Milnor number equals Tjurina number for functions on space curves *

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Abstract

We prove the equality of Milnor number and Tjurina number for functions on space curve singularities, as conjectured recently by V. Goryunov. As a consequence, the discriminant in such a situation is a free divisor.

1 Introduction

In many situations in complex geometry one can define a *Milnor number* that describes the rank of some vanishing homology, and a *Tjurina* number, the dimension of the base of a semi-universal deformation. For example, for a hypersurface singularity $X = V(f)$ one has

$$\mu = \dim \mathscr{O}/J_f \ , \quad \tau = \dim \mathscr{O}/(f, J_f) \ ,$$

so that $\mu \geq \tau$, with equality if f is quasi-homogeneous, see [Sa]. Recently, V. Goryunov [Go] has studied functions on curve singularities $X \subset \mathbb{C}^3$, and conjectured, somewhat surprisingly, that the equality $\mu = \tau$ holds in such a situation, *without any condition of quasi-homogeneity.* For functions on *smooth* curves, the equality of μ and τ is of course so evident that it passes without notice: every non-constant germ in one variable is equivalent to x^n for some $n > 0$. We show in this note that

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Goryunov's conjecture is correct. In fact, the result follows from known results in homological algebra. Because many of these results are scattered over the literature, we try to be self-contained. We can not, however, claim much originality.

2 The Milnor number

We will consider curve singularities X, by which we mean reduced, one-dimensional germs $X \subset (\mathbb{C}^N, 0)$. We let $\mathscr{O}_X = \mathscr{O}/I$, where $\mathscr{O} := \mathbb{C}\{x_1, x_2, \ldots, x_N\}$, be the local ring of X. Furthermore, we consider the *dualising module*

$$\omega_X := Ext^{N-1}(\mathscr{O}_X, \Omega^N) \ .$$

Here Ω^N is the module of N-forms on $(\mathbb{C}^N, 0)$. One can think of ω_X, *via* the so-called *class map* $cl : \Omega_X \longrightarrow \omega_X$, as consisting of certain meromorphic differential forms on the curve. For an explanation, we refer to [BG] and to Remark 2.4 and §4 of this note.

A function $f \in \mathscr{O}_X$ can be considered as a *map* $f : X \longrightarrow S$, where S is a smooth curve germ. We will always assume our f's to be non-constant on each branch, so that $\mathscr{O}_X$ becomes a finite and free $\mathscr{O}_S$-module via f.

The *ramification* of the map $f : X \longrightarrow S$ can be measured as follows: the composition of $\mathscr{O}_X \xrightarrow{df} \Omega_X$ with the class map $cl : \Omega_X \longrightarrow \omega_X$ produces an injective map $\mathscr{O}_X \longrightarrow \omega_X$, whose cokernel we denote by $\mathscr{M}_f$:

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \omega_X \longrightarrow \mathscr{M}_f \longrightarrow 0$$

It is a module of finite length. We will denote by $\mathscr{R}_f$ the short exact sequence by which we define $\mathscr{M}_f$.

Definition 2.1. We define the *Milnor number* of $f : X \longrightarrow S$ as

$$\mu(f) := \dim \mathscr{M}_f$$

Proposition 2.2. *μ is conserved under simultaneous deformation of f and X.*

Proof Consider a one-parameter deformation $\mathscr{X} \longrightarrow T$ of X over T, and a lift of f to a function $F : \mathscr{X} \longrightarrow S$. Because X is assumed to be reduced, $\omega_{\mathscr{X}}$

is T-flat, and the snake lemma, applied to the injective morphism of complexes $\mathscr{R}_F \xrightarrow{t\cdot} \mathscr{R}_F$ (with cokernel $\mathscr{R}_f$) gives the short exact sequence

$$0 \longrightarrow \mathscr{M}_F \xrightarrow{t\cdot} \mathscr{M}_F \longrightarrow M_f \longrightarrow 0.$$

Hence $\mathscr{M}_F$ is $\mathscr{O}_T$-flat, and therefore $\mathscr{O}_T$ free. It follows that μ is conserved in any 1-parameter deformation of $X \xrightarrow{f} S$, and thus in any deformation. □

Remark 2.3. If $\mathscr{X} \longrightarrow T$ is a smoothing of X, and F is generic, then $\mu(f)$ counts the number of critical points of $f_t = F_{| X_t}$. These can be thought of as the points where the smooth curve X_t is tangent to the level hypersurfaces of F. For at any such point, $\omega_{X_t} = \Omega^1_{X_t}$, and our module $\mathscr{M}_{f_t}$ is just $\Omega^1_{X_t}/df_t \wedge \mathscr{O}_{X_t}$, and thus isomorphic to the usual jacobian algebra $\mathscr{O}_{X_t,x}/J_{f_t}$. Hence an ordinary tangency of X_t to the level hypersurface of F will contribute 1 to the total. In fact these are the only points at which length$(\mathscr{M}_{f_t})$ is equal to 1.

Remark 2.4.

- The module ω_X can be identified with $Hom_S(\mathscr{O}_X, \omega_S)$ ([BH], Theorem 3.3.7 (b)). After a choice of generator for ω_S, $\omega_S = \mathscr{O}_S$, and so $\omega_X \simeq Hom_S(\mathscr{O}_X, \mathscr{O}_S)$. Let $Q(\mathscr{O}_X)$ and $Q(\mathscr{O}_S)$ denote the total rings of fractions of $\mathscr{O}_X$ and $\mathscr{O}_S$ respectively, and let $tr : \mathscr{O}_X \longrightarrow \mathscr{O}_S$ denote the trace map, which associates to each element g of $\mathscr{O}_X$ the trace of the $Q(\mathscr{O}_S)$-linear endomorphism of $Q(\mathscr{O}_X)$ induced by multiplication by g. (The trace of a function $g \in \mathscr{O}_X$ can also be thought of as the function $tr(g) \in \mathscr{O}_S$ whose value at a regular value $s \in S$ is the sum of the values of g at the preimages of s.) As ω_X is maximal Cohen Macaulay of rank 1, every element of $Hom_S(\mathscr{O}_X, \mathscr{O}_S)$ is a $Q(\mathscr{O}_X)$-multiple of tr; that is,

  $$Hom_S(\mathscr{O}_X, \mathscr{O}_S) = \{g \in Q(\mathscr{O}_X) \mid tr(gh) \in \mathscr{O}_S \ \forall h \in \mathscr{O}_X\} \cdot \ tr.$$

  The submodule of $Q(\mathscr{O}_X)$ defined here is the classical *complementary module*, usually denoted $\mathscr{L}_{X/S}$. Thus, we have shown

  $$\omega_X \simeq \mathscr{L}_{X/S}.$$

  Thought of as a module of meromorphic forms on X, $\omega_X = \mathscr{L}_{X/S} \cdot df$, and the quotient $\mathscr{M}_f$ with which we defined the Milnor number μ, is just $(\mathscr{L}_{X/S} \cdot df)/(\mathscr{O}_X \cdot df)$, or simply $\mathscr{L}_{X/S}/\mathscr{O}_X$.

- The module $(\mathscr{L}_{X/S}\cdot df)/(\mathscr{O}_X\cdot df)$ is also studied in [MS],under the name $R^+(df)$. It supports a natural quadratic form, induced by the residue pairing, and in [MS] the signature of this form is used to compute the number of branches of the curve-germ X and of its real part.

- There are many other ways to define numbers that are related to the ramification of the map $f : X \longrightarrow S$. In §5 we will see that if $X \subset \mathbb{C}^3$ is a space curve singularity, many of these numbers agree.

3 The Tjurina number

Given any map $f : X \longrightarrow S$ of analytic germs, there is a so-called (analytic) cotangent complex $\mathbb{L}_{X/S}$. The homology groups $T_k^{X/S}(M) := H_k(\mathbb{L}_{X/S} \otimes M)$ and cohomology groups $T^k_{X/S}(M) := H^k(Hom(\mathbb{L}_{X/S}, M))$ form a (co)-homology theory for $\mathscr{O}_X$-modules M. The modules $T_k^{X/S} := T_k^{X/S}(\mathscr{O}_X)$ and $T^k_{X/S} := T^k_{X/S}(\mathscr{O}_X)$ are important in the deformation theory of the map $f : X \longrightarrow S$.
$T_0^{X/S}$ is the module of relative Kähler 1-forms $\Omega_{X/S}$, $T^0_{X/S}$ is the module of relative vector fields $\Theta_{X/S}$ on X over S. The other T_k and T^k are to be thought of as higher derived versions of these, and are supported inside the critical locus of the map. For a good overview, we refer to [BC] and also [HW]. Most remarkable is the module $T^1_{X/S}$, which has the interpretation as space of *infinitesimal deformations* of the map $f : X \longrightarrow S$, where we keep S fixed, but are allowed to deform X in any flat way, and the map in an arbitrary way. Isomorphisms are defined by obvious diagrams, always with identity on S. This leads to the following very general

Definition 3.1. The *Tjurina number* of $f : X \longrightarrow S$ is

$$\tau(f) := \dim T^1_{X/S} \, .$$

Now let us return to the case of a function $f : X \longrightarrow S$ on a curve singularity. We remark that a space curve has a semi-universal deformation whose base is a smooth space of dimension $\tau = \dim\ T^1_{X/k}$ (see e.g. [Sch]). Similarly, standard arguments show that the base of a semi-universal deformation of the function $f : X \to S$ is smooth of dimension $\tau(f)$.

Example: For a Morse point, $\tau(f) = 1$. Once again, this is the only case where the value of the invariant is 1.

The modules T_k and T^k for $k \le 2$ can be computed using the Lichtenbaum-Schlessinger complex. One starts with $X \subset (\mathbb{C}^N, 0)$ and extends the map $X \longrightarrow S$

to $(\mathbb{C}^N, 0) \longrightarrow S$. Choose generators of the ideal $\mathscr{I}$ of X, and let $\mathscr{R}$ denote the module of relations between the generators. That is, we have an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{O}^k \longrightarrow \mathscr{I} \longrightarrow 0 \,.$$

Let $\mathscr{R}_0 \subset \mathscr{R}$ be the module of Koszul relations between the generators of $\mathscr{I}$. Consider the complex

$$\mathbb{L}^{\leq 2}: \quad \mathscr{R}/\mathscr{R}_0 \longrightarrow \mathscr{O}^k \otimes \mathscr{O}_X \longrightarrow \Omega_{\mathbb{C}^N/S} \otimes \mathscr{O}_X$$

The homology groups of this complex are the $T_k^{X/S}$, the cohomology groups of its $\mathscr{O}_X$-dual the $T^k_{X/S}$, for $k = 0, 1, 2$.

Proposition 3.2. *Assume that $T^2_{X/S} = 0$ and that X is smoothable. Then*

$$\mu(f) = \tau(f) \,.$$

Proof Consider a one-parameter smoothing $\mathscr{X} \longrightarrow T$, and a function $F : \mathscr{X} \longrightarrow S$. Let t be a local parameter on T. Put $\mathscr{S} = S \times T$; we have an obvious map $\mathscr{X} \longrightarrow \mathscr{S}$. The sequence

$$0 \longrightarrow \mathscr{O}_{\mathscr{X}} \xrightarrow{t\cdot} \mathscr{O}_{\mathscr{X}} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

induces a long exact sequence

$$\ldots \longrightarrow T^k_{\mathscr{X}/\mathscr{S}} \longrightarrow T^k_{\mathscr{X}/\mathscr{S}} \longrightarrow T^k_{X/S} \longrightarrow T^{k+1}_{\mathscr{X}/\mathscr{S}} \longrightarrow \ldots$$

However, $T^0_{X/S} = 0$, as there are no relative vector fields for $f : X \longrightarrow S$, as the fibres are discrete. By assumption, $T^2_{X/S} = 0$, so the long exact sequence contains the short exact sequence

$$0 \longrightarrow T^1_{\mathscr{X}/\mathscr{S}} \xrightarrow{t\cdot} T^1_{\mathscr{X}/\mathscr{S}} \longrightarrow T^1_{X/S} \longrightarrow 0$$

So, $T^1_{X/S}$ is T-flat, and by the same argument as in the proof of 2.2, τ is conserved in the family. At each point on the smooth curve X_t, $\tau(f_t)$ and $\mu(f_t)$ are equal, and by the conservation of each number in the deformation, the result follows. □

The reader will note that we do not exhibit an explicit isomorphism $\mathscr{M}_f \to T^1_{X/S}$.

Remark 3.3. • Associated to the composite of ring homomorphisms

$$\mathbb{C} \to \mathscr{O}_S \to \mathscr{O}_X$$

there is a long exact sequence of cotangent cohomology, the *Zariski-Jacobi* sequence, which runs

$$\cdots \longrightarrow T^i_{X/S} \longrightarrow T^i_{X/\mathbb{C}} \longrightarrow T^i_{S/\mathbb{C}} \longrightarrow T^{i+1}_{X/S} \longrightarrow \cdots .$$

Here $T^i_{X/\mathbb{C}}$ and $T^i_{S/\mathbb{C}}$ are just T^i_X and T^i_S respectively; since S is smooth, $T^i_S = 0$ for $i \geq 1$ (and recall that $T^0_S = \theta_S$), and it follows that $T^i_{X/S} = T^i_X$ for $i \geq 2$. So the condition in the theorem is equivalent to $T^2_X = 0$.

- It is known that $T^2_X = 0$ for space curves and Gorenstein curves in $\mathbb{C}^4$, so in these cases 3.2 holds, [He], [W]. See also §4 and in particular Corollary 4.3.

- In fact, the proposition holds more generally for all unobstructed curves (that is, curves in which the obstruction map $T^1_{X/S} \to T^2_{\mathscr{X}/\mathscr{S}}$ is equal to 0. For a general smoothable curve one can use a variant of [GL] to see that the *image* of $T^1_{\mathscr{X}/\mathscr{S}} \longrightarrow T^1_{X/S}$ has a dimension that is equal to the dimension of the *component of the deformation space on which the smoothing occurs.* In particular, this dimension is equal to $\mu(f)$, and thus all components have the same, computable dimension.

- In the same way one may prove that the relative T^1 is Cohen-Macaulay over the semi-universal deformation of an unobstructed curve. From this fact one deduces as in [St] that the discriminant is a free divisor.

- In general, one can use the long exact sequences for the relative T^k to find inequalities between μ and partial Euler characteristics of the T^i, see [HW].

4 The first Koszul homology

We will now take a closer look at *space curves* $X \subset \mathbb{C}^3$. The salient feature of this case is that one still has complete control over the structure of the equations of X, which are obtained as the $n \times n$ minors of some $(n+1) \times n$-matrix M. That is, writing $\mathscr{O}$ for the ring of functions on $\mathbb{C}^3$, $\mathscr{O}_X$ has a free $\mathscr{O}$-resolution of the form

$$0 \longrightarrow \mathscr{O}^n \xrightarrow{M} \mathscr{O}^{n+1} \xrightarrow{\Delta} \mathscr{O} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

Here Δ is the map with components Δ_i, which is the minor of M obtained by deleting the i-th row.

Note that $\mathscr{O}^n$ is the module $\mathscr{R}$ of relations between the Δ_i. The module $\mathscr{R}_0$ is generated by the Koszul (or trivial) relations $\Delta_i e_j - \Delta_j e_i$. To describe these elements inside $\mathscr{O}^n$, we use the map

$$\Gamma : \mathscr{O}^{\binom{n+1}{2}} = \wedge^2 \mathscr{O}^{n+1} \longrightarrow \mathscr{O}^n$$

whose entries consist of the $(n-1) \times (n-1)$ minors of M:

$$\Gamma(e_i \wedge e_j) = \sum_{k=1}^{n} \det \Gamma_{i,j}^k e_k,$$

where $\Gamma_{i,j}^k$ is the matrix obtained from M by deleting i-th and j-th rows and k-th column. The composition

$$\wedge^2 \mathscr{O}^{n+1} \xrightarrow{\Gamma} \mathscr{O}^n \xrightarrow{M} \mathscr{O}^{n+1}$$

maps $e_i \wedge e_j$ to the Koszul relation $\Delta_i e_j - \Delta_j e_i$.

The cokernel of the map Γ is the first Koszul homology module of the Δ_i:

$$\mathscr{R}/\mathscr{R}_0 = H_1(\mathscr{O}, \underline{\Delta}) =: H_1$$

It is a fundamental result that H_1 is a *torsion free* $\mathscr{O}_X$-module and this fact has some useful consequences. Let us see why this is the case.

As we are going to work with various minors, it is useful to describe the above resolution in a basis-free manner and work in complete generality. Let R be for the moment any ring, and let $\mathscr{F}$ and $\mathscr{G}$ be free R-modules of rank n and $(n+1)$ respectively. We fix isomorphisms

$$\wedge^n \mathscr{F} \cong R\,, \quad \wedge^{n+1} \mathscr{G} \cong R\,.$$

Note that with these fixed identifications, one has

$$\wedge^{n-p} \mathscr{F} = \wedge^p \mathscr{F}^*\,, \quad \wedge^{n+1-p} \mathscr{G} = \wedge^p \mathscr{G}^*$$

For any R-module N we denote its dual by $N^* := Hom_R(N, R)$.
Given any homomorphism $M : \mathscr{F} \longrightarrow \mathscr{G}$, one derives from it other maps like

$\wedge^p M : \wedge^p \mathscr{F} \longrightarrow \wedge^p \mathscr{G}$, $M^* : \mathscr{G}^* \longrightarrow \mathscr{F}^*$, $\wedge^p(M^*) : \wedge^p \mathscr{G}^* \longrightarrow \wedge^p \mathscr{F}^*$, etc. In particular, we get composable maps

$$\wedge^p \mathscr{F} \longrightarrow \wedge^p \mathscr{G} = \wedge^{n+1-p} \mathscr{G}^* \longrightarrow \wedge^{n+1-p} \mathscr{F}^* = \wedge^{p-1} \mathscr{F} .$$

It follows from the LAPLACE expansion rule of determinants that the composition of these two maps is zero, so that we get *complexes.* (We put the right most module at spot zero, so that the complex lives in negative degrees.)

For $p = 1$, we get the complex $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow R \longrightarrow 0$, for $p = n$ we get its dual $0 \longrightarrow R \longrightarrow \mathscr{G}^* \longrightarrow \mathscr{F}^* \longrightarrow 0$. We are most interested in the case $p = 2$:

$$0 \longrightarrow \wedge^2 \mathscr{F} \longrightarrow \wedge^2 \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow 0.$$

These complexes were first considered in [BuRi] and [AH].

Proposition 4.1. *(cf. [AH], lemma 1.6) Let $I \subset R$ be the ideal generated by the maximal minors of M. Assume that the local cohomologies $H^i_I(R) = 0$ for $i = 0, 1$ (i.e. $depth_I(R) \geq 2$). Then the above complexes are acyclic.*

Proof This is essentially the Lemme d'Acyclicité (see also A.2. and A.3. of [MS]).

For any R-module M, let

$$C^p(M) = \sum_{1 \leq i_1 < \cdots < i_p \leq 1} M_{f_{i_1}, \ldots, f_{i_p}}$$

be the p-th Cech module of M, with respect to a set of generators f_i of I, so that $C^\bullet(M)$ computes the local cohomology $H^\bullet_I(M)$ of M. Let $F_\bullet$ denote any of the complexes whose acyclicty we wish to prove, and consider the double complex $C^\bullet(F_\bullet)$. Since the cohomology of $F_\bullet$ is supported on $Spec(R/I)$, it follows that for any $f \in I$, the complex $(F_\bullet)_{(f)}$ is acyclic, and thus the same is true for all of the complexes $C^p(F_\bullet)$. So the total complex computes the cohomology of $F_\bullet$. On the other hand, the assumption of vanishing of local cohomology tells us that if we first take vertical cohomology, we get an E_1 term looking like

$$\begin{array}{ccc} * & * & * \\ 0 & 0 & * \\ 0 & 0 & * \end{array} \quad .$$

Because the associated spectral sequence also computes the cohomology of the total complex, the proposition follows immediately. $\square$

Corollary 4.2. *1. If R is Cohen-Macaulay, and I an ideal of codimension ≥ 2, then it follows that the codimension equals 2.*

2. Under the above assumptions, we have exact sequences

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

$$0 \longrightarrow \wedge^2 \mathscr{F} \longrightarrow \wedge^2 \mathscr{G} \longrightarrow F \longrightarrow H_1 \longrightarrow 0$$

In particular R/I and H_1 are Cohen-Macaulay.

□

Hence, for a space curve, the module $H_1 = \mathscr{R}/\mathscr{R}_0$ is *torsion free.*

Corollary 4.3. *If X is reduced Cohen-Macaulay of codimension 2, then $T_2^X = 0$.*

Proof From the Lichtenbaum-Schlessinger complex, $T_2^X = ker(H_1 \longrightarrow ...)$. But as T_2 is concentrated on the singular locus of X and H_1 is Cohen-Macaulay, it follows that T_2 must be zero. □

Remark 4.4.

1. The other complexes resolve the higher Koszul homologies H_i. So all of these are Cohen-Macaulay too.
2. It was shown by Huneke [Hu] that the Cohen-Macaulay property of all Koszul homology H_i is preserved under linkage. In particular, these results hold if X is *linked* to a complete interesection. This in particular applies to space curves: use induction and the fact that a curve defined by the minors of an $(n+1) \times n$ matrix is linked via the curve defined by two of these minors to the curve defined by the minors of their common $(n-1) \times n$ submatrix.

Another feature of space curves is the close relation between the $\mathscr{O}_X$- dual ω^* of the dualising module ω_X, and H_1, which implies that ω_X^* behaves well under flat deformation of X, and gives the vanishing of T_X^2.

Start with the resolution of ω_X

$$0 \longrightarrow R \longrightarrow \mathscr{G}^* \longrightarrow \mathscr{F}^* \longrightarrow \omega_X \longrightarrow 0$$

Applying $Hom_X(-, \mathscr{O}_X)$ to this, we get ω_X^* as kernel:

$$0 \longrightarrow \omega_X^* \longrightarrow \mathscr{F} \otimes \mathscr{O}_X \longrightarrow \mathscr{G} \otimes \mathscr{O}_X \ .$$

On the other hand, the composition

$$\wedge^2 \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G}$$

is the zero-map after tensoring with $\mathscr{O}_X$, hence the first map factors to ω_X^*. We let $\mathscr{C}$ be the cokernel:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \wedge^2\mathscr{G} & \longrightarrow & \wedge^2\mathscr{G} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega_X^* & \longrightarrow & \mathscr{F} \otimes \mathscr{O}_X & \longrightarrow & \mathscr{G} \otimes \mathscr{O}_X \\
 & & \twoheaddownarrow & & \twoheaddownarrow & & \\
0 & \longrightarrow & \mathscr{C} & \longrightarrow & H_1 & &
\end{array}$$

A local calculation shows that $\mathscr{C}$ is supported at the singular point; from the snake lemma $\mathscr{C} \longrightarrow H_1$ is injective. As H_1 is torsion free, we get $\mathscr{C} = 0$.

Hence:

$$\wedge^2 \mathscr{G} \longrightarrow \omega_X^* \longrightarrow 0$$

and

$$0 \longrightarrow \omega_X^* \longrightarrow \mathscr{F} \otimes \mathscr{O}_X \longrightarrow H_1 \longrightarrow 0$$

are exact sequences.

Corollary 4.5. *If X is reduced Cohen-Macaulay of codimension two, then $T_X^2 = 0$.*

Proof Dualising the exact sequence

$$0 \longrightarrow \omega_X^* \longrightarrow \mathscr{F} \otimes \mathscr{O}_X \longrightarrow H_1 \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow H_1^* \longrightarrow \mathscr{F}^* \otimes \mathscr{O}_X \longrightarrow \omega_X^{**} \longrightarrow Ext^1(H_1, \mathscr{O}_X) \longrightarrow 0.$$

But $\mathscr{F}^* \otimes \mathscr{O}_X$ has just ω_X as image in ω_X^{**}, hence we have an exact sequence

$$0 \longrightarrow H_1^* \longrightarrow \mathscr{F}^* \otimes \mathscr{O}_X \longrightarrow \omega_X \longrightarrow 0$$

On the other hand, tensoring the resolution of ω_X with $\mathscr{O}_X$ gives an exact sequence

$$\mathscr{G}^* \otimes \mathscr{O}_X \longrightarrow \mathscr{F}^* \otimes \mathscr{O}_X \longrightarrow \omega_X \longrightarrow 0$$

from which we conclude that we have a surjection

$$\mathscr{G}^* \otimes \mathscr{O}_X \twoheadrightarrow H_1^*$$

The cokernel of this map, however, is by definition T_X^2. $\square$

5 Kähler and Dedekind Different

In §2 we defined the Milnor number of a function on an arbitrary curve as the length of the cokernel $\mathscr{O}_X \longrightarrow \omega_X$. This cokernel has the advantage of being flat under any deformation, but has the disadvantage that in general it may have many generators. It would be computationally better to have a description of colength of some Jacobian type ideal $J \subset \mathscr{O}_X$. The first idea that comes to mind is to use the module Θ_X of all vector fields on X, and evalute them on f to get $\Theta_X(f) \subset \mathscr{O}_X$. In this way we get the *Bruce-Roberts* number, [BR]:

$$\mu_{BR}(f) := \dim \mathscr{O}_X/\Theta_X(f)\ .$$

As vector fields do not lift automatically over deformations, this will behave badly non-flat, and thus can not possibly be equal to the topological number $\mu(f)$.

A better idea is to look at the *transpose* of the map $\mathscr{O}_X \longrightarrow \omega_X$. It gives a map $\omega_X^* \longrightarrow \mathscr{O}_X$. Via the transpose of the class map, ω^* is mapped injectively into Θ_X; we denote its image by θ_X^s. The image of ω_X^* in $\mathscr{O}_X$ is the ideal obtained by evaluating these *special vector fields* (the elements of θ_X^s) on f. The ideal

$$\mathscr{D}_D(X/S) := \omega_X^*(f) \subset \mathscr{O}_X$$

is classically known as the *Dedekind different* of the map $f : X \longrightarrow S$; it is the inverse ideal of the complementary module $\mathscr{L}_{X/S}$.

Definition 5.1.

$$\mu_D(f) := \dim \mathscr{O}_X/\mathscr{D}_D(X/S)$$

Proposition 5.2. *Let $f : X \longrightarrow S$ be a function on a space curve singularity. Then:*

$$\mu(f) = \mu_D(f)$$

Proof It follows from §4 that ω_X^* behaves well under deformation. For let $\mathscr{X} \longrightarrow T$ be a flat deformation of X; applying $\mathrm{Hom}_{\mathscr{O}_\mathscr{X}}(\quad, \mathscr{O}_\mathscr{X})$ to the exact sequence

$$0 \longrightarrow \omega_\mathscr{X} \xrightarrow{t\cdot} \omega_\mathscr{X} \longrightarrow \omega_X \longrightarrow 0$$

we get the sequence

$$0 \longrightarrow \omega_\mathscr{X}^* \xrightarrow{t\cdot} \omega_\mathscr{X}^* \longrightarrow \omega_X^* \longrightarrow 0,$$

which we claim is also exact. The only point in doubt is the zero at the right hand side; this comes from the fact that we have a priori generators for ω_X^*: $\wedge^2\mathscr{G} \twoheadrightarrow \omega_X^*$. See also [St], where the result is deduced in a more complicated manner from $T_2^X = 0$.

By dualising the relative class map $\mathrm{cl}_{\mathscr{X}/T} : \Omega^1_{\mathscr{X}/T} \to \omega_{\mathscr{X}}$ we get an inclusion $\omega^*_{\mathscr{X}} \hookrightarrow \theta_{\mathscr{X}/T}$; denoting its image by $\theta^s_{\mathscr{X}/T}$, the exact sequence of ω^*'s becomes

$$0 \longrightarrow \theta^s_{\mathscr{X}/T} \xrightarrow{t\cdot} \theta^s_{\mathscr{X}/T} \longrightarrow \theta^s_X \to 0.$$

Now evaluate the special vector fields on a lifting F of f. Note that a function which is not constant on any branch of a curve cannot annihilate any non-torsion vector field, and that the modules of special vector fields are torsion-free. We therefore get an exact sequence

$$0 \to \mathscr{O}_{\mathscr{X}}/(\theta^s_{\mathscr{X}/T}\cdot F) \xrightarrow{t\cdot} \mathscr{O}_{\mathscr{X}}/(\theta^s_{\mathscr{X}/T}\cdot F) \longrightarrow \mathscr{O}_X/\mathscr{D}_D(X/S) \to 0$$

from which the proposition follows. □

As we know the generators of ω_X^*, we can be more explicit about the ideal $\mathscr{D}_D(X/S)$.

Proposition 5.3. *The element $e_i \wedge e_j$ of $\wedge^2\mathscr{G}$ maps to the vector field*

$$\det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \Delta_i}{\partial x} & \frac{\partial \Delta_i}{\partial y} & \frac{\partial \Delta_i}{\partial z} \\ \frac{\partial \Delta_j}{\partial x} & \frac{\partial \Delta_j}{\partial y} & \frac{\partial \Delta_j}{\partial z} \end{pmatrix}$$

in $\omega_X^* \subset \Theta_X$.

Proof According to [AL], the class map $\Omega_X \longrightarrow \omega_X = Ext^2_{\mathscr{O}}(\mathscr{O}_X, \Omega^3)$ is the following: let $\alpha \in \Omega^1$ be a lift of a 1-form from Ω_X. The element

$$\frac{1}{2}\alpha \wedge d\Delta \wedge dM$$

can be considered as an element of $\mathscr{F}^* \otimes \Omega^3$, which represents the element $cl(\alpha)$ in $Ext^2(\mathscr{O}_X, \Omega^3)$. In order to simplify notation, we will fix a volume form $dV = dx \wedge dy \wedge dz$ and freely divide and multiply by it, when necessary. An element $\phi \in \omega_X^*$ is a special element of $\mathscr{F} \otimes \mathscr{O}_X$; considered as vector field (i.e. in θ^s_X) it is

$$\phi(cl(dx))\frac{\partial}{\partial x} + \phi(cl(dy))\frac{\partial}{\partial y} + \phi(cl(dz))\frac{\partial}{\partial z}$$

Using a basis e_i for $\mathscr{G}$ and f_j for $\mathscr{F}$, the image of the element $e_i \wedge e_j$ under $\Gamma : \wedge^2\mathscr{G} \longrightarrow \mathscr{F}$ is $\Gamma^k_{i,j} f_k$, where $\Gamma^k_{i,j}$ is the minor obtained from M by deleting the i-th and j-th rows and k-th column. These elements generate ω^*_X. For these elements one has:

$$\Gamma(e_i \wedge e_j)(cl(\alpha)) = \frac{1}{2}\alpha \wedge d\Delta_p \wedge d(M^p_k)\Gamma^k_{i,j}$$

where we use the Einstein summation convention. Because $\Delta_p M^p_k = 0$ we have $d\Delta_p M^p_k + \Delta_p dM^p_k = 0$, so, modulo the ideal generated by the Δ_i, we have

$$d\Delta_p \wedge d(M^p_k \Gamma^k_{i,j}) = d\Delta_p \wedge d(M^p_k)\Gamma^k_{i,j}$$

Now we had $M^p_k \Gamma^k_{i,j} = \Delta_i \delta^p_j - \Delta_j \delta^p_i$, hence

$$d\Delta_p \wedge d(M^p_k \Gamma^k_{i,j}) = 2 d\Delta_j \wedge d\Delta_i$$

That is, one has

$$\Gamma(e_i \wedge e_j)(cl(\alpha)) = \alpha \wedge d\Delta_j \wedge d\Delta_i$$

The formula follows (up to a sign). □

The most obvious way to measure ramification, is to look at the module $\Omega_{X/S}$ of relative Kähler differentials. The problem with it is that is does not have flatness properties needed to have a topological meaning. So instead, we look at the Fitting ideal $F_0(\Omega_{X/S}) \subset \mathscr{O}_X$, generated by the maximal minors of a presentation matrix for $\Omega_{X/S}$. It is also known under the name of *Kähler different*: $\mathscr{D}_K(X/S) = F_0(\Omega_{X/S})$. We define

$$\mu_K(f) := \dim \mathscr{O}_X/\mathscr{D}_K(X/S)$$

In general, one has an inclusion

$$\mathscr{D}_K(X/S) \subset \mathscr{D}_D(X/S)$$

and hence an inequality $\mu_D(f) \leq \mu_K(f)$, [Be], [Bu].

For space curves, one has *equality*:

Proposition 5.4. *If $X \subset (\mathbb{C}^3, 0)$ is a space curve singularity, then*

$$\mathscr{D}_K(X/S) = \mathscr{D}_D(X/S)$$

hence

$$\mu(f) = \mu_D(f) = \mu_K(f)$$

Proof The sequence

$$\mathscr{O}_X^{n+1} \xrightarrow{J} \mathscr{O}_X^3 = \Omega_{\mathbb{C}^3} \otimes \mathscr{O}_X \longrightarrow \Omega_{X/S}$$

is a presentation of $\Omega_{X/S}$. Here the matrix J is

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial \Delta_1}{\partial x} & \cdots & \frac{\partial \Delta_{n+1}}{\partial x} \\ \frac{\partial f}{\partial y} & \frac{\partial \Delta_1}{\partial y} & \cdots & \frac{\partial \Delta_{n+1}}{\partial y} \\ \frac{\partial f}{\partial z} & \frac{\partial \Delta_1}{\partial z} & \cdots & \frac{\partial \Delta_{n+1}}{\partial z} \end{pmatrix}$$

So the ideal of 3×3 minors $F_0(\Omega_{X/S})$ is precisely $\mathscr{D}_D(X/S)$. □

Remark 5.5. One has

$$\dim \Omega_{X/S} - \dim T_1^{X/S} = \mu(f)$$

as one proves using $T_2 = 0$ and the argument of §2. Note that

$$T_1^{X/S} = torsion(I/I^2) = \int I/I^2$$

is the module of non-trivial hypersurfaces singular along X. For a complete intersection it is zero.

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