \longleftarrow 0
 \end{array} \tag{2.1}$$

where the vertical maps are isomorphisms. This diagram can be used to find a basis for $\Theta_S(-\log D)$. The germs $FdF(\partial/\partial u_i)$ generate $(F)\mathcal{O}_\Sigma$, therefore if

$$dF(\chi_i) = FdF\left(\frac{\partial}{\partial u_i}\right) \tag{2.2}$$

then the χ_i generate $\Theta_S(-\log D)$. This shows that $\Theta_S(-\log D)$ is a locally free module, so that D is a free divisor.

2.3. Stratification of D

The discriminant D is stratified in various ways. Of special relevance to us are the local \mathcal{R} and \mathcal{H} strata.

Suppose as before that $F : B \times S \rightarrow \mathbb{C}$ is a good representative of a versal deformation of f , where \mathcal{B} is open in \mathbb{C}^{n+1} and S is open in \mathbb{C}^μ . Write $f_u = F(-, u)$, and suppose that p_1, \dots, p_k are the critical points of f_u lying on $f_u^{-1}(0)$. For each point p_i , the germ

$$F : (B \times S, (p_i, u)) \rightarrow (\mathbb{C}, 0)$$

is an \mathcal{R}_e -versal deformation of the germ of f_u at p_i , by openness of versality. Hence there is a germ of submersion h_i from (S, u) to the base of an \mathcal{R}_e -miniversal deformation

$$G_i : (B \times \mathbb{C}^{\mu_i}, (x_i, 0)) \rightarrow (\mathbb{C}, 0)$$

of this germ, such that the germ of deformation $F : (B \times S, (p_i, u)) \rightarrow (\mathbb{C}, 0)$ is isomorphic to $h_i^*(G_i)$. We set

$$\mathcal{R}_i(u) = h_i^{-1}(0).$$

This is independent of the choice of miniversal deformation G_i and submersion h_i , since any two choices can be related by a commutative diagram of spaces and

maps. Again by openness of versality, the $\mathcal{R}_i(u)$, $i = 1, \dots, k$ are in general position with respect to one another, and we set

$$\mathcal{R}(u) = \bigcap_{i=1}^k \mathcal{R}_i(u).$$

This is the \mathcal{R} stratum through u . It is smooth of dimension $\mu - \sum_i \mu(f_u, p_i)$.

If in the above definition we replace $F : B \times S \rightarrow \mathbb{C}$ by the projection $V(F) \rightarrow S$, and replace each G_i by a \mathcal{K}_e -miniversal deformation H_i of the hypersurface singularity (C_u, p_i) , then we obtain the \mathcal{K} -strata $\mathcal{K}_i(u)$ and their intersection $\mathcal{K}(u)$, the \mathcal{K} -stratum through u , which is once again smooth, by openness of versality, and has dimension $\mu - \sum_i \tau(C_u, p_i)$. Since $\mathcal{R} \subset \mathcal{K}$, $\mathcal{R}(u) \subset \mathcal{K}(u)$.

If, for example, the fibre C_u has k A_1 singularities and no other singular points, then $\mathcal{R}(u) = \mathcal{K}(u)$ and its germ at u coincides with the germ at u of the set of points u' such that $C_{u'}$ has k A_1 points and no other singularities.

Definition 2.2. *The logarithmic tangent space $T_u^{\log D} S$ is the vector subspace of $T_u S$ spanned at u by the logarithmic vector fields.*

Proposition 2.3. *One has the equality of vector spaces*

$$T_u^{\log D} S = T_u \mathcal{K}(u).$$

Proof. We have the exact sequence

$$0 \rightarrow \Theta_S(-\log D) \rightarrow \Theta_S \rightarrow \pi_*(\mathcal{O}_{\tilde{D}}) \rightarrow 0$$

which gives

$$\frac{\Theta_S}{\Theta_S(-\log D)} \simeq \pi_*(\mathcal{O}_{\tilde{D}})$$

and so

$$\frac{T_u \mathbb{C}^\mu}{T_u^{\log D} S} \simeq \frac{\Theta_S}{\Theta_S(-\log D) + \mathfrak{m}_{S,u} \Theta_{S,u}} \simeq \bigoplus_i T_{\mathcal{K}_e}^1(f_u, x_i)$$

This means that to show

$$T_u^{\log D} S = T_u \mathcal{K}(u)$$

we need show only one inclusion. If $\vartheta \in \Theta_S(-\log D)_u$ then it has a lift $\tilde{\vartheta}$ tangent to $V(F)$. The integral flows of ϑ and $\tilde{\vartheta}$, φ_t on (S, u) and $\tilde{\varphi}_t$ on $V(F)$, satisfy $\pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi$. It follows that $\tilde{\varphi}_t$ maps C_u to $C_{\varphi_t(u)}$, and therefore for each singular point p_i in C_u , the curve germ $\{\varphi_t(u) : t \in (\mathbb{C}, 0)\}$ lies in the set $\mathcal{K}_i(u)$ defined above. Hence $\{\varphi_t(u) : t \in (\mathbb{C}, 0)\} \subset \bigcap_i \mathcal{K}_i(u) = \mathcal{K}(u)$, and $\vartheta(0) \in T_u \mathcal{K}(u)$. \square

2.4. Isomorphism $\mathcal{O}_\Sigma \rightarrow \Omega_F$

A choice of a nowhere-vanishing relative $(n+1)$ -form $\omega \in \Omega_{B \times S/S}^{n+1}$ determines an isomorphism

$$\mathcal{O}_\Sigma \simeq \Omega_F^{n+1}, \quad g \mapsto g\omega$$

where

$$\Omega_F^{n+1} := \Omega_{B \times S/S}^{n+1} / dF \wedge \Omega_{B \times S/S}^n.$$

Such an isomorphism leads to many additional structures. First of all, there is a canonical perfect pairing, the *residue pairing*,

$$\text{Res} : \Omega_F^{n+1} \times \Omega_F^{n+1} \rightarrow \mathcal{O}_S,$$

from which one obtains a perfect pairing on \mathcal{O}_Σ .

$$\langle -, - \rangle : \mathcal{O}_\Sigma \times \mathcal{O}_\Sigma \rightarrow \mathcal{O}_S.$$

Furthermore, because Ω_S^1 and $\Omega_S^1(\log D)$ are \mathcal{O}_S -dual to Θ_S and $\Theta_S(-\log D)$, such a choice of ω also determines isomorphisms

$$\alpha : \Omega_S^1 \rightarrow \mathcal{O}_\Sigma \quad \text{and} \quad \beta : \Omega_S^1(\log D) \rightarrow \mathcal{O}_\Sigma$$

via the formulas

$$\langle dF(\vartheta), \alpha(\omega) \rangle = \omega(\vartheta), \quad \text{and} \quad \left\langle \frac{dF}{F}(\vartheta), \beta(\omega) \right\rangle = \omega(\vartheta).$$

As a result $\Theta_S, \Theta_S(-\log D), \Omega_S^1$ and $\Omega_S^1(\log D)$ are all identified with \mathcal{O}_Σ and hence with one another. For example we have the isomorphism

$$k^{-1} \circ \beta : \Omega_S^1(\log D) \rightarrow \Theta_S,$$

where $k : \Theta_S \rightarrow \mathcal{O}_\Sigma$ is the Kodaira-Spencer map dF .

Note that for any $a, b, c \in \mathcal{O}_\Sigma$, the pairing satisfies

$$\langle a, bc \rangle = \langle ab, c \rangle,$$

and so multiplication by F on \mathcal{O}_Σ is self-adjoint:

$$\langle g, Fh \rangle = \langle Fg, h \rangle.$$

As a result, if $\check{g}_i, i = 1, \dots, \mu$ denotes the \mathcal{O}_S basis of \mathcal{O}_Σ dual to the basis $g_i = \partial F / \partial u_i, i = 1, \dots, \mu$, then replacing $F dF(\partial / \partial u_i)$ in (2.2) by \check{g}_i , we get generators χ_1, \dots, χ_μ for $\Theta_S(-\log D)$ whose matrix of coefficients with respect to the $\partial / \partial u_j$ is the symmetric matrix with i, j entry $\chi_{ij} = \langle \check{g}_i, F \check{g}_j \rangle$.

In our calculations in section 7 we always use such a basis. We note that if $\omega_1, \dots, \omega_\mu$ is the basis for $\Omega^1(\log D)$ dual to χ_1, \dots, χ_μ then

$$k^{-1} \beta(\omega_i) = \frac{\partial}{\partial u_i}, \quad \text{and} \quad k^{-1} \alpha(du_i) = \chi_i. \quad (2.3)$$

3. The Gauß-Manin connection

The study of the Gauß-Manin connection for hypersurface singularities was initiated by BRIESKORN in [Bri70] and has since then developed into a very detailed theory. We can only outline the parts of the theory that are relevant to our application. For a more detailed accounts we refer to [Loo84], [AGZV88], [Kul98], [Her02] and the original papers quoted there.

3.1. The cohomology bundle and its extensions

The spaces $H^n(X_u) = H^n(X_u; \mathbb{C})$ fit together into the cohomology bundle

$$H^* = \bigcup_{u \in S \setminus D} H^n(X_u)$$

over $S \setminus D$. It is a flat vector bundle and the associated sheaf of holomorphic sections

$$\mathcal{H}^* = H^* \otimes_{\mathbb{C}} \mathcal{O}_{S \setminus D}$$

is equipped with a natural flat connection, the Gauss Manin connection,

$$\nabla : \mathcal{H}^* \rightarrow \mathcal{H}^* \otimes_{\mathcal{O}_S} \Omega_{S \setminus D}^1 \quad (3.1)$$

As usual, we write

$$\nabla_{\vartheta} : \mathcal{H}^* \rightarrow \mathcal{H}^*$$

for the action of a vector field $\vartheta \in \Theta_{S \setminus D}$. The sheaf \mathcal{H}^* over $S \setminus D$ has various extensions to S . Most relevant to us is the parameterised version of Brieskorn's module \mathcal{H}' :

$$\mathcal{H}' := \pi_*(\Omega_{X/S}^n) / d\pi_*(\Omega_{X/S}^{n-1}). \quad (3.2)$$

A section of \mathcal{H}' over $U \subset S$ is represented by a (relative) holomorphic n -form η on $\pi^{-1}(U) \subset X$. If $U \subset S \setminus D$ and $u \in U$, the restriction of η to the smooth fibre X_u is a closed form n -form and thus determines a cohomology class

$$[\eta|_{X_u}] \in H^n(X_u)$$

In this way one obtains an isomorphism $\mathcal{H}'(U) \rightarrow \mathcal{H}^*(U)$ and thus \mathcal{H}' can be considered as an extension of \mathcal{H}^* , that is, there is a map of \mathcal{O}_S -modules

$$\mathcal{H}' \rightarrow j_* \mathcal{H}^*,$$

which is an isomorphism over $S \setminus D$, where $j : S \setminus D \hookrightarrow S$ is the inclusion. The sheaf \mathcal{H}' is a locally free sheaf of rank μ , but for a general $\vartheta \in \Theta_S$ the Gauß-Manin connection maps \mathcal{H}' into a bigger extension $\mathcal{H}'' \supset \mathcal{H}'$. This second Brieskorn module \mathcal{H}'' can be defined as

$$\mathcal{H}'' := \pi_* \omega_{X/S} / d\pi_*(d\Omega_{X/S}^{n-1}),$$

where $\omega_{X/S}$ denotes the relative dualising module, [Loo84, page 158]. Elements from $\omega_{X/S}$ are most conveniently described as residues of $n+1$ -forms, that is, as Gelfand-Leray forms. There is an exact sequence

$$0 \rightarrow \mathcal{H}' \rightarrow \mathcal{H}'' \rightarrow \Omega_{X/S}^{n+1} \rightarrow 0 \quad (3.3)$$

When we restrict to logarithmic vector fields, the connection maps \mathcal{H}' and \mathcal{H}'' to themselves, so we have logarithmic connections

$$\nabla : \mathcal{H}' \rightarrow \mathcal{H}' \otimes_{\mathcal{O}_S} \Omega_S^1(\log D)$$

$$\nabla : \mathcal{H}'' \rightarrow \mathcal{H}'' \otimes_{\mathcal{O}_S} \Omega_S^1(\log D)$$

extending the Gauss-Manin connection (3.1). (As there is no possibility of confusion, we denote all these maps by the same symbol ∇)

The action of $\chi \in \Theta_S(-\log D)$ on a local section $[\eta]$ represented by a relative n -form η is given by the Lie derivative with respect to a lift $\tilde{\chi}$ of χ :

$$\nabla_{\chi}\eta = [Lie_{\tilde{\chi}}(\eta)]$$

([Loo84, p. 148]).

3.2. \mathcal{H}' and the cohomology of singular fibres

We have seen that for $u \in S \setminus D$, the restriction of a global relative n -form η to a smooth fibre X_u determines a cohomology class

$$[\eta|_{X_u}] \in H^n(X_u)$$

If $u \in D$ then the fiber X_u is singular, but the form η still can be integrated over n -cycles in X_u and gives rise to a well defined cohomology class in $H^n(X_u)$. We sketch the argument. Suppose γ_1 and γ_2 are n -cycles in C_u and Γ is a $n+1$ -chain in X_u with $\partial\Gamma = \gamma_1 - \gamma_2$. After subdivision, we can write $\Gamma = \Gamma' + \Gamma''$ where Γ' is a $n+1$ -chain in the smooth part of C_u and $\Gamma'' = \Gamma \cap \bigcup_i B_\varepsilon(p_i)$, where the p_i are the singular points of C_u . Then

$$\int_{\gamma_1} \eta - \int_{\gamma_2} \eta = \int_{\partial\Gamma'} \eta + \int_{\partial\Gamma''} \eta.$$

The first integral on the right hand side vanishes by Stokes's Theorem. The contribution $\int_{\partial\Gamma''} \eta$ tends to 0 as $\varepsilon \rightarrow 0$, as the integrand is regular and one is integrating over ever smaller sets.

In general, if Z is any analytic space with singularities we can look at the de Rham complex (Ω_Z^\bullet, d) of Kähler forms, and integration over p -cycles is well-defined and determines a *de Rham evaluation map*

$$DR : H^p(\Gamma(Z, \Omega_Z^\bullet)) \rightarrow H^p(Z, \mathbb{C})$$

If Z is a Stein space, then this map is even *surjective*. The reason is the following: because Z is Stein, the group at the left hand side is equal to the p -th hypercohomology group \mathbb{H}^p of the de Rham complex (Ω_Z^\bullet, d) . The map of complexes $\mathbb{C}_Z \rightarrow (\Omega_Z^\bullet, d)$ (induced by the inclusion map $\mathbb{C}_Z \rightarrow \mathcal{O}_Z$) induces a map

$$\alpha : H^p(Z; \mathbb{C}) = \mathbb{H}^p(\mathbb{C}_Z) \rightarrow \mathbb{H}^p((\Omega_Z^\bullet, d)) = H^p(\Gamma(Z, \Omega_Z^\bullet))$$

and it is shown in [Loo84], p.141, that DR is a *section* of the map α , i.e. $DR \circ \alpha = Id$. In particular, DR is surjective.

Proposition (8.5) of [Loo84] provides a relative version of this argument, that we specialise to our situation of $\pi : X \rightarrow S$. For this we look at the (truncated) relative de Rham complex

$$0 \rightarrow \mathcal{O}_X \rightarrow \Omega_{X/S}^1 \rightarrow \dots \rightarrow \Omega_{X/S}^{n-1} \rightarrow \Omega_{X/S}^n$$

The cohomology sheaves are $\pi^{-1}\mathcal{O}_S$ in degree 0 and

$$\mathcal{H}^n := \Omega_{X/S}^n / d\Omega_{X/S}^{n-1},$$

a sheaf supported on \tilde{D} , in degree n . The direct image $(\pi_*\Omega_{X/S}^\bullet, d)$ also has two non-vanishing cohomologies, namely $\pi_*\pi^{-1}\mathcal{O}_S$ in degree 0 and \mathcal{H}' in degree n . The two hypercohomology spectral sequences now produces a short exact sequence

$$0 \longrightarrow R^n\pi_*(\mathbb{C}_X) \otimes \mathcal{O}_S \xrightarrow{\alpha} \mathcal{H}' \xrightarrow{\beta} \pi_*\mathcal{H}^n \longrightarrow 0$$

([Loo84, Proposition 8.5]). Restriction to a (geometrical) fibre over u gives an exact sequence

$$0 \longrightarrow H^n(X_u) \longrightarrow \mathcal{H}'|_u \longrightarrow \pi_*\mathcal{H}_u^n \longrightarrow 0$$

In the middle we have a vector space of dimension μ , at the right hand side a direct sum of vector spaces of dimension μ_i , the Milnor numbers of the singularities appearing in the fibre over u . So indeed

$$\dim H^n(X_u) = \mu - \sum \mu_i$$

The composition

$$R^n\pi_*(\mathbb{C}_X) \longrightarrow R^n\pi_*(\mathbb{C}_X) \otimes \mathcal{O}_S \xrightarrow{\alpha} \mathcal{H}'$$

is for any $u \in S$ a *section* to the deRham-evaluation map

$$DR_u : \mathcal{H}'_u \longrightarrow H^n(X_u, \mathbb{C})$$

Corollary 3.1. *For all $u \in S$, the deRham evaluation map*

$$\mathcal{H}'_u \rightarrow H^n(X_u); \eta \rightarrow [\eta|_{X_u}]$$

is surjective.

3.3. The period map

The theory of the period map was developed independently by VARCHENKO and K. SAITO around the same time and has numerous applications. The basic idea is quite simple. Let us first fix a relative n -form η and a point $u \in S \setminus D$ and a horizontal basis $\gamma_1(s), \gamma_2(s), \dots, \gamma_\mu(s) \in H_n(X_s)$ for points s in a neighbourhood U of u . The *period map*

$$P_\eta : U \longrightarrow \mathbb{C}^\mu, \quad s \mapsto \left(\int_{\gamma_1(s)} \eta, \int_{\gamma_2(s)} \eta, \dots, \int_{\gamma_\mu(s)} \eta \right)$$

send a point s to the tuple of periods of the form η . By further parallel transport one extends P_η to a (multi-valued) map

$$P_\eta : S \setminus D \longrightarrow \mathbb{C}^\mu$$

between spaces of the same dimension μ . The form η is called *non-degenerate* if it is non-degenerate at all points $u \in S \setminus D$, which means that P_η is a local isomorphism near u . Of course, this can be tested by looking at the derivative of this map, which can be identified with the map

$$\nabla P_{\eta,u} : T_u S \rightarrow H^1(X_u), \quad \vartheta \mapsto [\nabla_{\vartheta}\eta|_{X_u}] \in H^n(X_u)$$

which is the geometrical fibre at u of the sheaf map

$$\Theta_{S \setminus D} \longrightarrow \mathcal{H}^*, \quad \vartheta \mapsto \nabla_{\vartheta}\eta$$

This map extends to a sheaf map

$$\Theta_S \longrightarrow \mathcal{H}'', \quad \vartheta \mapsto \nabla_{\vartheta}\eta$$

which is an *isomorphism* in case η is non-degenerate.

Proposition 3.2. *A non-degenerate relative n -form η gives rise to a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}' & \longrightarrow & \mathcal{H}'' & \longrightarrow & \Omega_{X/S}^{n+1} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Theta_S(-\log D) & \longrightarrow & \Theta_S & \longrightarrow & \mathcal{O}_{\tilde{D}} \longrightarrow 0 \end{array}$$

with exact rows and where the vertical maps are the isomorphisms described in the last paragraph and where the map at the right hand side is induced by multiplication by $\omega = d\eta$.

This diagram can be found in [Sai83, p. 1248].

From this we get immediately the following

Theorem 3.3. *If η is non-degenerate, then for each point $u \in S$ one obtains an isomorphism*

$$\nabla P_{\eta,u} : T_u^{\log D} S \longrightarrow \mathcal{H}'_u$$

The composition with the de Rham evaluation map gives a surjection

$$DR \circ \nabla P_{\eta,u} : T_u^{\log D} S \longrightarrow H^n(X_u)$$

In fact the restriction of $DR \circ \nabla P_{\eta,u}$ to $T_u \mathcal{R}(u)$ is an isomorphism. This statement was shown by Varchenko to hold in special cases and conjectured to hold in general, [Var87]. A proof basically along these lines was sketched to us in a letter by HERTLING, [Her14].

4. The case of curves

We specialise to the case $n = 1$, so $C := X_0$ is a plane curve singularity. If C has r branches then by the formula of MILNOR

$$\mu = 2\delta - r + 1,$$

and for $u \in S \setminus D$ the fibre $C_u := X_u$ is a smooth Riemann surface of genus $\delta - r + 1$ with r boundary circles. In the case where C is irreducible, then $\mu = 2\delta$ and for $u \notin D$, C_u is a smooth Riemann surface of genus δ . For $u \in D$ the curve C_u is a singular, say with singularities (C_u, p_i) , $i = 1, 2, \dots, N$ then its normalisation \tilde{C}_u has genus

$$\delta(C) - \delta(C_u)$$

where $\delta(C_u) = \sum_{i=1}^N \delta(C_u, p_i)$.

4.1. Intersection form

Now assume that C is irreducible. For fixed $u \in S$ let $C_u^* = C_u/\partial C_u$ be the closed Riemann surface obtained by shrinking ∂C_u to a point, and let \tilde{C}_u and \tilde{C}_u^* be the normalisations of C_u and C_u^* .

The diagram

$$\begin{array}{ccc} \tilde{C}_u \hookrightarrow \tilde{C}_u^* & \text{gives rise to the diagram} & H^1(\tilde{C}_u) \xleftarrow{\cong} H^1(\tilde{C}_u^*) \\ \downarrow n & & \uparrow n^* \\ C_u \xhookrightarrow{j} C_u^* & & H^1(C_u) \xleftarrow{\cong} H^1(C_u^*) \end{array}$$

in which the vertical arrows are surjections. Write I_u and \tilde{I}_u for the intersection forms on C_u and \tilde{C}_u . These are pulled back from the intersection forms on the closed curves C_u^* and \tilde{C}_u^* by means of the isomorphisms in the preceding diagram. Because $n_* : H_2(\tilde{C}_u, \partial \tilde{C}_u) \simeq H_2(C, \partial C)$, it follows by functoriality that

$$\tilde{I}_u(n^*a, n^*b) = I_u(a, b), \quad (4.1)$$

Note that the form \tilde{I}_u is non-degenerate.

4.2. de Rham version of I_u

The pairing I_u has the following DE RHAM description. We choose a pair of collars $U \subset V \subset C_u$ around the boundary ∂C_u and a C^∞ bump-function ρ , equal to 1 on U and 0 outside V . If η is a holomorphic (Kähler) 1-form on C_u , it follows from Stokes theorem that

$$\int_{\partial C} \eta = 0$$

By integration we can therefore find a holomorphic function α on V with $d\alpha = \eta$ on V . The form η is cohomologous to $\tilde{\eta} := \eta - d\rho\alpha$ and as $\rho = 1$ on U and there $d\alpha = \eta$, it follows that $\tilde{\eta}$ is a form with compact support, contained in $C \setminus U$. It is holomorphic and equal to η outside V , but only C^∞ on the annulus $V \setminus U$. One then has, using Stokes theorem

$$I_u([\eta], [\eta']) = I_u([\tilde{\eta}], [\eta']) = - \int_{\partial C} \alpha \eta'$$

More details are given in Section 7.

4.3. Extension to \mathcal{H}^* and \mathcal{H}'

The pairings I_u on $H^1(C_u)$ combine to give a perfect duality

$$I : \mathcal{H}^* \times \mathcal{H}^* \rightarrow \mathcal{O}_S$$

over $S \setminus D$. Because of its topological origin, the intersection form is *horizontal* with respect to the Gauss-Manin connection: for any two sections s_1, s_2 of \mathcal{H}^* ,

$$d(I(s_1, s_2)) = I(\nabla s_1, s_2) + I(s_1, \nabla s_2).$$

Using a relative version of the above DE RHAM-description of the intersection pairing one obtains an extension of I , still called I , to \mathcal{H}' :

$$I : \mathcal{H}' \times \mathcal{H}' \rightarrow \mathcal{O}_S$$

For two sections η_1, η_2 of \mathcal{H}' one has

$$I(\eta_1, \eta_2)(u) = I_u([\eta_1|C_u], [\eta_2|C_u]) \quad (4.2)$$

4.4. Pulling back the intersection form

Using the period map one can pull-back the intersection form on $H^1(C_u)$ to obtain a 2-form on S . Let us first start with an arbitrary section $s \in \mathcal{H}^*$ over $S \setminus D$. From it we obtain a 2-form

$$\Omega = s^*I \in \Omega_{S \setminus D}^2$$

on $S \setminus D$ by the formula

$$\Omega(\vartheta_1, \vartheta_2) := I(\nabla_{\vartheta_1}s, \nabla_{\vartheta_2}s)$$

Proposition 4.1. *The form Ω is closed.*

Proof. This is ‘clear’ as we are pulling back the ‘constant form I ’, but here is a nice direct calculation: if a, b and c are germs of commuting vector fields on S then

$$\begin{aligned} d(s^*I)(a, b, c) &= d(I(a, b))(c) - d(I(a, c))(b) + d(I(b, c))(a) \\ &= I(\nabla_c \nabla_a s, \nabla_b s) + I(\nabla_a s, \nabla_c \nabla_b s) \\ &\quad - I(\nabla_b \nabla_a s, \nabla_c s) - I(\nabla_a s, \nabla_b \nabla_c s) \\ &\quad + I(\nabla_a \nabla_b s, \nabla_c s) + I(\nabla_b s, \nabla_a \nabla_c s) \end{aligned} \quad (4.3)$$

Because a and b commute and ∇ is flat, $\nabla_a \nabla_b = \nabla_b \nabla_a$, and similarly for $\nabla_a \nabla_c$ and $\nabla_b \nabla_c$. This means that all terms on the right hand side in (4.3) cancel, except the first and last. These cancel because of the anti-symmetry of I . \square

Theorem 4.2. ([GV82]) *If $s = \eta$ is a non-degenerate section of \mathcal{H}' , then Ω is itself non-degenerate and hence symplectic, and moreover extends to all of S as a symplectic form.*

5. Results

In [GV82] one find the formulation of a *principle* that the types of degeneration that occur in the fibres C_u are reflected in the lagrangian properties of the corresponding strata. Our results can be seen as a vindication of this principle in some special cases.

As before, we will consider the versal deformation $\pi : X \rightarrow S$ of an irreducible curve singularity, a non-degenerate section η of the Brieskorn-module \mathcal{H}' and the resulting symplectic form Ω on S , obtained by pulling back the intersection form on the fibres $H^1(C_u)$.

5.1. The rank of Ω on the logarithmic tangent space

Recall that for a point $u \in S$, the logarithmic tangent space $T_u^{\log D} S \subset T_u S$ is the sub-space spanned by the logarithmic vector fields at u .

Theorem 5.1. *The rank of Ω restricted to $T_u^{\log D} S$ is equal to the rank of I_u on $H^1(C_u)$, which is equal to $\dim H^1(\tilde{C}_u) = 2(\delta(C) - \delta(C_u))$.*

Proof. Let $\mathcal{R}(u)$ and $\mathcal{K}(u)$ denote, respectively, the right-equivalence stratum and the \mathcal{K} -equivalence stratum containing u . Recall that by 3.3 the period map maps the space $T_u \mathcal{K}(u)$ surjectively to $H^1(C_u)$; its restriction to $T_u \mathcal{R}(u) \subseteq T_u \mathcal{K}(u)$ maps isomorphically to $H^1(C_u)$. From (4.2) it follows that the rank of Ω on $T_u^{\log D} S$ at u is equal to the rank of the intersection form I_u on $H^1(C_u)$, which is equal to the rank of $H^1(\tilde{C}_u)$, and therefore is equal to $\mu(C) - 2\delta(C_u) = 2\delta(C) - 2\delta(C_u)$. \square

5.2. Coisotropy of the Severi strata

Recall that a subspace V of a symplectic vector space $(W, \langle -, - \rangle)$ is *coisotropic* if $V^\perp \subset V$, where $V^\perp = \{w \in W : \langle v, w \rangle = 0 \text{ for all } v \in V\}$. A submanifold X of a symplectic manifold M is coisotropic if for all $x \in X$, $T_x X$ is a coisotropic subspace of $T_x M$. A singular subset X of the symplectic manifold M is coisotropic if X_{reg} is coisotropic.

Theorem 5.2. *All the Severi strata*

$$D(\delta) \subset D(\delta - 1) \subset \dots \subset D(1) = D$$

are coisotropic with respect to Ω .

Proof. Suppose that u is a regular point of $D(k)$, so C_u has exactly k ordinary double points as singularities. As $\mathcal{R}(u) = \mathcal{K}(u) = D(k)$ near u , the tangent space $T_u D(k)$ is the same as $T_u^{\log D} S$. From theorem 5.1 the rank of $\Omega|_{T_u D(k)}$ is equal to $\mu - 2k$, hence $\dim \ker \Omega|_{T_u D(k)} = k$. But from the non-degeneracy of Ω it follows that $T_u D(k)^\perp$ has dimension equal to the codimension of $D(k)$, namely k . Thus both sides in the relation

$$T_u D(k)^\perp \supset \ker(\Omega_u|_{T_u D(k)})$$

have dimension k , and are therefore equal. It follows that $T_u D(k)^\perp \subset T_u D(k)$. That is, $D(k)$ is coisotropic. \square

The principle mentioned above explains this result by simply saying the near a regular point $u \in D(k)$ there are k mutually non-intersecting cycles vanishing at u , which make up an isotropic subspace of H_1 . However, making this into an honest proof is another matter and leads to the considerations outlined above. The form Ω is not unique, and moreover is determined globally rather than locally. One cannot prove anything by using a local normal form $N(\ell) := \{u_1 \cdots u_\ell = 0\}$ for D at a generic point u_0 of a Severi stratum $D(\ell)$, since the symplectic form one picks there will not in general coincide with the pullback of the form Ω by an isomorphism identifying (D, u_0) with $(N(\ell), 0)$.

5.3. Equations for the $D(k)$

Let χ_1, \dots, χ_μ be a basis for $\Theta_S(-\log D)$, and let $\omega_1, \dots, \omega_\mu$ be the dual basis for $\Omega_S^1(\log D)$. Considering Ω as an element of $\Omega_S^2(\log D)$, it can be expressed as the sum

$$\Omega = \sum_{i < j} \Omega(\chi_i, \chi_j) \omega_i \wedge \omega_j.$$

We denote the skew matrix with i, j 'th entry $\Omega(\chi_i, \chi_j)$ by $\chi^t \Omega \chi$. Then

$$\wedge^k \Omega = \sum_{1 \leq i_1 < \dots < i_{2k} \leq \mu} \text{Pf}(\chi^t \Omega \chi(i_1, \dots, i_{2k})) \omega_{i_1} \wedge \dots \wedge \omega_{i_{2k}}. \quad (5.1)$$

where $\chi^t \Omega \chi(i_1, \dots, i_{2k})$ is the submatrix of $\chi^t \Omega \chi$ consisting of rows and columns i_1, \dots, i_{2k} and Pf denotes its Pfaffian. The ideal generated by the coefficients of $\wedge^k \Omega$ with respect to the basis $\omega_{i_1} \wedge \dots \wedge \omega_{i_{2k}}$ of $\Omega^{2k}(\log D)$ is the same as the ideal $\text{Pf}_{2k}(\chi \Omega \chi)$ of $2k \times 2k$ Pfaffians of $\chi^t \Omega \chi$.

Theorem 5.3. $D(k) = V \left(\text{Pf}_{2(\delta-k+1)}(\chi^t \Omega \chi) \right)$. In particular, the δ -constant stratum $D(\delta)$ is defined by the entries of $\chi^t \Omega \chi$.

Proof. Consider an arbitrary $u \in S$. The rank of the matrix $\chi^t \Omega \chi$ at u is the rank of Ω restricted to the space of evaluations at u of the vector fields in $\Theta_S(-\log D)_u$, which is precisely the logarithmic tangent space $T_u^{\log D} S$. Theorem 5.1 states that the rank of Ω on $T_u^{\log D}$ at u is equal to $2\delta(C) - 2\delta(C_u)$. As the rank of a skew-symmetric matrix is always even and equal to the size of the largest non-vanishing Pfaffian, it follows that $D(k)$ is precisely cut out by the Pfaffians of size $2(\delta - k + 1)$ of the matrix $\chi^t \Omega \chi$, i.e. $D(k) = V \left(\text{Pf}_{2(\delta-k+1)}(\chi^t \Omega \chi) \right)$. \square

A symplectic form Ω on a manifold S gives rise to a Poisson bracket $\{ -, - \}$ on the sheaf of functions on S , as follows: Ω determines an isomorphism $\Omega_S^1 \rightarrow \Theta_S$ sending a 1-form α to a vector field α^\flat . Then for functions f, g ,

$$\{f, g\} = \Omega((df)^\flat, (dg)^\flat).$$

The vector field $\chi_f := (df)^\flat$ is called the *Hamiltonian vector field* associated to f . If $V \subset S$ is a sub-variety and $I(V) \subset \mathcal{O}_S$ the ideal of functions vanishing on V , then it is easy to show that for a regular point $x \in V$ one has

$$T_x V^\perp = \{\chi_f(x) : f \in I(V)_x\}. \quad (5.2)$$

The following is well-known:

Proposition 5.4. $V \subset S$ is coisotropic if and only if the ideal $I(V)$ is Poisson-closed:

$$\{I(V), I(V)\} \subset I(V).$$

For the convenience of the reader we include a proof.

Proof. Let $x \in V$ be a regular point, $v, w \in T_x V^\perp$, and $f, g \in I(V)$ two functions with $\chi_f(x) = v$, $\chi_g(x) = w$ (using (5.2)). Then

$$\Omega(v, w) = \Omega(\chi_f(x), \chi_g(x)) = \{f, g\}(x)$$

From this we see that $\{f, g\}$ vanishes at x if and only if $\Omega(v, w) = 0$, which means that $T_x V^\perp \subset (T_x V^\perp)^\perp = T_x V$, that is, V is coisotropic. \square

Thus, for each of the Severi strata $D(k)$, the ideal $I(D(k))$ is involutive. But note that an ideal defining a coisotropic subvariety is not necessarily involutive; the proof only shows that this holds if the ideal is radical.

Conjecture 5.5. *For all $k = 1, 2, \dots, \delta$ (a) $\text{Pf}_{2k}(\chi^t \Omega \chi)$ is involutive. (b) $\text{Pf}_{2k}(\chi^t \Omega \chi)$ is radical.*

By the theorem 1.2, (b) \implies (a), as vanishing ideals of coisotropic varieties are involutive. Nevertheless, involutivity of the ideals $\text{Pf}_{2k}(\chi^t \Omega \chi)$ may hold even without their being radical.

Problem: How to write the Poisson bracket of two Pfaffians of $\chi^t \Omega \chi$ as a linear combination of Pfaffians? Is there a universal formula?

6. The symplectic form as Extension

The matrix $\chi^t \Omega \chi$ can be considered as an endomorphism of \mathcal{O}_S^μ and its cokernel N_Ω defines a rank 2 Cohen-Macaulay module on \mathcal{O}_D . If the basis χ of $\Theta_S(-\log D)$ is chosen to be *symmetric*, as described in Subsection 2.4, then N_Ω sits in an exact sequence

$$0 \longleftarrow \mathcal{O}_{\tilde{D}} \longleftarrow N_\Omega \longleftarrow \mathcal{O}_{\tilde{D}} \longleftarrow 0 \quad (6.1)$$

In fact, we show that the extension (6.1) has a coordinate-independent meaning, depending only on the choice of ω used in the definition of the period map. As such it represents an element in the $\Omega_{\tilde{D}}$ -module

$$\text{Ext}_D^1(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}})$$

and therefore an infinitesimal deformation of $\mathcal{O}_{\tilde{D}}$ as \mathcal{O}_D -module. We refer to N_Ω as the *intersection module*. For a vector field ϑ , let $\vartheta^\#$ denote the contraction of Ω by ϑ . Begin with the exact sequence

$$0 \longleftarrow \frac{\Omega_S^1(\log D)}{\Omega_S^1} \longleftarrow \frac{\Omega_S^1(\log D)}{\Theta_S(-\log D)^\#} \longleftarrow \frac{\Theta_S}{\Theta_S(-\log D)} \longleftarrow 0. \quad (6.2)$$

This exists for every divisor D and non-degenerate 2-form Ω . Here the first arrow is induced by contraction with Ω , which maps Θ_S to Ω_S^1 and $\Theta_S(-\log D)$ to $\Theta_S(-\log D)^\#$. Then the exact sequence we consider is obtained from (6.2) by composing the last arrow with the isomorphism $k^{-1} \circ \beta : \Omega_S^1(\log D) \rightarrow \Theta_S$ described in Subsection 2.4, inducing an isomorphism

$$\frac{\Theta_S}{\Theta_S(-\log D)} \longleftarrow \frac{\Omega_S^1(\log D)}{\Omega_S^1}.$$

Since we have a canonical isomorphism $\Theta_S/\Theta_S(-\log D) \rightarrow \mathcal{O}_{\tilde{D}}$ defined by dF , we obtain the exact sequence (6.1). Thus provided the pairing on \mathcal{O}_Σ is chosen canonically, the extension class of (6.1) depends only on F and on the symplectic form.

Remark 6.1. If we apply $k^{-1} \circ \beta$ also to the middle term of the sequence (6.2) as well as the third, we obtain (6.1) in the slightly different form

$$0 \leftarrow \frac{\Theta_S}{\Theta_S(-\log D)} \leftarrow \frac{\Theta_S}{k^{-1} \circ \beta(\Theta_S(-\log D)^\#)} \leftarrow \frac{\Theta_S}{\Theta_S(-\log D)} \leftarrow 0. \quad (6.3)$$

Note that $k^{-1} \circ \beta(\Theta_S(-\log D)^\#)$ is generated over \mathcal{O}_S by vector fields whose components with respect to the usual basis $\partial/\partial u_1, \dots, \partial/\partial u_\mu$ are given by the columns of the matrix $\chi\Omega\chi$. It is interesting that in all of the examples where we have made the calculations, $k^{-1} \circ \beta(\Theta_S(-\log D)^\#) \subset \Theta_S$ is a Lie sub-algebra, evidently contained in $\Theta_S(-\log D)$. We cannot at present prove this or explain it.

6.1. Calculation of Ext groups

We state without proof the results of some relatively straightforward calculation of Ext groups. Let \mathcal{C} denote the conductor ideal of the projection $n = \pi| : \tilde{D} \rightarrow D$.

Lemma 6.2. (i) Both $\text{Ext}_D^1(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}})$ and $\text{Ext}_D^2(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}})$ are $\mathcal{O}_{\tilde{D}}/\mathcal{C}$ -modules.

$$(ii) \quad \text{Ext}_{\mathcal{O}_D}^1(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}}) \simeq \frac{\{\mathcal{O}_{\tilde{D}}\text{-syzygies of } g_1, \dots, g_\mu\}}{\mathcal{O}_{\tilde{D}} \cdot \{\mathcal{O}_D\text{-syzygies of } g_1, \dots, g_\mu\}};$$

$$(iii) \quad \text{Ext}_{\mathcal{O}_D}^2(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}}) \simeq \mathcal{O}_{\tilde{D}}/\mathcal{C}.$$

Proposition 6.3. $\text{Ext}_D^1(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}})$ is a maximal Cohen-Macaulay module over $\mathcal{O}_{\tilde{D}}/\mathcal{C}$ presented by the matrix $\tilde{\chi}$ obtained from the symmetric matrix χ of the basis for $\Theta_S(-\log D)$ by deleting its last row and column.

In [AM13] it is shown that if $n : \tilde{D} \rightarrow D$ has corank 1 then $\text{Coker} \tilde{\chi} \simeq \pi_* \mathcal{O}_{D^2(n)}$, where, by $D^2(n)$, we mean the double-point scheme of the map n :

$$D^2(n) = \text{closure}\{(x_1, x_2) \in \tilde{D} \times \tilde{D} : x_1 \neq x_2, n(x_1) = n(x_2)\}.$$

The isomorphism holds only if n has corank 1. The map $n : \tilde{D} \rightarrow D$, normalising the discriminant in the base of a versal deformation, has corank 1 exactly for the A_μ singularities. Thus, for the A_μ , and only for these, $\text{Ext}_D^1(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}}) \simeq \mathcal{O}_{D^2(n)}$.

7. Computations and Examples

It was described in [Giv88] how the symplectic form Ω can be computed in the case of irreducible quasi-homogeneous curve singularities. The projective closure of such a curve has a unique point at infinity ∞ .

Proposition 7.1. *Let C be a curve, $\infty \in C$ a smooth point and ω, η two meromorphic differential forms, holomorphic on $C \setminus \{\infty\}$. Then the intersection form of the cohomology classes $[\omega], [\eta] \in H^1(C)$ is*

$$I([\omega], [\eta]) = 2\pi i \operatorname{Res}_\infty(\alpha\eta)$$

where α is a meromorphic function in a neighbourhood of ∞ with $d\alpha = \omega$.

Proof. Choose two small open discs $U \subset V \subset C$ around ∞ , and a C^∞ bump function ρ on C , equal to 1 on U and 0 outside V . Choose a function α meromorphic on V with $d\alpha = \omega$. Then $\omega - d(\rho\alpha)$ is a C^∞ compactly supported form, cohomologous to $[\omega]$. Using $\omega \wedge \eta = 0$, we find

$$I([\omega], [\eta]) = - \int_C d(\rho\alpha) \wedge \eta = - \int_U d(\rho\alpha \cdot \eta)$$

and by Stokes theorem

$$- \int_U d(\rho\alpha \cdot \eta) = - \int_{\partial U} \alpha\eta$$

which, noticing the reverse of orientation in the boundary, gives the above formula. \square

This proposition can be used to calculate intersections using Laurent-series expansions. If the curve C is given by an affine equation $f(x, y) = 0$ and has a single point at infinity, we can find a Laurent parametrisation of C around ∞

$$x(t), y(t) \in \mathbb{C}[[t]][1/t]$$

If $\omega = A(x, y)dx$ and $\eta = B(x, y)dy$ are the differential forms on C , then by substitution we obtain expansions

$$\omega = a(t)dt, \eta = b(t)dt$$

where $a(t), b(t) \in \mathbb{C}[[t]][1/t]$ are Laurent series. Integrating up one we find

$$\alpha(t) = \int a(t)dt \in \mathbb{C}[[t]][1/t]$$

and we can compute the cohomological intersection as:

$$I([\omega], [\eta]) = \operatorname{Res}_0 \alpha(t)b(t)dt$$

Proposition 7.2. ([GV82]) *Suppose that f is quasihomogeneous. Then for $\omega = dx \wedge dy$, the period map P_ω is non-degenerate.*

Case A_4

We consider the versal deformation of A_4 given by

$$F(x, a, b, c, d) = y^2 + x^5 + ax^3 + bx^2 + cx + d.$$

We take the symmetric basis for $\Theta_S(-\log D)$ with Saito matrix

$$\chi := \begin{pmatrix} 10a & 15b & 20c & 25d \\ 15b & -6a^2 + 20c & -4ab + 25d & -2ac \\ 20c & -4ab + 25d & -6b^2 + 10ac & -3bc + 15ad \\ 25d & -2ac & -3bc + 15ad & -4c^2 + 10bd \end{pmatrix} \quad (7.1)$$

The symplectic form pulled back by the period mapping induced by the 1-form ydx is

$$\Omega = ada \wedge db + da \wedge dd + 3db \wedge dc. \quad (7.2)$$

Therefore the ideal of entries of the matrix $\chi\Omega\chi$, defining the δ -constant stratum $D(2)$, is generated by

$$a^4 + \frac{27}{4}ab^2 - 9a^2c + 20c^2 - \frac{25}{2}ad, \quad a^3b + \frac{27}{4}b^3 - 9abc - 10a^2d + 50cd \quad (7.3)$$

and

$$a^3c + \frac{27}{4}b^2c - 4ac^2 - 20abd + \frac{125}{4}d^2$$

Case A_6

A versal deformation of A_6 is given by

$$F(x, a, b, c, d, e, f) = x^7 + ax^5 + bx^4 + cx^3 + dx^2 + ex.$$

We take the basis of $\Theta_S(-\log D)$ with Saito matrix

$$\begin{pmatrix} 2a & 3b & 4c & 5d & 6e & 7f \\ 3b & -\frac{10}{7}a^2 + 4c & -\frac{8}{7}ab + 5d & -\frac{6}{7}ac + 6e & -\frac{4}{7}ad + 7f & -\frac{2}{7}ae \\ 4c & -\frac{8}{7}ab + 5d & -\frac{12}{7}b^2 + 2ac + 6e & -\frac{9}{7}bc + 3ad + 7f & -\frac{6}{7}bd + 4ae & -\frac{3}{7}be + 5af \\ 5d & -\frac{6}{7}ac + 6e & -\frac{6}{7}bc + 3ad + 7f & -\frac{12}{7}c^2 + 2bd + 4ae & -\frac{8}{7}cd + 3be + 5af & -\frac{4}{7}ce + 4bf \\ 6e & -\frac{4}{7}ad + 7f & -\frac{6}{7}bd + 4ae & -\frac{8}{7}cd + 3be + 5af & -\frac{10}{7}d^2 + 2ce + 4bf & -\frac{5}{7}de + 3cf \\ 7f & -\frac{2}{7}ae & -\frac{3}{7}be + 5af & -\frac{4}{7}ce + 4bf & -\frac{5}{7}de + 3cf & -\frac{6}{7}e^2 + 2df \end{pmatrix}$$

and symplectic form

$$\Omega = \begin{pmatrix} 0 & -3a^2 - c & -6b & 9a & 0 & -3 \\ 3a^2 + c & 0 & -5a & 0 & -5 & 0 \\ 6b & 5a & 0 & -15 & 0 & 0 \\ -9a & 0 & 15 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Each of the ideals $\text{Pf}_{2\ell}$ is Poisson-closed, and defines a Cohen-Macaulay variety of codimension $3 - \ell + 1$.

3. For A_8 , each of the ideals $\text{Pf}_{2\ell}$ is Poisson-closed, and defines a Cohen-Macaulay variety of codimension $4 - \ell + 1$.

Case E_6 and E_8

A versal deformation of E_6 is given by

$$F(x, y, a, b, c, d, e, f) = x^3 + y^4 + axy^2 + bxy + cy^2 + dx + ey + f.$$

We take the basis of $\Theta_S(-\log D)$ with symmetric Saito matrix χ equal to

$$\begin{pmatrix} 2a & 5b & 6c & 8d & 9e & 12f \\ 5b & \frac{-a^4}{6} - 4ac + 8d & \frac{a^2b}{2} + 9e & -\frac{a^3b}{12} - \frac{3bc+ae}{2} & \frac{ab^2-a^3c}{6} - \frac{a^2d-9c^2}{3} + 12f & \frac{abd}{6} - \frac{a^3e}{12} - \frac{3ce}{2} \\ 6c & \frac{a^2b}{2} + 9e & -\frac{5b^2+2a^2c+10ad}{3} & \frac{7ab^2}{12} - \frac{4a^2d}{3} + 12f & \frac{7abc}{6} - \frac{13bd+4a^2e}{3} & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f \\ 8d & -\frac{a^3b}{12} - \frac{3bc+ae}{2} & -\frac{5b^2+2a^2c+10ad}{3} & \frac{7ab^2}{12} - \frac{4a^2d}{3} + 12f & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf \\ 9e & \frac{ab^2-a^3c}{6} - \frac{a^2d-9c^2}{3} + 12f & \frac{7abc}{6} - \frac{13bd+4a^2e}{3} & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf \\ 12f & \frac{abd}{6} - \frac{a^3e}{12} - \frac{3ce}{2} & -\frac{8d^2}{3} + \frac{7abe}{12} - 2a^2f & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf & \frac{10b^2d-a^2be}{24} - \frac{4ad^2}{3} - \frac{9e^2}{4} + 6cf \end{pmatrix} \quad (7.4)$$

The symplectic form Ω has matrix

$$\begin{pmatrix} 0 & -\frac{1}{15}ab & \frac{1}{5}c & \frac{2}{15}a^2 & 0 & \frac{1}{5} \\ \frac{1}{15}ab & 0 & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{5}c & 0 & 0 & 1 & 0 & 0 \\ -\frac{2}{15}a^2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.5)$$

The ideal of 2×2 Pfaffians (i.e. the ideal of entries) of $\chi\Omega\chi$, defining the δ -constant stratum, is Cohen-Macaulay of codimension 3, and Poisson-closed. Below we comment on the computations involved in proving Cohen-Macaulayness. The ideal J of 4×4 Pfaffians is also Poisson closed, and has codimension 2 but projective dimension 3.

For both E_6 and E_8 we check the Cohen Macaulay property for the ideal generated by the entries in the matrix $\chi\Omega\chi$ using the *Depth* package of *Macaulay 2*. To show that this ideal is radical, we use the result of [FGvS99], that the geometric degree of $D(\delta)$ is equal to the Euler characteristic of the compactified Jacobian. This Euler characteristic is calculated in [Pio07]: for E_6 it is 5 and for E_8 7. Using *Singular* we computed the algebraic degree of $\mathcal{O}_{D(\delta)}$, as defined by the ideal of entries of $\chi\Omega\chi$, and found that it took these values, showing, in view of Cohen-Macaulayness, that this is the reduced structure.

Betti numbers of the Severi strata for A_{2k}

The following table shows the non-zero betti numbers of minimal free resolutions of the ideals of Pfaffians, $\text{Pf}_{2\ell}$, of the matrix $\chi\Omega\chi$ for singularities of type A_{2k} for $1 \leq k \leq 4$.

	A_2	A_4		A_6			A_8			
ℓ	β_0	β_0	β_1	β_0	β_1	β_2	β_0	β_1	β_2	β_3
1	1	3	2	6	8	3	10	20	15	4
2	—	1	—	5	4	—	15	24	10	—
3	—	—	—	1	—	—	7	6	—	—
4	—	—	—	—	—	—	1	—	—	—

(7.6)

Since depth + projective dimension = dimension S and $\text{codim } D(j) = j$, it follows from the data in the table that for A_{2k} with $k \leq 4$, each of the rings $\mathcal{O}_S / \text{Pf}_{2\ell}$, and therefore each of the Severi strata $D(k - \ell + 1) = V(\text{Pf}_{2\ell}) \subset S$, is Cohen-Macaulay.

Conjecture 7.3. *For all ℓ and k with $\ell \leq k$, each of the Severi strata $D(\ell)$ in the base of a miniversal deformation of A_{2k} is Cohen Macaulay.*

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